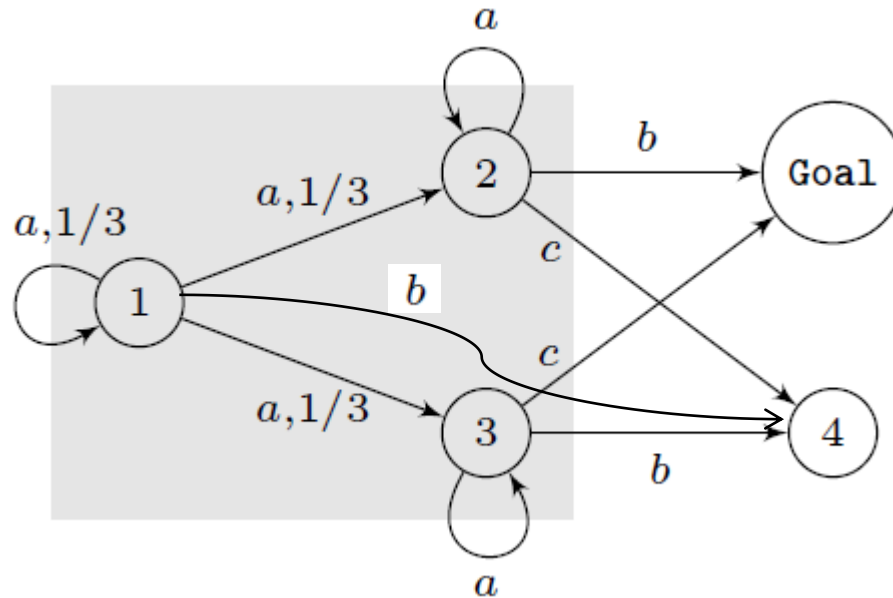


Language of a **unary** PFA
(**u**PFA=Markov Chain)

MDP:

Controller chooses action. Sees state the system in (full information).
Stochastic moves once action is chosen.



Strategy: play a long enough till end up in state 2 or 3.

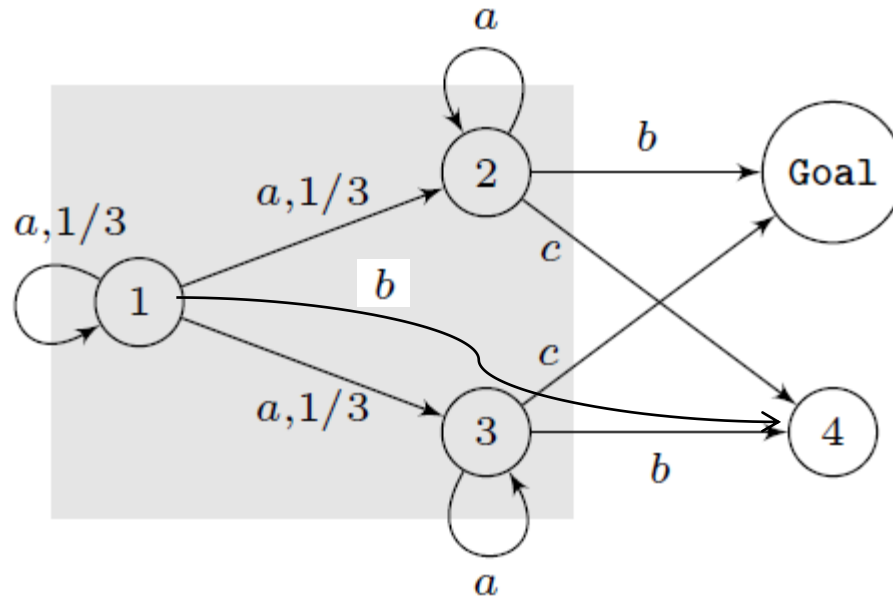
If in 2 play b

If in 3 play c .

State Goal reached with probability 1.

Probabilistic Finite Automata (PFA)

PFA = Blind (PO)MDP = MDP where we don't know the state



No Strategy to reach Goal with proba 1.

Don't know if not in 1. Don't know if in 2 or in 3.

Strategy only based on time, a word. e.g. $a^n b$.

Proba of acceptance: after a^n , probability $(1 - 1/3^n)/2$ to be in 2.

Hence proba $(1 - 1/3^n)/2$ to reach Goal with $a^n b$.

Decidability for PFA?

Undecidable to know whether there exists a strategy ensuring probability 0.5 to reach a state . [Bertoni'71]

Worse: cannot approximate the probability to reach a state.
[Madani, Hanks, Condon AI 2003]

Decidability only for some qualitative questions
(reachability proba=1,safety >0...).

What about **Unary** PFA?

[Turakainen 68, Chadha et al QEST'14...]

Only **one letter**: a. Only one strategy: a....a. It is much simpler.
Correspond to having a **Markov Chain**.

Does there exists n such that the probability to be in Goal after a^n
is at least 0.5?

Complexity?

What about **Unary** PFA?

Does there exist n such that the probability to be in Goal after a^n is at least 0.5?

Surprisingly, **we do not know** if it is decidable or not!
... and not because we are stupid 😊.

Reduction with **Skolem Problem**, decidability open for 40 years.
[Akshay, Antonopoulos, Ouaknine, Worell IPL 2014]

Note: problem orthogonal with **PCTL** (linear complexity on MC)

Decidability for **Unary** PFA?

Does there exist n such that the probability to be in Goal after a^n is at least 0.5?

Surprisingly, **we do not know** if it is decidable or not!

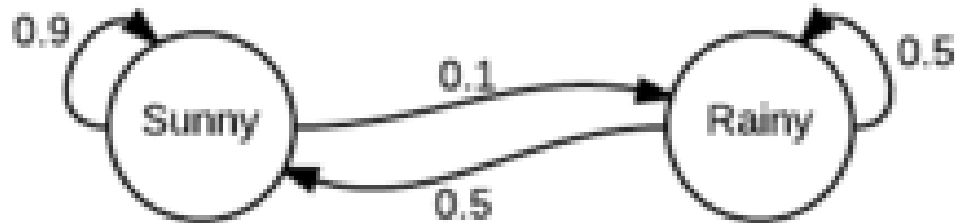
Approximation:

For all ε , does there exist n_ε such that the probability to be in Goal after a^{n_ε} is at least $0.5 - \varepsilon$?

Complexity: **NP** for uPFA. [Chadha et al QEST'14]

More general approximations of **trajectories**, valid for all questions
[Akshay et al. LICS'12]

Trajectories



Setting:

Markov Chain.

Give initial distribution on states (e.g. $P(\text{Rainy}, t=0) = 1$).

Give set of Goal states (e.g. Goal = {Sunny})

Give a probability threshold (e.g. 0.75).

Letters A=above 0.75, B=Below 0.75

$P(\text{Sunny}, 0) = 0$, $P(\text{Sunny}, 1) = 0.5$, $P(\text{Sunny}, 2) = 0.7$, $P(\text{Sunny}, 3) = 0.78$

Trajectory from $P(\text{Rainy}) = 1$ is B B B A A...

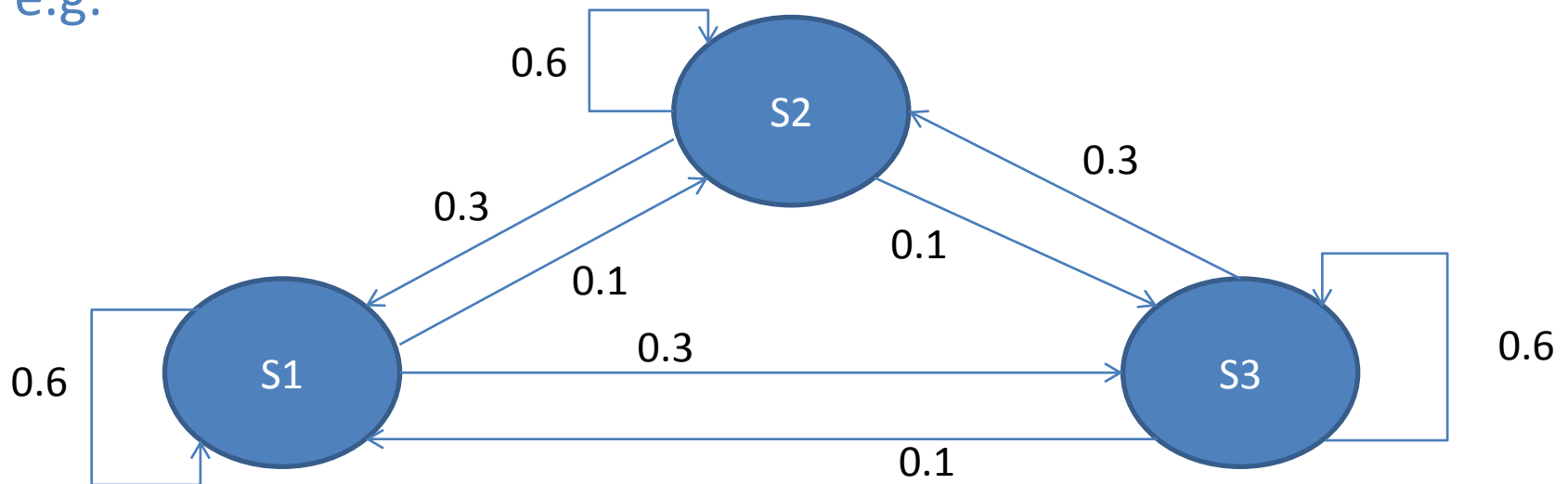
Using trajectories

From trajectory, can answer many possible questions:
Can i reach A? Can it infinitely switch between A and B?...

We do not know if it is decidable or not!

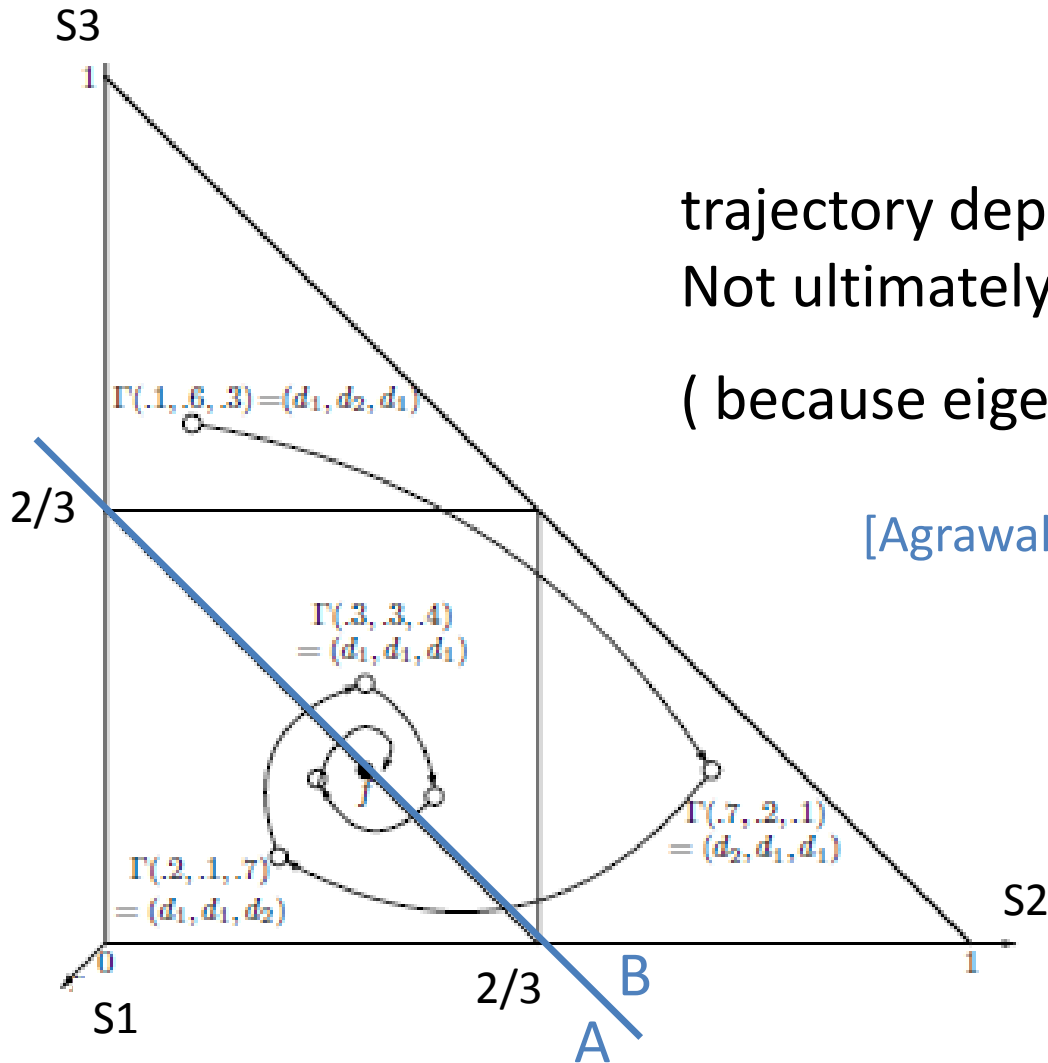
Reason: Trajectories are **not always ultimately periodic**.

e.g:



Threshold $1/3$, Goal = {state S1}, init: $P(S1,t=0)=P(S2,t=0)=1/4$.

What is happening?



trajectory depicted: **BB A BB A (B A...)**
 Not ultimately periodic!

(because eigen values: 1, — —)

[Agrawal, Akshay, G., Thiagarajan, JACM'15]

Trajectories for **Unary** PFA

In general, trajectories of uPFA **not ultimately periodic**.

Results: Every trajectory from any initial distribution **ultimately periodic**
if

All eigen values of uPFA are **distinct roots of real numbers**

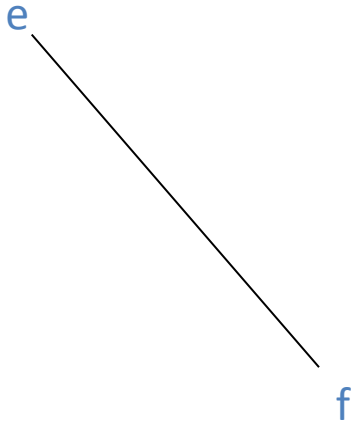
Question: what if (infinite) set of initial distributions?

e.g.: check the behavior vs **small perturbation** of the initial distribution?

Does there **exists an initial distribution** in the set s.t. some property?

Problem: « ultimate periodic » is **not uniform** over all initial distributions

Language



e.g. : Set of Initial distributions:
 $\{\lambda e + (1-\lambda)f \mid \lambda \in [0,1]\}$.

Language = Set of trajectories over all initial distributions.

Result: if all eigen values are **distincts positive real numbers**,
Then language is **regular**.

[Submitted. With Bruno Karelavic and S. Akshay.]

First, under these conditions, all trajectories are **ultimately constant**.

Language

Language = Set of trajectories over all initial distributions.

Result: $1 = \text{ev}_1 > \text{ev}_2 > \text{ev}_3 > \dots > \text{ev}_k > 0$ eigen values ,
Then language is **regular**.

Distincts eigen values \Rightarrow eigen vectors basis.

Decompose any Initial distribution D on e.v. basis
and use $M^n(\text{e_vec.}) = (\text{e_value})^n \text{e_vec.}$

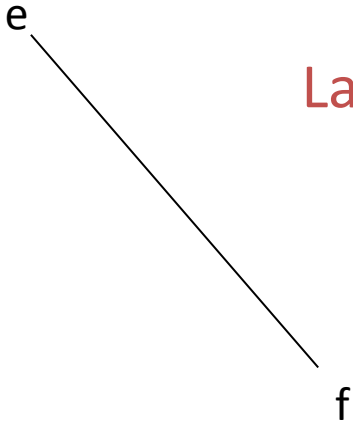
Compute function $u_D(n)$ 0 iff n -th letter of traject is A.

$$U_D(n) = a_1(D) \text{ev}_1^n + a_2(D) \text{ev}_2^n + \dots + a_k(D) \text{ev}_k^n \quad \leftarrow$$

$$\text{Linearity: } u_{\lambda e + (1-\lambda)f} = \lambda u_e + (1-\lambda) u_f,$$

$$\text{That is } a_i(\lambda e + (1-\lambda)f) = \lambda a_i(e) + (1-\lambda) a_i(f).$$

Ultimately constant



Ultimate Language

e A^ω
 $a_1(e) > 0$

Let N such that after N steps,
the trajectories from e, f are $A^\omega B^\omega$

The set of trajectories in (e, f) **after** N steps:

Lemma 1: Included into $B^* A^\omega$

Lemma 2: for all i , exists starting point with $B^i A^\omega$



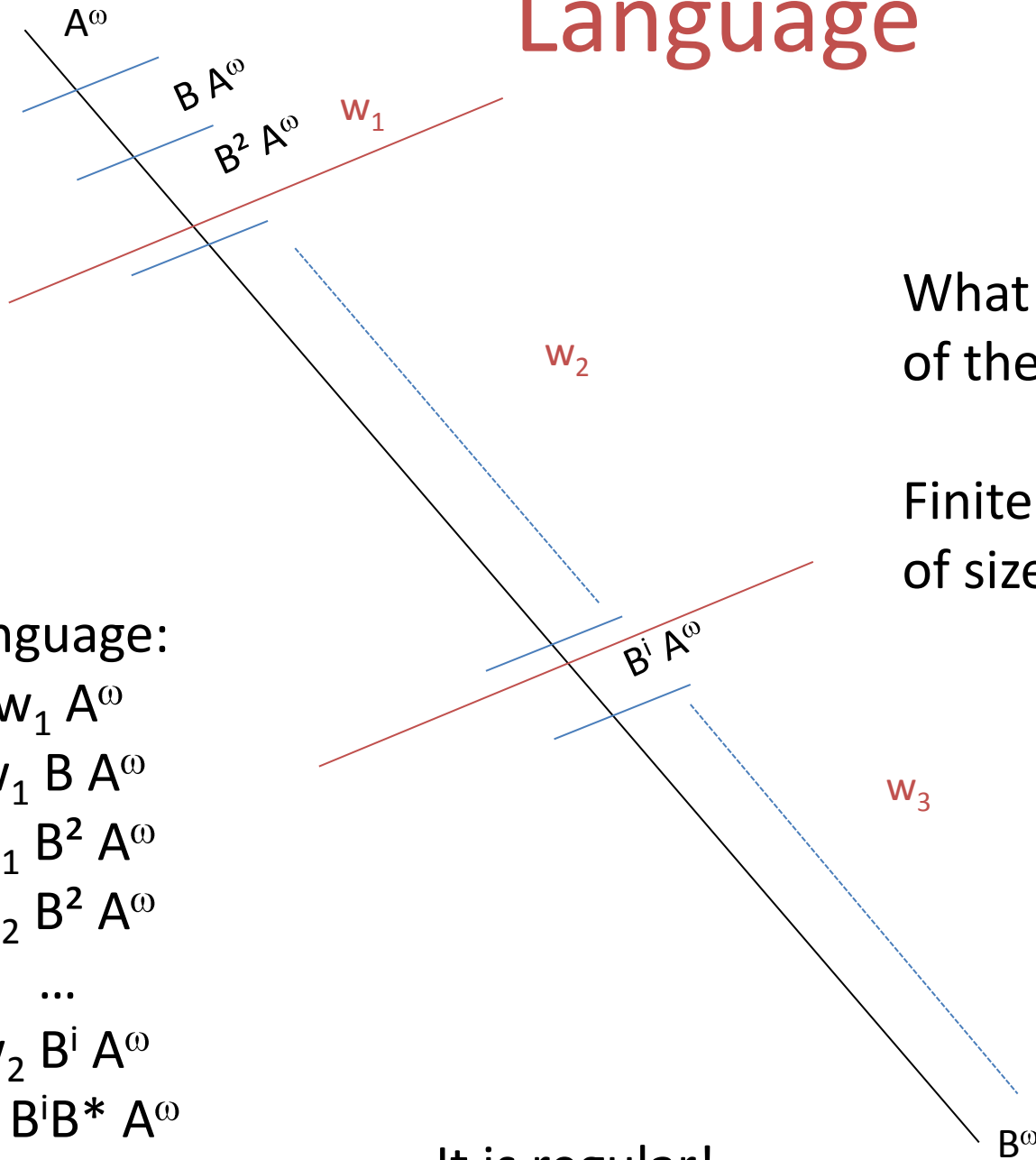
Ultimately constant after $N+i$ steps.
tends to infinity, non uniform

$a_1(f) = 0$
 $a_2(f) < 0$ f B^ω

Language

Language:

$w_1 A^\omega$
 $w_1 B A^\omega$
 $w_1 B^2 A^\omega$
 $w_2 B^2 A^\omega$
...
 $w_2 B^i A^\omega$
 $w_3 B^i B^* A^\omega$

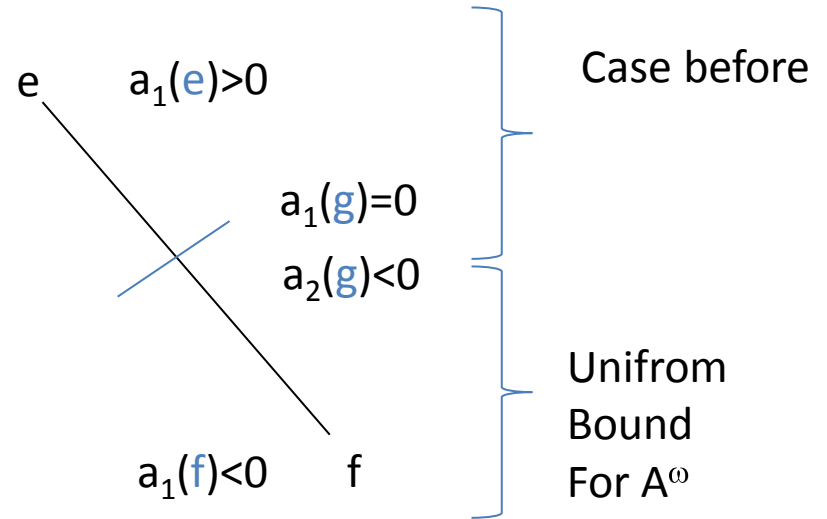
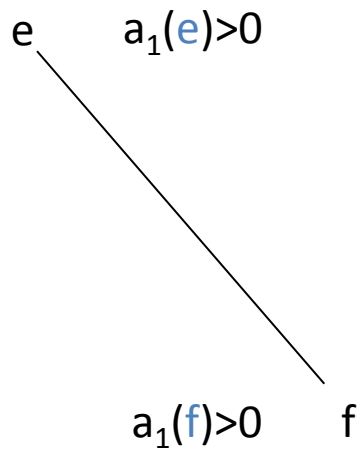


What about the **prefixes** of the N first steps?

Finite number of **prefixes** of size N.

It is regular!

Language in general



max bound e, bound f
is a uniform bound
for ultimately constant.

Language in general

Case of $e_1..e_z$ with

$$a_1(e_1) > 0$$

$$a_1(e_2)=0, a_2(e_2)<0$$

$$a_1(e_3)=a_2(e_3)=0, a_3(e_3)>0$$

...

....

$$\text{Sign}(a_k(e_k))=(-1)^k$$

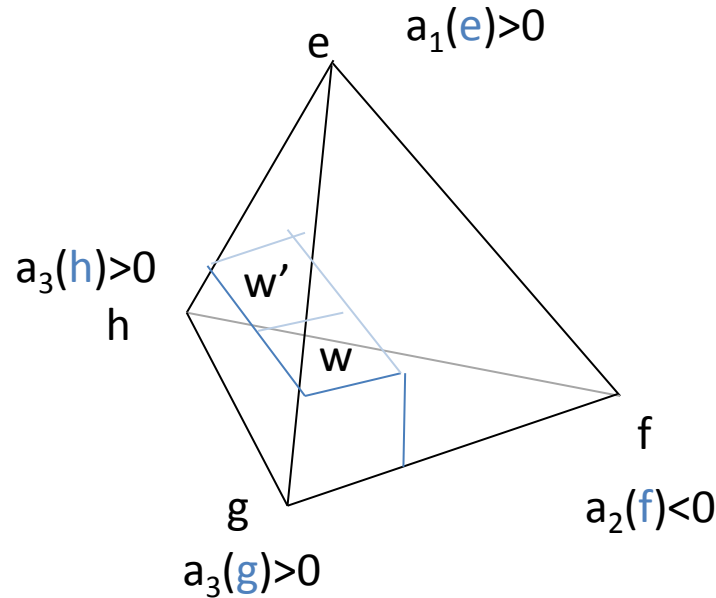
N is the max of the ultimately constant bound for $e_1..e_z$

The set of trajectories in $(e_1..e_z)$ **after** N steps:

Lemma 1: At most z switch, i.e. Included into $B^*A^*B^*A^*... B^* A^\omega$

Lemma 2: for all $i_1..i_z$, exists starting point with $B^{i_1} A^{i_2} .. B^{i_z} A^\omega$

Language in general



Induction on the highest « z » in the space.

In the picture, $z=3$, $n(\text{dimension})=4$

Take w touching (h,g) and touching (h,g,f) with a point not touching h or g

And touching (h,g,f,e) with a point not touching (hfg) .

We can prove that for some i ,

$w A^i B A^\omega$ is a trajectory

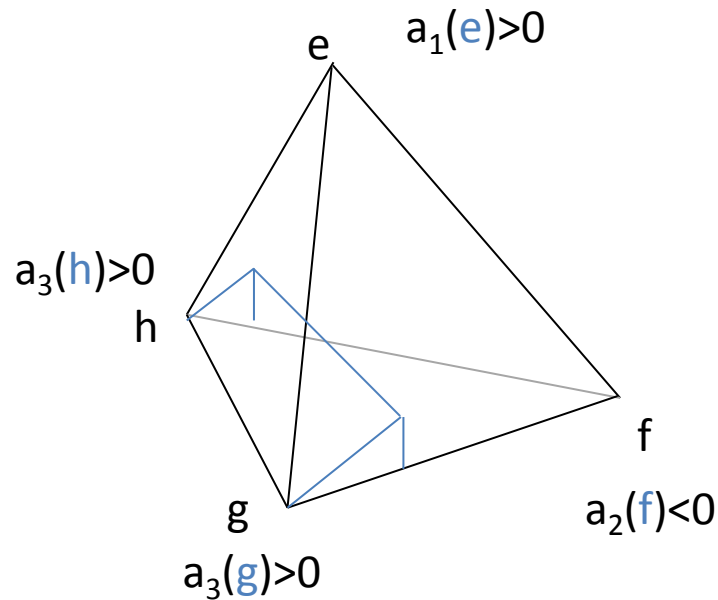
$w A^i B^\omega$ is a trajectory

$w A^\omega$ is a trajectory

continuity argument

$\Rightarrow w A^i A^* B^* A^\omega$ included into trajectory

Language in general



Induction on the highest « z » in the space.

Remove points with trajectory $wA^iA^*B^*A^\omega$ and $w'A^iA^*B^*A^\omega$

It remains a finite union of convex polyhedra with lower « z »

Hence the language is a finite union of regular set, hence it is regular.

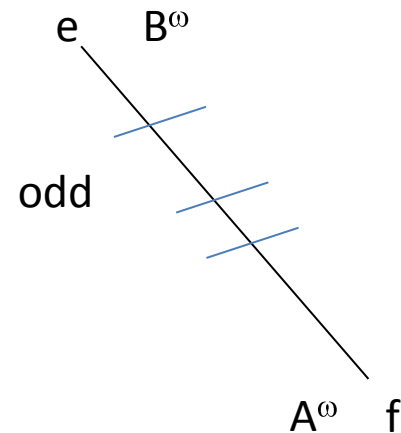
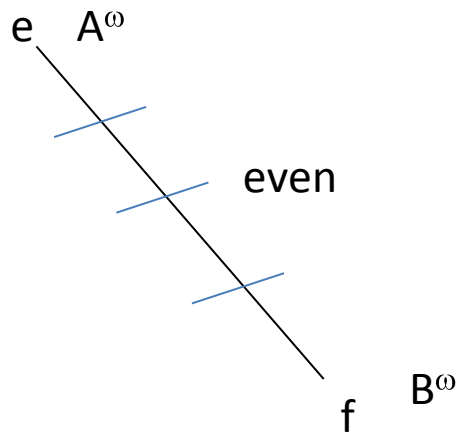
Conclusion

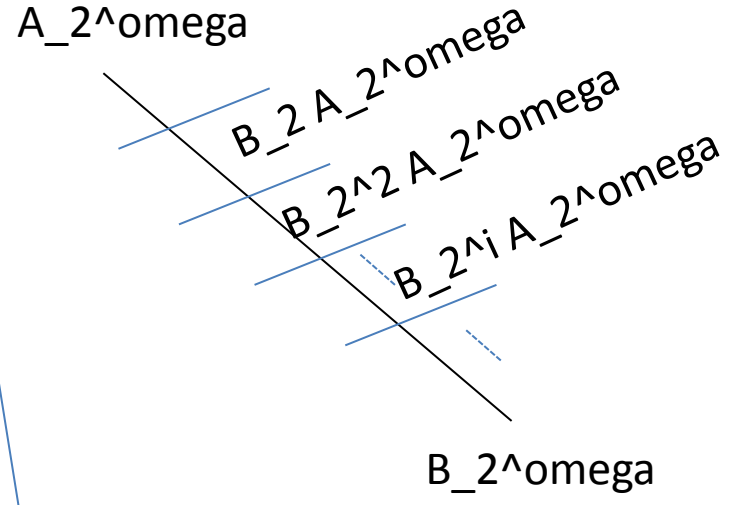
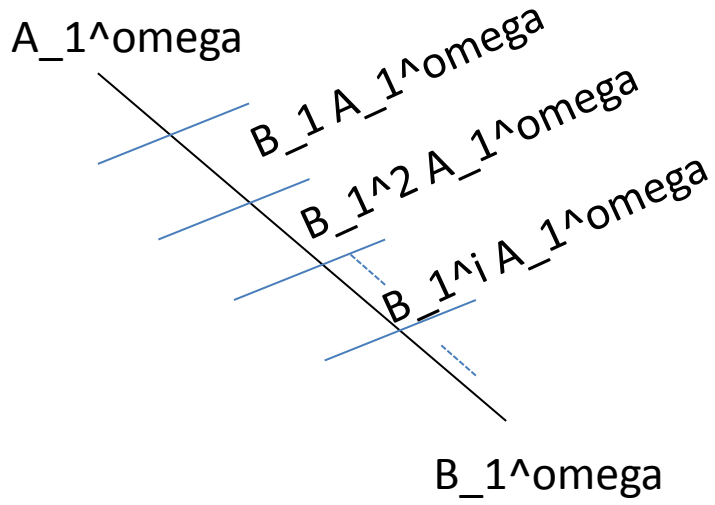
Unary PFA/ Markov Chain:

Simplistic formalism but still **many open problems**

Even taking **strong hypothesis**, not easy to describe their behavior.

For instance, what happens with negative eigen values
(or roots of reals)?

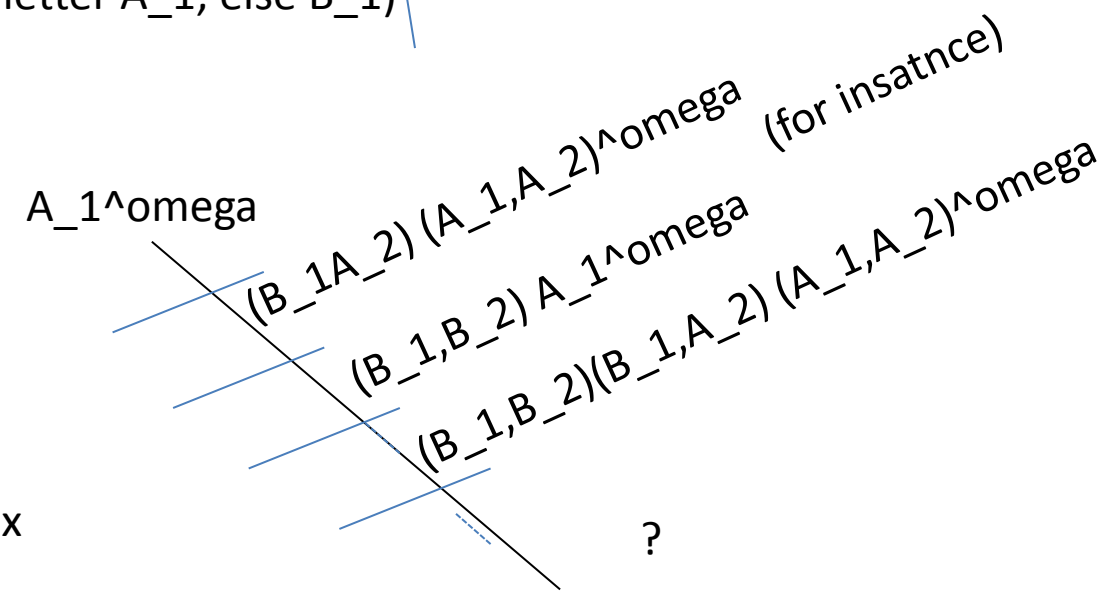




On variable $x_2 > \alpha_2$ (= letter A_2 , else B_2)

On variable $x_1 > \alpha_1$ (= letter A_1 , else B_1)

Together:



On variable x_1, x_2

Prefix is finite, ok, but need to show that suffix (close to $B_1^\omega = B^n A^\omega$ for n big) is regular

