

# A string diagrammatic point of view on symbolic dynamics

Vivien Ducros

`vivien.ducros@ens-paris-saclay.fr`

DER Informatique, ENS Paris-Saclay

Internship at LIX, École Polytechnique

Supervised by Titouan Carette, joint work with Etienne Moutot

Journées SDA2, Orléans

May 31, 2024



# Drawings for Wang tilesets (1)

$$\boxed{T} = \left\{ \begin{array}{c} \text{Red/Blue diagonal} \\ \text{Yellow/Red diagonal} \end{array} \right\}$$

$$\triangleleft T' = \left\{ \begin{array}{c} \text{Red/Blue triangle} \\ \text{Yellow/Red triangle} \end{array} \right\}$$

$$\boxed{T} \triangleleft T' = \left\{ \begin{array}{c} \text{Composite shapes 1, 2, 3} \\ \text{Composite shapes 4, 5} \end{array} \right\}$$

Composition (contraction)

## Drawings for Wang tilesets (2)



$$\boxed{T} = \left\{ \begin{array}{c} \text{Red/Teal/Yellow square} \\ \text{Teal/Red/Yellow square} \\ \text{Teal/Yellow/Red square} \end{array} \right\}$$

$$\triangleleft T' = \left\{ \begin{array}{c} \text{Teal/Red/Yellow triangle} \\ \text{Yellow/Red/Teal triangle} \\ \text{Teal/Yellow/Red triangle} \end{array} \right\}$$

$$\boxed{T} \triangleleft T' = \left\{ \begin{array}{l} \begin{array}{c} \text{Red/Teal/Yellow square} \\ \text{Teal/Red/Yellow triangle} \end{array}, \begin{array}{c} \text{Teal/Red/Yellow square} \\ \text{Yellow/Red/Teal triangle} \end{array}, \begin{array}{c} \text{Teal/Yellow/Red square} \\ \text{Teal/Yellow/Red triangle} \end{array}, \\ \begin{array}{c} \text{Red/Teal/Yellow square} \\ \text{Teal/Red/Yellow triangle} \end{array}, \dots, \\ \begin{array}{c} \text{Teal/Red/Yellow square} \\ \text{Teal/Yellow/Red triangle} \end{array}, \dots \end{array} \right\}$$


Cartesian product (tensor product)

# Our diagrams

- Boxes with wires:  , 

- Contraction

- Tensor product

- Identity:  =  $\left\{ \begin{array}{c} \text{Red triangles} \\ \text{Teal triangles} \\ \text{Yellow triangles} \end{array} \right\}$

(Strict monoidal category)

## Reminders on subshifts

- $\mathcal{A}$  finite alphabet of symbols,  $\mathcal{A}^{\mathbb{Z}}$  set of configurations
- Shift operation  $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ ,
  - ▶  $\forall w \in \mathcal{A}^{\mathbb{Z}}, k \in \mathbb{Z}, \sigma(w)_k = w_{k+1}$
- Cantor metric on configurations:  $\forall v, w \in \mathcal{A}^{\mathbb{Z}}$ ,

$$d(v, w) = \begin{cases} 2^{-\min\{k \in \mathbb{N} \mid v_{-k} \neq w_{-k} \vee v_k \neq w_k\}} & \text{if } v \neq w \\ 0 & \text{else} \end{cases}$$

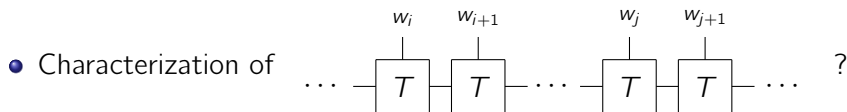
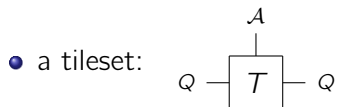
### Definition

$X \subseteq \mathcal{A}^{\mathbb{Z}}$  is a *subshift* if it is:

- shift-invariant ( $\sigma(X) = X$ )
- closed in the Cantor topology

# Describing subshifts with diagrams

- $Q$  horizontal colors "constraints",  $\mathcal{A}$  finite set of symbols

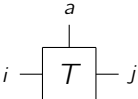


- $\mathbb{Z} \cdot T$  set of visible configurations

## More formally...

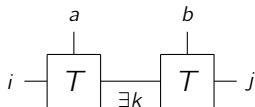
- Boxes are relations:  $T \subseteq Q \times Q \times \mathcal{A}$

- $T_{i,j}^a \iff (i,j,a) \in T$



A square box labeled 'T' has a horizontal line on the left labeled 'i', a horizontal line on the right labeled 'j', and a vertical line on top labeled 'a'.

- Composition of relations:



$$T^2 = \{(i,j,ab) \in Q \times Q \times \mathcal{A}^2 : \exists k \in Q, T_{i,k}^a \wedge T_{k,j}^b\}$$

and so on for all  $n \in \mathbb{N}$ ,  $T^n$

### Definition

$$\mathbb{Z} \cdot T = \bigcap_{n \in \mathbb{N}} \{w \in \mathcal{A}^{\mathbb{Z}} : (T^{2n+1})^{w_{[-n,n]}} \neq \emptyset\}$$

## Let's go back to subshifts

We describe exactly the subshifts...

- $\forall T \subseteq Q \times Q \times \mathcal{A}$ ,  $\mathbb{Z} \cdot T$  is a subshift
- For all subshift  $X \subseteq \mathcal{A}^{\mathbb{Z}}$ ,  $\exists Q, T \subseteq Q \times Q \times \mathcal{A}$  such that  $\mathbb{Z} \cdot T = X$

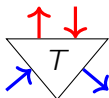
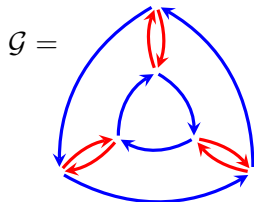
Question: if we impose  $Q$  finite, what is  $\mathbb{Z} \cdot T$  ?

...and sofic subshifts

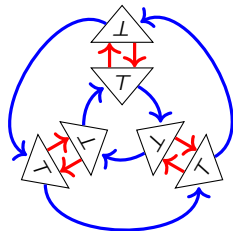
- If  $Q$  is finite,  $\mathbb{Z} \cdot T$  is sofic
- Conversely, if  $X$  is a sofic subshift, there exists  $T \subseteq Q \times Q \times \mathcal{A}$  with  $Q$  finite such that  $\mathbb{Z} \cdot T = X$

# Subshifts on Cayley Graphs

- $G$  a group, with presentation  $\langle S|R \rangle = G$
- Cayley graph: a colored and oriented graph  $\mathcal{G} = (V, E)$  where  $V = G$ ,  $E = \{(g, sg, s) : g \in G, s \in S\}$
- Example:  ~~$\mathbb{Z}_2$~~   $D_3 = \langle a, c | a^3 = c^2 = e, ca^2 = ac \rangle$



$\mathcal{G} \cdot T =$



# An other (finite) approach!

- Wires are configurations in  $Q^{\mathbb{Z}}$  and  $\mathcal{A}^{\mathbb{Z}}$

- New box: the shift  $\text{---} \diamond \sigma \text{---} = \{(x, \sigma(x)) : x \in Q^{\mathbb{Z}}\}$

- The infinite contraction box:

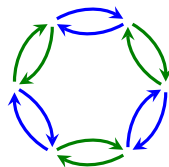
$$\begin{array}{c} \mathcal{A}^{\mathbb{Z}} \\ | \\ \boxed{T^{\mathbb{Z}}} \\ | \\ Q^{\mathbb{Z}} \text{---} \boxed{T^{\mathbb{Z}}} \text{---} Q^{\mathbb{Z}} \end{array} = \{(x, y, w) : \forall k \in \mathbb{Z}, T_{x_k, y_k}^{w_k}\}$$

- When tracing with shift:

$$\begin{array}{c} | \\ \boxed{T^{\mathbb{Z}}} \\ | \\ \text{---} \diamond \sigma \text{---} \end{array} = \{(x, \sigma(x), w) : \forall k \in \mathbb{Z}, T_{x_k, x_{k+1}}^{w_k}\}$$

## Application: independence of the Cayley Graph

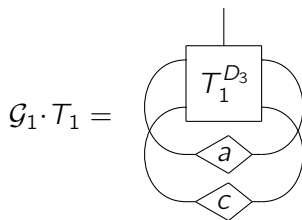
- an other Cayley Graph for  $D_3 = \langle b, c \mid b^2 = c^2 = e, bcb = cbc \rangle$



- Known: subshifts on groups don't depend on the Cayley Graph

## Application, example with $D_3$

- $D_3 = \langle a, c \mid a^3 = c^2 = e, ca^2 = ac \rangle = \langle b, c \mid b^2 = c^2 = e, bcb = cbc \rangle$   
(here  $a = bc$ )
- Question:  $\mathcal{G}_1, \mathcal{G}_2$  and  $T_1$  on  $\mathcal{G}_1$ .  $\exists T_2$  such that  $\mathcal{G}_1 \cdot T_1 = \mathcal{G}_2 \cdot T_2$  ?
- Proof with diagrams!



$\mathcal{G}_2 \cdot T_2 = ???$

## Conclusion and further work

- New way to make symbolic dynamics!?
- What about SFTs?
- (a)Periodicity property
- Probabilistic tilings, quantum tilings
- Category theory: completeness of our graphical language

Thank you for listening

Please ask your questions !