

Graphical Representation of a Conjugacy Invariant for Discrete Dynamical Systems

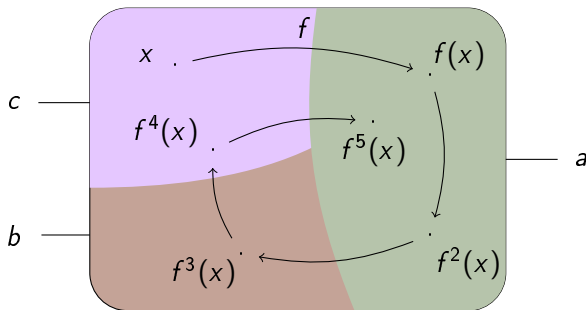
Vivien Ducros
Supervised by Emmanuel Jeandel

Internship at team MOCQUA, Loria

June 19, 2025
EJCIFM, Caen

Dynamical Systems

Dynamical system : (X, f) , X state space, $f : X \rightarrow X$ continuous.



Orbit: $x \quad f(x) \quad f^2(x) \quad f^3(x) \quad f^4(x) \quad f^5(x) \quad \dots$
 $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \dots$
 $c \quad a \quad a \quad b \quad c \quad a \quad \dots$

$$x \simeq caabca \dots$$

$$f(x) \simeq aabca \dots = \sigma(caabca \dots)$$

Symbolic Dynamics

- \mathcal{A} : finite alphabet, \mathcal{A}^* : finite words, $\mathcal{A}^{\mathbb{N}}$: semi-infinite words,
- σ : shift operator, $\forall x \in \mathcal{A}^{\mathbb{N}}, n \in \mathbb{N}$,

$$\sigma(x)_n = x_{n+1}$$

- Cantor topology on $\mathcal{A}^{\mathbb{N}}$:

$$d(x, y) = \begin{cases} 2^{-\min\{i \in \mathbb{N} : x_i \neq y_i\}} & \text{if } x \neq y \\ 0 & \text{else.} \end{cases}$$

Example: $d(100 \dots, 101 \dots) = 2^{-2} = 1/4$.

- Basis of open sets: the cylinders $[u] = \{u \cdot x \mid x \in \mathcal{A}^{\mathbb{N}}\}$, $u \in \mathcal{A}^*$.

Subshifts

$X \subseteq \mathcal{A}^{\mathbb{N}}$ is a *subshift* when:

- X is shift-invariant, $\sigma(X) = X$,
- X is closed for the Cantor topology.

Conjugacy

Radius $r \in \mathbb{N}$, encoding $f : \mathcal{A}^r \rightarrow \mathcal{A}$.

Cellular automaton: $\forall x \in \mathcal{A}^{\mathbb{N}}, n \in \mathbb{N}$,

$$\bar{f}(x)_n = f(x_n \cdots x_{n+r-1})$$

Example: $r = 2, f : \begin{cases} aa \mapsto 1 & ca \mapsto 0 \\ ab \mapsto 0 & cb \mapsto 0 \\ bc \mapsto 1 \end{cases}$

$x = c \ b \ \boxed{c \ a} \ a \ b \ c \ b \ c \ a \ \cdots$

\downarrow
 $\bar{f}(x) = 0 \ 1 \ \boxed{0} \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots$

Conjugacy

Two subshifts X and Y are *conjugated* if there exist a cellular automaton $\phi : X \rightarrow Y$ which is reversible.

Notation: $X \cong Y$.

Conjugacy Invariants

Conjugacy invariant

An *invariant* is a property P of a subshift that is conserved for conjugated subshifts:

$$X \cong Y \implies P(X) = P(Y)$$

Examples:

- $P_n(X) = \#\{x \in X : x \text{ of period } n\}$
- The continuous full group $\Gamma_X = \{\tau : X \rightarrow X : \tau \text{ cylinder map}\}$

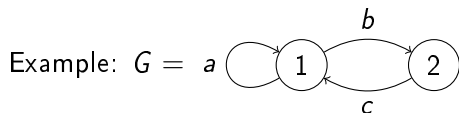
Edge Shifts: a Class of Subshifts

Graph $G = (Q, E, s, t)$, source/target maps $s, t : E \rightarrow Q$,

Edge shift of G

$$X_G = \{x \in E^{\mathbb{N}} : \forall n \in \mathbb{N}, s(x_{n+1}) = t(x_n)\}$$

Set of semi-infite paths on G .



$$x = cbc \ aa \ bcbc \ a \ bcbcbc \ aaa \cdots \in X_G$$

$$X_G = [s^{-1}(1)] \uplus [s^{-1}(2)]$$

Cylinder Maps

X subshift, a cylinder map $\tau : X \rightarrow X$ is a function that:

- 1 Read a prefix of the input,
- 2 Replace this prefix by another one,
- 3 The new prefix is uniquely determined by the prefix read.

Example: $G = a \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \begin{array}{c} \text{1} \\ \circlearrowleft \end{array} \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{c} \end{array} \begin{array}{c} \text{2} \\ \circlearrowleft \end{array}, X = X_G, \tau : \begin{cases} aaz \mapsto bcz \\ abc z \mapsto ca z \\ bz \mapsto cbz \\ cz \mapsto az \end{cases}$

Remarks:

- Two cylinder partitions of X :

$$X = \bigsqcup \{ [aa]_X, [abc]_X, [b]_X, [c]_X \} = \bigsqcup \{ [a]_X, [bc]_X, [ca]_X, [cb]_X \}$$

- Follower sets must match:

$$(aa)^{-1}X = [s^{-1}(1)] = (bc)^{-1}X$$

Graphical Representation of Cylinder Maps?

$$G = a \text{ (loop on 1)} \quad b \text{ (loop on 2)} \quad c \text{ (loop from 2 to 1)}, \quad X = X_G = [s^{-1}(1)]_X \uplus [s^{-1}(2)]_X,$$













- k input wire,
- $x \in [s^{-1}(1)]_X$ or $[s^{-1}(2)]_X$ according to the color of k -th wire.

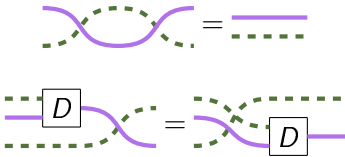
We can generalize cylinder maps!

Colored PROP

Diagrams constructed with:

- Colored wires: , ,
- Sequential composition,   = ,
- Parallel composition,   = ,
- Permutation of wires (swaps): , , ...

Such that:



Drawings for Generalized Cylinder Maps!

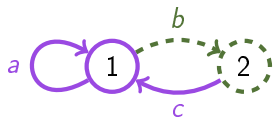


Colors: one for each state $q \in Q$ (here: $1 \leftrightarrow$ ———, $2 \leftrightarrow$ - - - -).

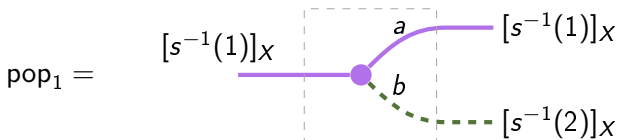
We add two generators for each $q \in Q$:

- Pop:
- Push:

Interpretation: Intuition pop



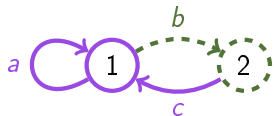
$$[s^{-1}(1)]_X = \{ax' \in X\} \cup \{bx' \in X\},$$
$$[s^{-1}(2)]_X = \{cx' \in X\}$$



$x \in [s^{-1}(1)]_X$:

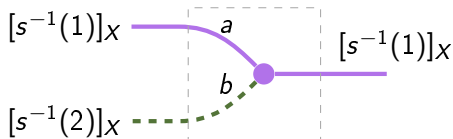
- if $x = ax'$, $\llbracket \text{pop}_1 \rrbracket (ax') = x' \in [s^{-1}(1)]_X$
- if $x = bx'$, $\llbracket \text{pop}_1 \rrbracket (bx') = x' \in [s^{-1}(2)]_X$

Interpretation: Intuition push



$$[s^{-1}(1)]_X = \{ax' \in X\} \cup \{bx' \in X\},$$
$$[s^{-1}(2)]_X = \{cx' \in X\}$$

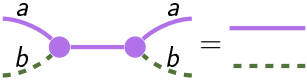
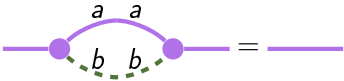
$\text{push}_1 =$



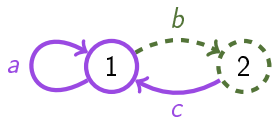
$x \in [s^{-1}(1)]_X \uplus [s^{-1}(2)]_X$:

- if $x \in [s^{-1}(1)]_X$, $\llbracket \text{push}_1 \rrbracket (x) = ax$
- if $x \in [s^{-1}(2)]_X$, $\llbracket \text{push}_1 \rrbracket (x) = bx$

Relations between Generators

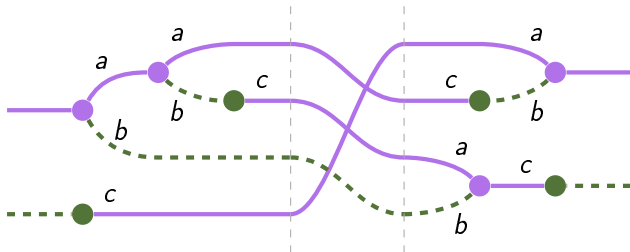


Going Back to our Example



$$\tau : \begin{cases} aaZ \mapsto bcZ & bz \mapsto cbz \\ abcZ \mapsto caZ & cz \mapsto aZ \end{cases}$$

$$X = \bigsqcup \{ [aa]_X, [abc]_X, [b]_X, [c]_X \} = \bigsqcup \{ [a]_X, [bc]_X, [ca]_X, [cb]_X \}$$



Conclusion

Graphical way to represent cylinder maps:

- Easy to construct,
- Sound and complete relations.

The continuous full group:

- Is a conjugacy invariant,
- Generalizes the Brin-Thompson groups.

Further work:

- Generalization to other classes of subshifts (SFT, sofic),
- Extension to bi-infinite words.

Thank you for listening

Please ask your questions !

Continuous Full Group

The continuous full group $\Gamma_X = \{\tau : X \rightarrow X : \tau \text{ cylinder map}\}$

Cylinder map

X subshift, $\tau : X \rightarrow X$ is a *cylinder map* if there exist L finite set, $(\mu^{(l)})_{l \in L}$, and $(\nu^{(l)})_{l \in L}$ such that:

- 1 $\{[\mu^{(l)}]_X\}_{l \in L}$ and $\{[\nu^{(l)}]_X\}_{l \in L}$ are two partitions of X ,
- 2 $\forall l \in L, (\mu^{(l)})^{-1}X = (\nu^{(l)})^{-1}X$,
- 3 $\forall l \in L, z \in (\mu^{(l)})^{-1}X, \tau(\mu^{(l)}z) = \nu^{(l)}z$.

Formal Interpretation

$G = (Q, E, s, t)$, $X = X_G$. For $q \in Q$, define φ_q , α_q such that:

$$\forall k \in \llbracket 1, \deg(q) \rrbracket, q \xrightarrow{\alpha_q(k)} \varphi_q(k)$$

For $D \in \mathcal{D}$, $D : u \rightarrow v$,

$$\llbracket D \rrbracket : \bigoplus_{i=0}^{|u|} [s^{-1}(u_i)]_X \rightarrow \bigoplus_{j=0}^{|v|} [s^{-1}(v_j)]_X$$

- $\llbracket \text{id}_q \rrbracket : (1, w) \mapsto (1, w)$
- $\llbracket \text{swap}_{q, q'} \rrbracket : (k, w) \mapsto (3 - k, w)$
- $\llbracket D' \circ D \rrbracket : (k, w) \mapsto \llbracket D' \rrbracket \circ \llbracket D \rrbracket (k, w)$
- $\llbracket D \otimes D' \rrbracket : (k, w) \mapsto \begin{cases} \llbracket D \rrbracket (k, w) & \text{if } k \leq |u| \\ +^{|v|} \llbracket D' \rrbracket (k - |u|, w) & \text{else} \end{cases}$
- $\llbracket \text{pop}_q \rrbracket : (1, \alpha_q(k) \cdot z) \mapsto (k, z)$
- $\llbracket \text{push}_q \rrbracket : (k, z) \mapsto (1, \alpha_q(k) \cdot z)$