Network allometry

A. Maritan,1 R. Rigon,2 J.B. Banavar,3 A. Rinaldo,4

Abstract. We derive a new allometric scaling law for loopless networks, which we confirm with studies on rivers, exact network results and computer simulations. We provide evidence suggesting that ensemble averaging of the allometric property (where individual realizations are different networks) induces remarkably little scatter compared to the known limit scaling of efficient and compact networks. Our results complement recent work suggesting that network-related allometric scaling in living organisms is regulated by metabolic supply-demand balance, because we show that scaling features are robust to geometrical fluctuations of network properties.

1. Introduction

We analyze data from digital terrain maps (DTM) of river basins [e.g. Rodríguez-Iturbe and Rinaldo, 1997], complemented by studies of a variety of computationally derived or exact recursive constructs, well-established in network studies [Doyle and Snell, 1989; Huber, 1991; Mandelbrot, 1983; Marani et al., 1991; Maritan et al., 1995; Piana, 1890; Scheidegger, 1967; Takayasu et al., 1991; Rodríguez-Iturbe et al., 1992; Rinaldo et al., 1996]. Our focus is on a generalized allometric rule that links the sum of total contributing areas at every site within the basin (which we term M, i.e a proxy of ‘mass’ derived [Banavar et al., 1999, 2001] by the sum of fluxes (and thus flow volumes) within the entire network) to the basin area (termed B, a proxy of metabolic rates).

We postulate that the study of the above relationship is an analog of allometric scaling in biology [McMahon and Bonner, 1983; Calder,
1984; Schmidt-Nielsen, 1994; Damuth, 1998], i.e. any of the large number of relationships that link body size to organ sizes, rates of physiological processes or biological cycle times. These relationships usually take the form of power laws $M = bB^\alpha$, where $M$ is body mass, $B$ is the biological property of interest and $b$ and $\alpha$ are constants specific to the relationship. In fact, West et al. [1997] have suggested that the above relationship can be linked to the geometrical and topological properties of a distribution network sustaining the supply for metabolic activity. Banavar et al. [1999] have shown, such being the case, that directed networks would yield exactly $\alpha = (D + 1)/D$ where $D$ is the dimension of the underlying space ($D = 2$ for planar networks, and $D = 3$ for a network in space). They also showed that arbitrary loopless networks have $\alpha \geq (D+1)/D$ hence suggesting that the purported ubiquity of the value $\alpha \approx 4/3$ in nature [McMahon and Bonner, 1983; Calder, 1984; Schmidt-Nielsen, 1994], though recently questioned [Dodds et al., 2001], may stem from a tendency of networks in nature to attain directed and optimal conformations whatever their ontogeny.

In this paper we predict the extent of the deviations of the exponent $\alpha$ from its exact lower bound depending of the basic scaling properties that characterize a loopless network. We also investigate the effects of ensemble averaging on the allometric property. The issues are deemed of importance because it has recently been shown [Banavar et al., 2001] that metabolic rates depend upon both the capacity of the biological system to deliver metabolites to the tissues and the rate at which they can be taken in, whereas the lack of exact balance has a profound effect on the basic allometric scaling. In particular, the value $\alpha = 4/3$ (for $D = 3$ i.e. in three-dimensional space) is predicted when supply and demand rates balance. This implies that the central allometric tendency would re-
spond to biological necessity. We thus wonder what is the effect of geometrical constraints in living organisms, epitomized by noise in network geometry. In this Letter we show how the basic scaling features change owing to geometrical fluctuations in network shape.

2. Allometry

A river network [see e.g., the literature review in Rodriguez-Iturbe and Rinaldo, 1997] can be defined from a suitable assessment of channelized discrete unit areas where the average elevation, and thus drainage directions, is provided (Fig. 1a). Typically [Dietrich et al., 1992; Montgomery and Dietrich, 1992] one defines from topography the extent of topographically convex areas, where flow directions are multiple, attributing them to hillslopes. The channelized portion of the watershed requires concave topography and the exceedence of a threshold depending on contributing area and slope at a site. The total contributing area $A_X$ associated with an arbitrary $X$-th 'site' of a digital terrain map (DTM) (where source areas are pixels of a DTM grid and the links are determined by drainage directions) is computed by the recursive equation:

$$A_X = \sum_Y W_{XY} A_X + I$$

(1)

where the sum is over all sites $Y$ and $W_{XY}$ is the element of a connectivity matrix $W$ which is non-zero (and positive) only if $Y$ drains into $X$. Notice that $\sum_Y W_{XY} = 1$ for every site $X$ by the requirement of flow continuity. When the connectivity matrix has zero or unit elements, and thus by continuity only one non-zero element per column, one obtains a tree i.e. a unique flow path exists from any site to the outlet. Noninteger values of $W_{XY}$ imply multiple flow directions, appropriate for convex areas, and Eq. 1 may still hold provided the max-
imum eigenvalue of the transpose of the matrix $I - \mathcal{W}$ is zero ($I$ is the identity matrix, see e.g. [Rodriguez-Iturbe and Rinaldo, 1997]). The term $I$ in Eq. 1 is an injection term, usually taken as constant. This is reasonable when considering hydrologic networks developing within the so-called runoff-producing areas [e.g. Rodriguez-Iturbe and Rinaldo, 1997].

One should observe that the assumption of constant injection will play some role in the ensuing derivation. It is also reasonable, owing to the ventilated proportionality of area and fluxes that derives from continuity, that $I$ is proportional to the unit of area, i.e. the 'pixel' area.

Note that the computation of total contributing area from DTM{s} involves considerable computational machinery [Costa-Cabral and Burges, 1994; for a general reference and suitable algorithms see e.g. Turboton, 2001] and that Eq. 1 is a general network equation where, depending on the connectivity specified by $\mathcal{W}$ and the nature of the injection $I$, $A_X$ represents mass aggregation with injection, a problem which has a longstanding scientific tradition that produced several exact results directly relevant this work [e.g. Doyle and Snell, 1989; ; Huber, 1991; Maruni et al., 1991; Takayasu et al., 1991; Colaiori et al., 1997].

The area $A_X$ at any site $X$ within the basin plays the role of the basin metabolic rate $B$, whereas the analog of mass, $M$, is defined by the quantity

$$M = \sum_{Y(X)} A_Y$$

where $Y(X)$ indexes the collection of all sites $Y(X)$ connected to $X$ [Banavar et al., 1999]. Note that $\sum_{Y(X)} A_Y$ does not add to the basin area $A_X$. We now specify the nature of the scaling exponent of allometric plots relating $M$ and $B$ via a power law, $M \propto B^\alpha$. We recall that for directed networks in two dimensions ($D = 2$), the key prediction [Banavar et
al., 1999] is that the lowest attainable value is \( \alpha = (D + 1)/D = 3/2 \). The fractal nature of river networks stems from the fact that embedded within any basin are other sub-basins with similar features reflected in linked scaling exponents [Maritan et al., 1996; Rigon et al., 1996; Rinaldo et al., 1999; Dodds and Rodriguez, 2000]. The mass \( M \) in Eq. 2 associated with any site \( X \) relates to its area \( A_X \) via:

\[
M = A_X < L_X > I
\tag{3}
\]

where \( < L_X > \) is the mean distance of the sites within the sub-basin to its outlet \( X \) measured along the network. Here \( I \) needs be constant. In fact, from Eq. 1 one gets:

\[
A_Y = \sum_{Z} \sum_{n=0}^{\infty} (W^n)_{YZ} I
\tag{4}
\]

Notice that \( W^n_{YZ} = 0 \) if \( n > L_{ZY} \), where \( L_{ZY} \) is the longest distance between \( Z \) and \( Y \) measured along the network in the flow directions. This distance is well defined once we have assumed that no loops are allowed. Since \( \sum_Y W_{YZ} = 1 \) for all \( Z \), then \( \sum_{Y \in \text{basin}} (W^n)_{YZ} = 1 \) (or 0) is \( n \leq L_{ZX} \) (or conversely if \( n > L_{ZX} \)). Thus from Eq. 3 one has \( \sum_{Y \in \text{basin}} A_Y = I \sum_{Z \in \text{basin}} L_{ZX} \). Eq. 2 follows since \( \sum_{Z \in \text{basin}} 1 = A_X \), i.e. the number of all 'sites' connected to \( X \) times the unit area is its contributing area, and \( < L_X > = \sum_{Z \in \text{basin}} L_{ZX} / \sum_{Z \in \text{basin}} 1 \). Alternatively one might have shown the same result by defining \( \gamma_{Z\rightarrow Y} = 1 \), if \( Z \) is upstream of \( Y \) and zero otherwise. Thus \( \sum_Y \gamma_{Z\rightarrow Y} \gamma_{Y\rightarrow X} = L_{ZX} \) where \( L_{ZX} \) is the distance along the network between \( Z \) and \( X \).

Eq. 2 follows noting that \( A_Y = \sum_Z \gamma_{Z\rightarrow Y} \).

It should be noticed that only from this point we restrict our attention to the case of spanning trees, i.e. networks where \( W_{XY} = 1 \) or 0 and the network formed by the links with \( W_{XY} = 1 \) is spanning the entire area. In this case if \( Y \) is upstream of \( X \) there is only one path joining \( Y \) to \( X \) along the network. Such being the case,
we claim that:

\[ < L_X > \propto A_X^h \]  \hspace{1cm} (5)

where \( h \) is the so-called Hack’s exponent relating the upstream length to the total contributing area [Hack, 1957]. Hack’s 'law', whose validity and meaning have been much debated in the scientific literature [e.g. Mandelbrot, 1983; Mesa and Gupta, 1987; Marilán et al., 1996; Rigon et al., 1996; Rodríguez-Iturbe and Rinaldo, 1997; Rinaldo et al., 1999], is commonly defined relating mainstream length, say \( L \), to drainage area \( A \) at the closure rather than everywhere within the basin. The validity of Eq. (5) also within nested subbasins has been suggested to be a strong version of Hack’s law and a proof of the embedded similarity in the network structure – hence of the fractal structure of river basins [Rigon et al., 1996]. We therefore assume, following [Rigon et al., 1996], that the main stream, sometimes rather arbitrarily defined but most commonly taken as the longest flow path length and thereby a single flow path, is proportional to the mean length upstream of \( X \), i.e. \( L \propto < L_X > \).

From Eqs. (3) and (5), owing to the allometric scaling \( M \propto B^\alpha \), one obtains our final result:

\[ \alpha = 1 + h \]  \hspace{1cm} (6)

which exceeds the limit scaling \( \alpha = 3/2 \) whenever \( h > 1/2 \). Notice that Hack’s exponent \( h \) would be equal to \( 1/2 \) only if geometric similarity is to be preserved as a basin increases in area while preserving its shape. Typical observational values range about \( h \sim 0.57 \) [Hack, 1957]. Fig. 2 shows typical allometric plots for four river basins of various sizes, geology, vegetational state and digital terrain map properties. The observed values of \( \alpha \) range from 1.50 to 1.59, and the scatter of the individual curves (which we term intra-network scaling to suggest that the noise within same ‘species’ is
studied), relative to the nested subbasins of the same basin, is remarkably small. Table 1 shows a summary of direct calculations of the $\alpha$ exponent with error estimates for many more natural basins. The Table also provides independent verifications of Eq. (6) because we did measure directly $h$ from data [e.g. Rigon et al., 1996]. Notice that the scaling exponents shown in the Table provide quantitative measures for directedness and fractality, and that their consistent linkage provides an exhaustive statistical description [Maritan et al., 1996; Rinaldo et al., 1999; Dodds and Rothman, 2000]. From the results of Table 1 we observe a near perfect computational match. It is also interesting to note that individual networks conform to scaling laws that can significantly differ from the lower bound $\alpha = 3/2$.

To investigate the extent of the deviations of $\alpha$ from its lower limit we have also carried out studies on a broad class of statistical and deterministic network models, some amenable to exact solution. These include stochastic constructs such as the Scheidegger network [Scheidegger, 1967](Fig. 1b), whose scaling exponents are known exactly [Huber, 1991]; Peano's network [Peano, 1890; Mandelbrot, 1983] (Fig. 1c), which is a deterministic fractal, whose exact multiscaling properties have been addressed [Manini et al., 1991; Colaiori et al., 1997]; optimal channel networks [Rodriguez-Iturbe et al., 1992; Rinaldo et al., 1992; Rodriguez-Iturbe and Rinaldo, 1997] (Fig. 1d) whose fractal characteristics are obtained through a specific network selection process. We find (Table 2) robust allometric scaling, with the exponent detectably different from the limiting value of $\alpha = 3/2$.

Tables 1 and 2 contain our estimates for both $\alpha$ and $h$, along with a set of related scaling exponents described in the captions. Excellent agreement is found between the directly determined value of the allometric scaling exponent
and its relationship to Hack’s exponent. In all cases $\alpha \geq 3/2$ as predicted [Banavar et al., 1999].

Inter-species allometric plots correspond to an ensemble average of data from different populations of networks. In river basins, $\alpha$ varies in the range $1.50 - 1.60$ with relatively small scatter in the individual curves. The ensemble average built by mixing different sub-basins nested in the same basin with other basins and their sub-basins is shown in Fig. 3. The scatter is higher, mimicking that of most macroecological data sets, and the mean value of $\alpha$ is statistically indistinguishable from $3/2$. This result matches an earlier, probably overlooked result of the geomorphological literature. It has been shown that the ensemble average of Hack’s exponent from different basins and nested sub-basins covering over 11 orders of magnitude is indistinguishably close to $h = 1/2$ [Montgomery and Dietrich, 1992], a fact that puzzled investigators for a long time [see e.g. Rodriguez-Iturbe and Rinaldo, 1997]. This would suggest that the inter-species allometric scaling exponent would be $\alpha = 3/2$. Notice that topologically random networks also have asymptotically $h = 0.5$ [Mesa and Gupta, 1987] and hence $\alpha = 3/2$.

We have also studied the effect of ensemble averaging of networks both in the bulk or at the boundaries of multiple-outlet optimal networks where competition for drainage occurs because of the constraint of the fixed total area being drained. The allometric exponent in the bulk is consistent with our previous results while that of areas seeded in the boundaries is systematically lower and approaches the $3/2$ value which is the theoretical limit for infinite size.

3. Conclusions

Our results demonstrate that individual network forms (referred to as intra-species scaling)
have values of the allometric exponent that are sensitive probes of the network structure, which are directly related to the underlying fractal structure of the network. Ensemble averages, the analog of inter-species scalings, smooth out details, enhance the scatter and lead to an $\alpha$ exponent that approaches the limiting value obtained for directed networks.

Our results demonstrate the robustness of the central tendency of allometric scaling in network structures. However, the sensitivity in probing the geometrical variability of network shapes is much refined when studying homogeneous geometries reflected in consistent deviations of the allometric scaling exponent from the limit values $\alpha = 3/2 (D = 2)$ for planar networks or $\alpha = 4/3 (D = 3)$ in plants and living organisms.

We thus conclude that our results complement nicely recent results [Banavar et al., 2001] showing that the central tendency in allometric scaling is regulated by metabolic supply-demand balance. In fact, we suggest that unavoidable noise in the geometrical arrangements of the parts and the whole of a living network does not alter the basic tendency provided by biological needs. Thus the ubiquity of the so-called quarter-power law may be a consequence of the robustness of network properties with respect to geometrical fluctuations in systems where supply rates are independent of body mass. Thus the purported recurrence of the so-called 3/4 law may consist of chance, owing to the robustness to noisy geometry and topology, and necessity dictated by supply-demand balance.

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Figure Captions

Figure 1. A sample of networks whose allometry has been measured: a) an example of a river network extracted from a digital terrain map, using the criterion of critical slope-dependent support area [Dietrich et al., 1992] (Dry Fork river, West Virginia, 1442 km², mainstream length $L = 97.8$ km, characteristic longitudinal length (i.e. the maximum eurian distance from a point on the boundary to the outlet [Rigon et al., 1996] $L = 66.9$ km, extracted from a $30 \times 30$ DTM). The drainage density is uneven owing to the choice of slope-dependent geomorphic threshold [Dietrich et al., 1992]; b) an example of Scheidegger’s network [Scheidegger, 1967, Takayasu et al., 1991; Huber, 1991]; c) Peano’s basin [Mandelbrot, 1983; Maruni et al., 1991; Peano, 1890]; d) an optimal channel network (OCN) within a rectangular domain [Rodriguez-Iturbe et al., 1992; Rodriguez-Iturbe and Rinaldo, 1997]; e) a non-directed network [Rinaldo et al., 1996]. For a brief description of the (stochastic or deterministic) network generators, see the caption of Table 2.

Figure 2. Allometric plots of $M$ versus $B$ for real river networks. Double logarithmic plots of $\sum_{Y \in X} A_Y$ versus $A_X$ for four river networks characterized by different climates, geology, geographic locations and coarse-grained topographic information (Dry Fork, West Virginia, 1442 [km²], digital terrain map (DTM) size $30 \times 30$ m²; Guyandotte, West Virginia, 586 [km²], DTM size $30 \times 30$ m²; Island Creek, Idaho, 260 [km²], DTM size $20 \times 20$ m²; Tirso, Italy, 2024 [km²], DTM size $237 \times 237$ m²). The circles denote the mean values obtained from the experimental points by binning total contributing areas and computing the ensemble average of the sum of the inner areas for each subbasin within the binned interval. The units chosen for areas are km². Unlike in previous analyses, in calculating $M = \sum_{Y \in X} A_Y$ we did not include the area $A_X$ seeded in the site $X$, which may be thought of as a correction to scaling contribution. Notice that the curves are arbitrarily offset vertically (with proper scale for the lowest) to distinguish the different basins. Notice also that averages binning several subbasins of roughly the same area are stopped when binned areas are too few.

Figure 3. Ensemble average of several river basins, including both nested subbasins and different basins. A line with slope $\alpha = 3/2$ is shown as a guide to the eye.
Table Captions

Table 1. A summary of scaling exponents of river networks, which exhaustively [Rinaldo et al., 1999] characterize in a quantitative manner the fractal structure of river networks. The sinuosity exponent $d_t$ is measured through log-log plots of the fluvial path (mainstream) length $L$ (the measure of the longest flow path from source to outlet) vs the characteristic Eulerian basin size $L_0$ (the longest distance from a point on the boundary to the outlet) [see for details Maritan et al., 1995]; Hack’s exponent $h$ is evaluated from the slope of log-log plots of the ratio of consecutive moments of the distribution of fluvial lengths, $<L_X^h>/ <L_X^{h-1}>$ with $n$ equal to 2, 3, $\ldots$, versus the basin size $A_X$, postulated by a finite-size scaling ansatz [Rigon et al., 1996]; the Hurst coefficient $H$ is evaluated through finite-size scaling for the distribution of total contributing areas [Maritan et al., 1995]; and the allometric exponent $\alpha$ is directly evaluated as discussed in the text (see Fig. 2). The exponents in both this Table and Table 2 confirm the validity of two scaling relations: $\alpha = 1+h$ and $h = d_t/(1+H)$ [Maritan et al., 1995]. Note that the exponents in river basins and in the computationally generated networks are obtained by fitting the data and may not correspond to the values that one obtains in the (computationally inaccessible) limit of infinite size.

Table 2. Scaling exponents for several networks. Schneidegger’s directed network (Fig. 1b) is constructed by a stochastic rule – with even probability, a walker chooses between right or left forward sites only. The model was devised with reference to drainage patterns of an intramontane trench [Schneidegger, 1967] and maps exactly into a model of random aggregation with injection Takayasu et al., 1991] and the time activity [Dhar, 1999] of a self-organized critical avalanche [Bak et al., 1987; Bak, 1996] and an exact solution is known [Huber, 1991]. Peano’s network (Fig. 1c) is a deterministic recursive construct whose main topological and scaling features, some involving exact multifractals, have been solved analytically [Maruni et al., 1991; Colaiori et al., 1997]. The basic prefractal is a square cross seeded at an angle, and all subsequent subdivisions cut in half each branch to reproduce the prefractal on four, equal subbasins. Here the process is shown at the 11-th stage of iteration. An optimal channel network [Rodriguez-Iturbe et al., 1992] (Fig. 1d) is obtained by selecting the spanning network configuration that minimizes the total potential energy (or, equivalently, the total energy dissipation) of the system given by $\sum_X A_X^{1/2}$ where the
sum is over all sites $X$ of the basin and, as defined earlier, $A_X$ is the total contributing area at $X$. One obtains a rich structure of scaling optimal forms that are known \cite{Rinaldo et al., 1992} to closely conform to the scaling of real networks, even in the case of unrealistic geometric boundaries. We have also designed truly non-directed networks (Fig. 1e) by considering optimal channel networks at very high thermodynamic temperatures \cite{Rinaldo et al., 1996} using a Metropolis algorithm \cite{Metropolis et al., 1953}. These are called hot OCNs in the legend. The superscripts $(r)$ and $(f)$ denote networks grown within rectangular domains and domains with fractal boundaries respectively and show that the effects of the constraint on the overall organization of the network are rather mild.
Table 1.

<table>
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<tr>
<th>River basin (location)</th>
<th>Area [km²]</th>
<th>L [km]</th>
<th>L₁ [km]</th>
<th>d₀</th>
<th>h</th>
<th>H</th>
<th>1 + $rac{h}{L}$</th>
<th>1 + h</th>
<th>α</th>
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<td>0.55</td>
<td>0.92</td>
<td>1.55</td>
<td>1.55</td>
<td>1.56 ±0.02</td>
</tr>
</tbody>
</table>
Table 2.

<table>
<thead>
<tr>
<th>Network</th>
<th>$d_i$</th>
<th>$h$</th>
<th>$H$</th>
<th>$\frac{1 + \mu}{\mu}$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scheidegger</td>
<td>1</td>
<td>2/3</td>
<td>1</td>
<td>5/3</td>
<td>1.67 ±0.01</td>
</tr>
<tr>
<td>Peano</td>
<td>1</td>
<td>1/2</td>
<td>1</td>
<td>3/2</td>
<td>1.50 ±0.02</td>
</tr>
<tr>
<td>OCN (r)</td>
<td>1.05</td>
<td>0.57</td>
<td>0.84</td>
<td>1.57</td>
<td>1.57 ±0.02</td>
</tr>
<tr>
<td>OCN (f)</td>
<td>1.05</td>
<td>0.56</td>
<td>0.88</td>
<td>1.56</td>
<td>1.56 ±0.02</td>
</tr>
<tr>
<td>Hot OCN (r)</td>
<td>1.23</td>
<td>0.67</td>
<td>0.84</td>
<td>1.67</td>
<td>1.67 ±0.01</td>
</tr>
<tr>
<td>Hot OCN (f)</td>
<td>1.24</td>
<td>0.67</td>
<td>0.85</td>
<td>1.67</td>
<td>1.67 ±0.02</td>
</tr>
</tbody>
</table>