

Mending Partial Solutions with Few Changes

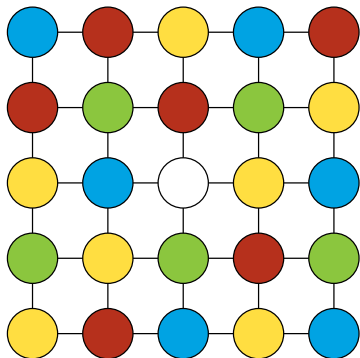
Darya Melnyk¹ and Jukka Suomela¹ and Neven Villani^{1,2}

¹ Aalto University, Espoo, Finland

² ENS Paris-Saclay, Université Paris-Saclay, France

OPODIS 2022, December 14th, UCLouvain

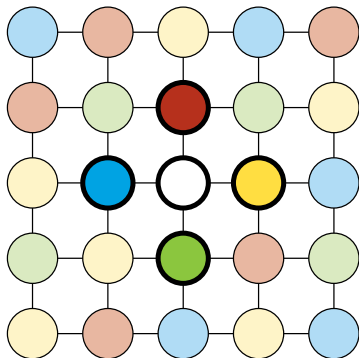
Fixing a partial solution



Partial 4-coloring:

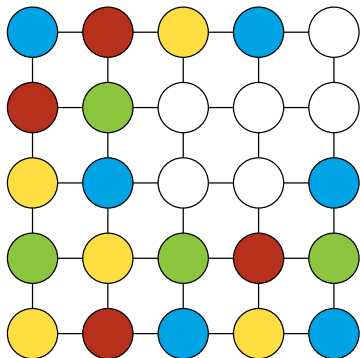
- 4 colors + unlabeled
- no two adjacent vertices of the same color
- anything is allowed next to an unlabeled vertex

Fixing a partial solution



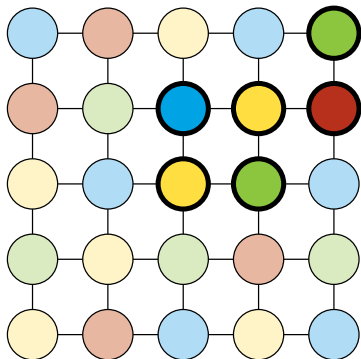
Impossible to complete the partial solution : no color available for the unlabeled vertex

Fixing a partial solution



Erase some labels...

Fixing a partial solution

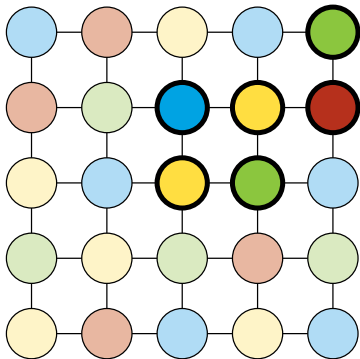


...then recolor.

Number of erased labels
→ volume of the mend

Greatest distance to erased
label → radius of the mend

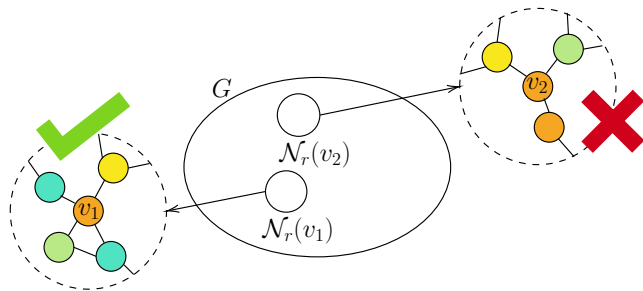
Fixing a partial solution



Mend with volume 5
and radius 4

LCL problems (Locally Checkable Labelings)

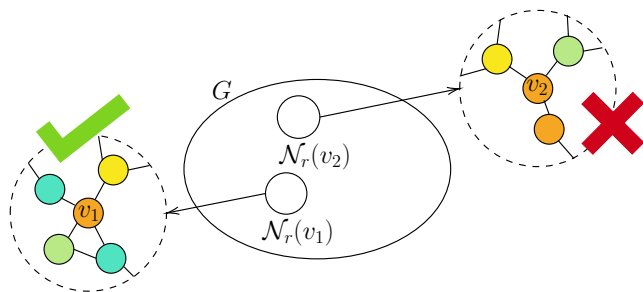
- input is a graph $G = (V, E)$
- assign to each vertex v a label $\lambda(v) \in \Sigma$
- φ a verifier is local with radius r
- all labeled neighborhoods $\mathcal{N}_r(v)$ must be accepted by φ



LCL problems (Locally Checkable Labelings)

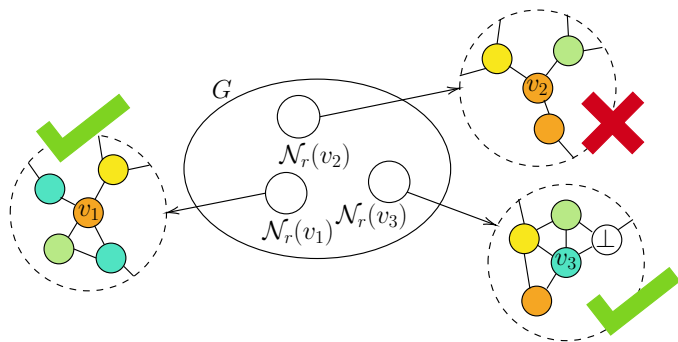
LCLs: n -vertex-coloring, n -edge-coloring, sinkless orientation, maximal independent set, minimal edge cover

Not LCLs: maximum independent set, minimum cut, tree height



Partial solutions of LCLs

- fresh label \perp : “unlabeled”
- any neighborhood that contains \perp is accepted



Mending LCLs

For an LCL problem Π ,

An instance of a mending problem is (G, λ, u)

- G is a graph
- λ is a partial solution on G
- u is unlabeled in λ

Task: produce λ'

- λ' must be a partial solution
- u must be labeled in λ'
- λ' must not introduce new unlabeled vertices
- λ' should minimize the cost $|\{v : \perp \neq \lambda(v) \neq \lambda'(v)\}|$

The mending volume of a problem is the optimal cost of the worst-case instance

Why mend ?

Studied a lot: given an LCL, how hard is it to *solve* it ?

Our question: given an LCL, how hard is it to *mend* it ?

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For distributed systems

- less costly than to compute a new solution from scratch
- less disruption

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For distributed systems

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For centralized parallel algorithms

- mending procedure provides sequential solving algorithm
- efficient mendability implies efficient parallel solvability

Prior work: landscape of the mending radius

Setting	$O(1)$	$\Theta(\log n)$	$\Theta(n^\alpha)$	$\Theta(n)$

Landscape of the mending radius

From “Local Mending” (2021) by A. Balliu, J. Hirvonen, D. Melnyk, D. Olivetti, J. Rybicki, J. Suomela; <https://arxiv.org/abs/2102.08703>

Prior work: landscape of the mending radius

Setting	$O(1)$		$\Theta(\log n)$		$\Theta(n^\alpha)$	$\Theta(n)$
Paths, Cycles	✓	×	×	×	×	✓

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Paths, Cycles	✓	×	×	×	×	✓
Trees	✓	×	✓	×	×	✓

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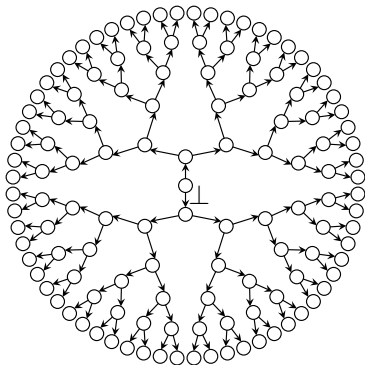
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Trees	✓	×	✓	×	×	✓
General graphs	✓	×	✓	?	✓	✓

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Same radius, different volumes



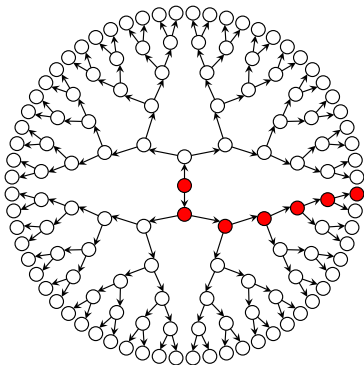
On rooted binary trees:

- $\Sigma = \{\text{red, white}\}$
- root must be red
- a red vertex must have i red children

Radius ?

Volume ?

Same radius, different volumes



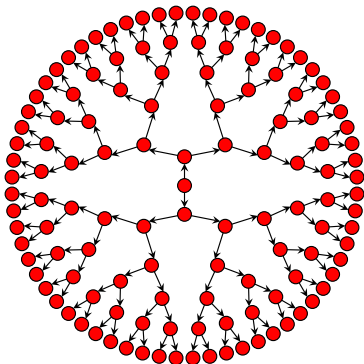
On rooted binary trees:

- $\Sigma = \{\text{red, white}\}$
- root must be red
- a red vertex must have $i = 1$ red children

Radius $\log_2 n$

Volume $\log_2 n$

Same radius, different volumes



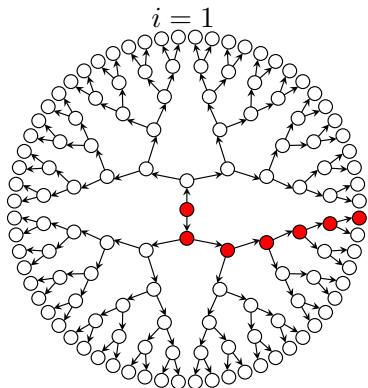
On rooted binary trees:

- $\Sigma = \{\text{red, white}\}$
- root must be red
- a red vertex must have $i = 2$ red children

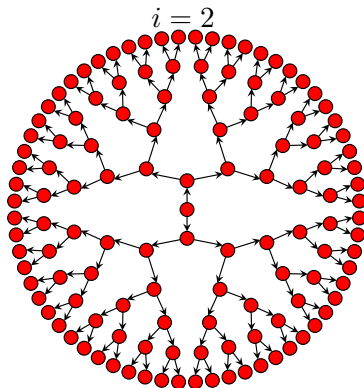
Radius $\log_2 n$

Volume n

Same radius, different volumes



Volume $\Theta(\log n)$



Volume $\Theta(n)$

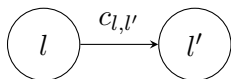
- problems with the same radius can have different volumes

Propagation problems in infinite regular trees

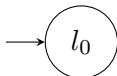
Problem restrictions:

- infinite Δ -regular rooted trees
- only labeling constraints of the form

any vertex labeled l must have at least $c_{l,l'}$ children labeled l'

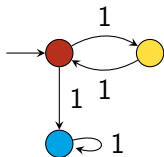


the root must be labeled l_0

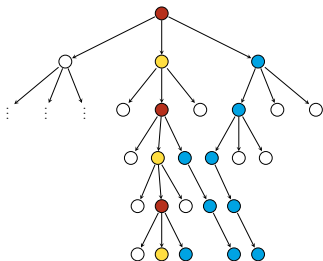
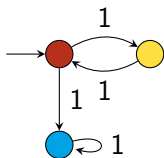


This class is complex enough to exhibit previously unobserved complexities

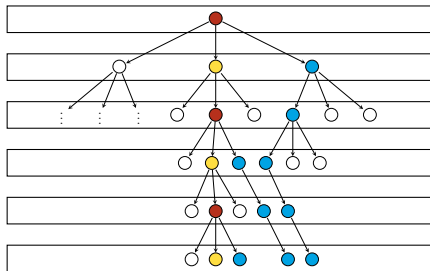
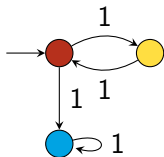
Layer partition of a mend



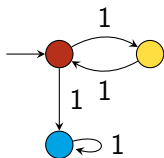
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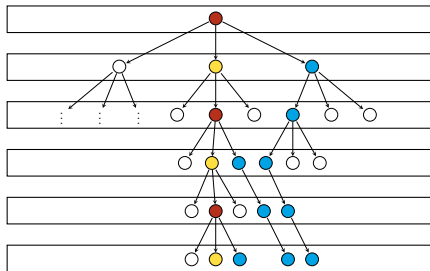
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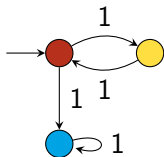
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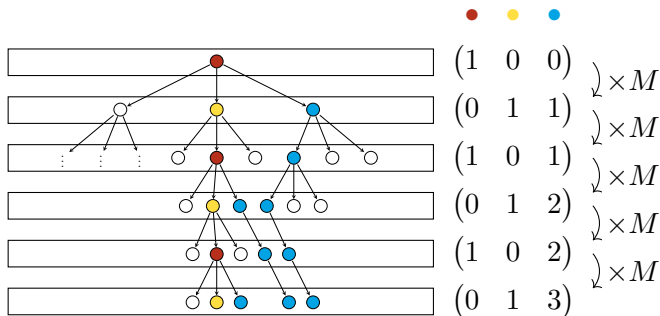
$$M = \begin{pmatrix} \nearrow & \bullet & \bullet & \bullet \\ \bullet & 0 & 1 & 1 \\ \bullet & 1 & 0 & 0 \\ \bullet & 0 & 0 & 1 \end{pmatrix}$$



Layer partition of a mend



$$M = \begin{pmatrix} \nearrow & \bullet & \bullet & \bullet \\ \bullet & 0 & 1 & 1 \\ \bullet & 1 & 0 & 0 \\ \bullet & 0 & 0 & 1 \end{pmatrix}$$



Complexity landscape

Matrix growth	$\rightarrow 0$	$\Theta(1)$	$\text{poly}(n)$	$\exp(n)$
Mending volume	$\Theta(1)$	$\Theta(\log n)$	$\text{poly}(\log n)$	$n^\alpha, 0 < \alpha \leq 1$

For a well-chosen degree Δ

Overview

Setting	$O(1)$		$\Theta(\log n)$	$\text{poly}(\log n)$		$\Theta(n^\alpha)$	$\Theta(n)$
Paths & Cycles	✓	×	×				✓
Rooted trees	✓	×	✓				✓
Trees	✓	×	✓				✓
General graphs	✓	×	✓				✓

Landscape of the mending volume

Overview

Setting	$O(1)$		$\Theta(\log n)$	$\text{poly}(\log n)$		$\Theta(n^\alpha)$	$\Theta(n)$
Paths & Cycles	✓	×	×			×	✓
Rooted trees	✓	×	✓			✓	✓
Trees	✓	×	✓			✓	✓
General graphs	✓	×	✓			✓	✓

Landscape of the mending volume

Polynomial complexities

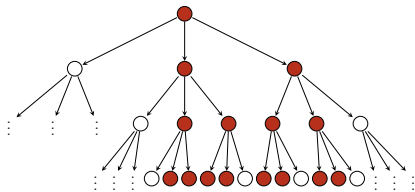


$$M = (2)$$

Polynomial complexities



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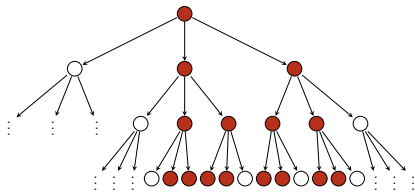
Polynomial complexities



$$M = (2)$$

Matrix growth rate:
exponential

For $\Delta = 3$,
volume $\Theta(n^{\ln 2 / \ln 3})$



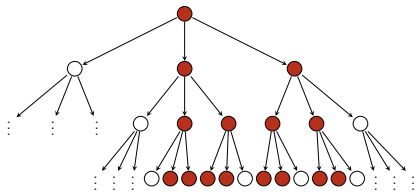
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


$$M = (2)$$

Matrix growth rate:
exponential

For $\Delta = 3$,
volume $\Theta(n^{\ln 2 / \ln 3})$



Generalization:


for $\Delta = 2^q$, volume $\Theta(n^{p/q})$

Overview

Setting	$O(1)$		$\Theta(\log n)$	$\text{poly}(\log n)$		$\Theta(n^\alpha)$	$\Theta(n)$
Paths & Cycles	✓	×	×			×	✓
Rooted trees	✓	×	✓			✓	✓
Trees	✓	×	✓			✓	✓
General graphs	✓	×	✓			✓	✓

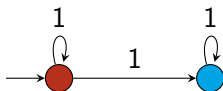
Landscape of the mending volume

Overview

Setting	$O(1)$		$\Theta(\log n)$	$\text{poly}(\log n)$		$\Theta(n^\alpha)$	$\Theta(n)$
Paths & Cycles	✓	✗	✗	✗		✗	✓
Rooted trees	✓	✗	✓	✓		✓	✓
Trees	✓	✗	✓	✓		✓	✓
General graphs	✓	✗	✓	✓		✓	✓

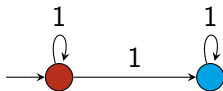
Landscape of the mending volume

Polylogarithmic complexities

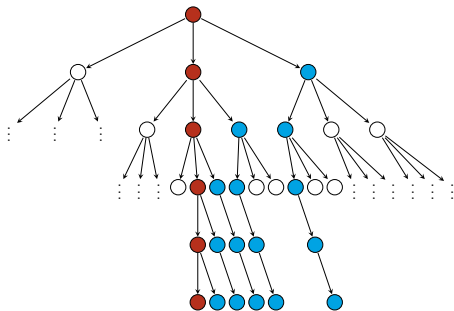


$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

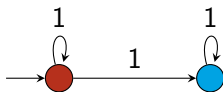
Polylogarithmic complexities



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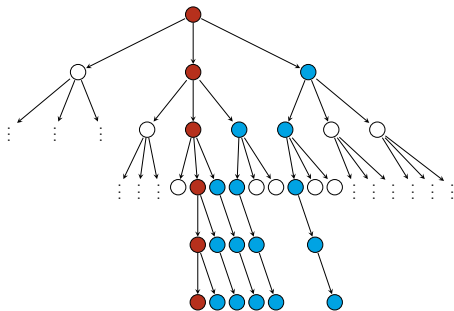
Polylogarithmic complexities



$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Matrix growth rate:
linear

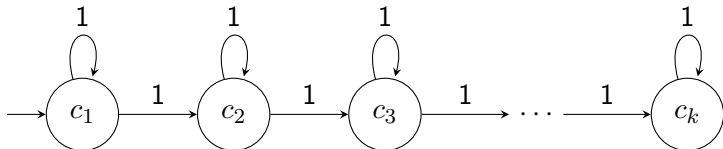
For $\Delta = 3$,
volume $\Theta(\log^2 n)$



Polylogarithmic complexities

Generalization:

$$M = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$



for $\Delta = 3$, volume $\Theta(\log^k n)$

Overview

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Paths & Cycles	✓	✗	✗	✗		✗	✓
Rooted trees	✓	✗	✓	✓		✓	✓
Trees	✓	✗	✓	✓		✓	✓
General graphs	✓	✗	✓	✓		✓	✓

Landscape of the mending volume

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Trees	✓	×	✓	✓	?	✓	✓
General graphs	✓	×	✓	✓	?	✓	✓

Landscape of the mending volume

Future research directions

- Can we design alternative (algorithmic) measures of the mending volume that take into account the difficulty of finding a mend ? <https://arxiv.org/abs/2209.05363>

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- How do those algorithmic measures compare to the mending volume ?

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- Can we design alternative (algorithmic) measures of the mending volume that take into account the difficulty of finding a mend ? <https://arxiv.org/abs/2209.05363>
- How do those algorithmic measures compare to the mending volume ?
- Is the landscape of the mending volume complete ? (gap between polylogarithmic and polynomial)

Summary

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Rooted trees	✓	×	✓	×	✓
Trees	✓	×	✓	?	✓
General graphs	✓	×	✓	?	✓