

# Generalizing Symbolic Dynamics: Toward Quantum Tilings

Vivien Ducros Titouan Carette

vivien.ducros@ens-paris-saclay.fr  
perso.crans.org/v-ducros/

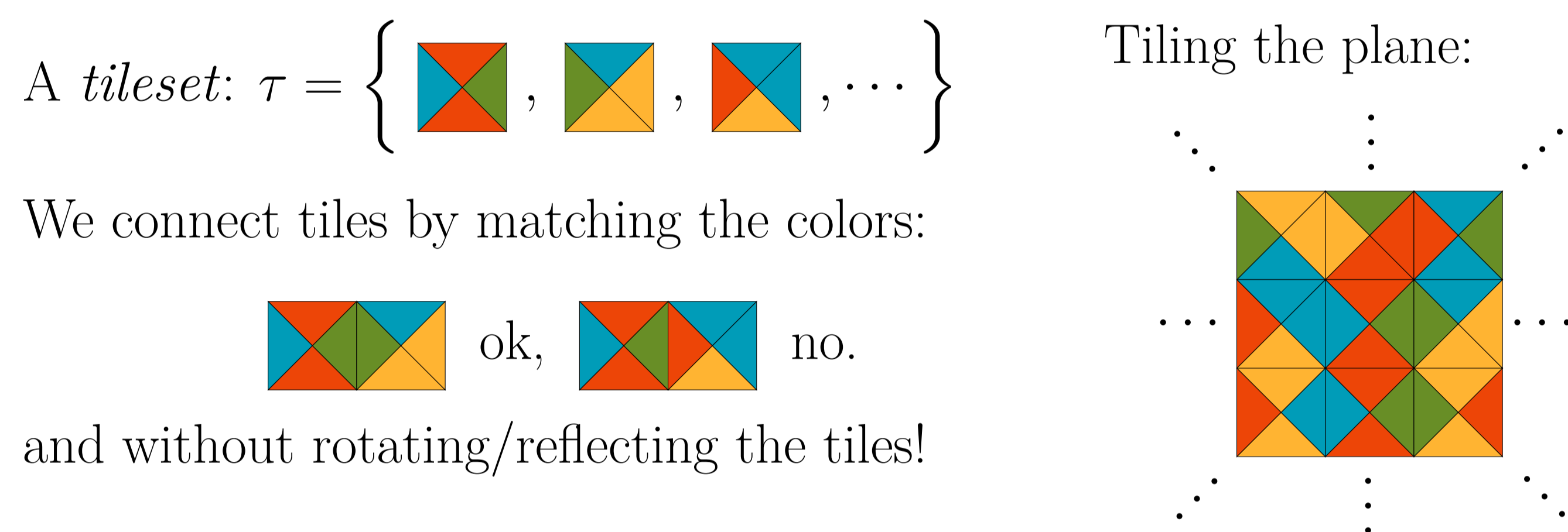
carette@lix.polytechnique.fr  
lmf.cnrs.fr/Person/TitouanCarette

## Abstract

We design a new formalism that describes Wang tilings using diagrams. This graphical language is able to describe some notions in symbolic dynamics and can be extended to subshifts on any groups. Moreover, it allows to define quantum superposition for a tile. Then the notion of tiling is more expressive as we observe interference phenomena. In the long term, we hope this formalism will give an other approach for condensed matter physics.

## Introduction to Wang Tilings

Wang tiles are a model of computation created in 1961. As other tessellations models, we are interested in the *tileability* and *periodicity* problems.



## Notions of Symbolic Dynamics

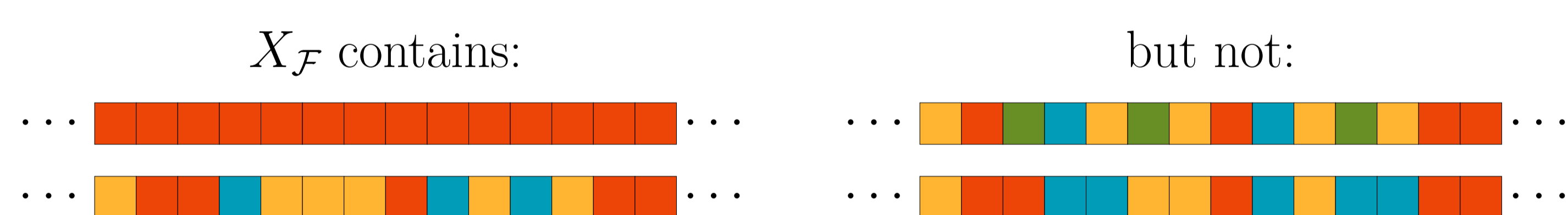
The field of symbolic dynamics studies among others tilings and cellular automata. The mathematical objects we consider are called *subshifts*. The following define these sets of configurations in the one-dimensional case.

Given a finite set of *symbols* (=colors):  $\mathcal{A} = \{ \text{[Red]}, \text{[Green]}, \text{[Blue]}, \text{[Yellow]} \}$ , we call *configurations* the strings of symbols in  $\mathcal{A}^{\mathbb{Z}}$ .

$$w = \dots \text{[Red][Green][Blue][Yellow]} \dots \in \mathcal{A}^{\mathbb{Z}}$$

With a set  $\mathcal{F}$  of *forbidden patterns* we define a *subshift*  $X_{\mathcal{F}}$  containing all configurations without these patterns.

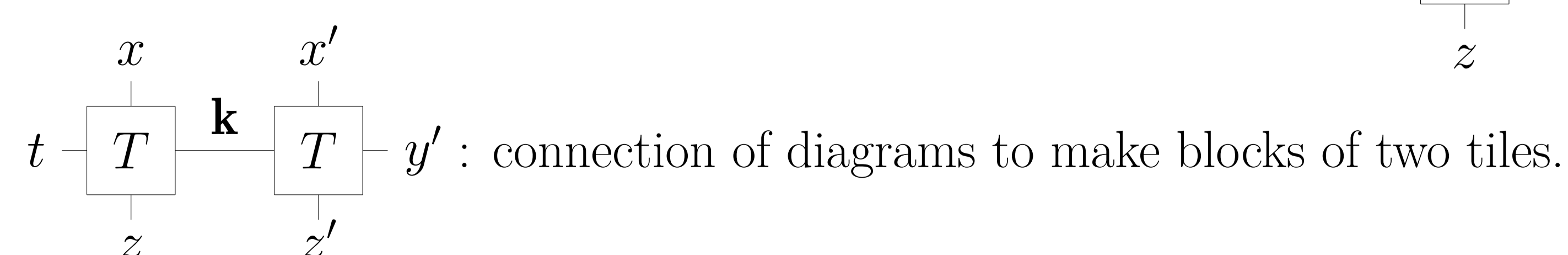
**Example:** with  $\mathcal{F} = \{ \text{[Green][Blue]}, \text{[Red][Blue]}, \text{[Red][Yellow]} \}$ , then:



## Our Graphical Language

We introduce tensors to represent tilings: the tensor  $T_{x,y,z,t}$  embodies the tiling  $\tau$  if,  $\forall x, y, z, t \in \mathcal{A}$ ,  $T_{x,y,z,t} = \begin{cases} 1 & \text{if } \begin{array}{c} x \\ t \times y \\ z \end{array} \in \tau \\ 0 & \text{else} \end{cases}$

Our diagrams are a graphical representation of the tensors:  $T = t \begin{array}{c} x \\ \boxed{T} \\ z \end{array} y$ .



For tensors, it's equivalent to their multiplication:  $T^2 = \sum_{\mathbf{k} \in \mathcal{A}} T_{x,\mathbf{k},z,t} T_{x',y',z',\mathbf{k}}$

## Extension to the Quantum Case

In this framework, the tensors count the number of ways to tile a pattern imposing colors on the borders.

By putting other coefficients, we define probabilistic and quantum tilings!

Coefficients	Normalisation	Count
$\in \{0, 1\}$	/	Number of tilings having a certain border
$\in [0, 1]$	$\sum_{\mathbf{u}} T_{\mathbf{u}} = 1$	Probability of having a certain border
$\in \mathbb{C}$	$\sum_{\mathbf{u}}  T_{\mathbf{u}} ^2 = 1$	Amplitude of the observed state

## Quantum interferences in the 1D case

Results in the classical case don't hold: example of interference phenomenon.

In one dimension, Wang tiles are dominos: . The tensors are matrices,  $T = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$  represents  $|\tau\rangle = |\text{[Red][Blue]}\rangle + |\text{[Green][Blue]}\rangle - |\text{[Red][Yellow]}\rangle - |\text{[Green][Yellow]}\rangle$ .

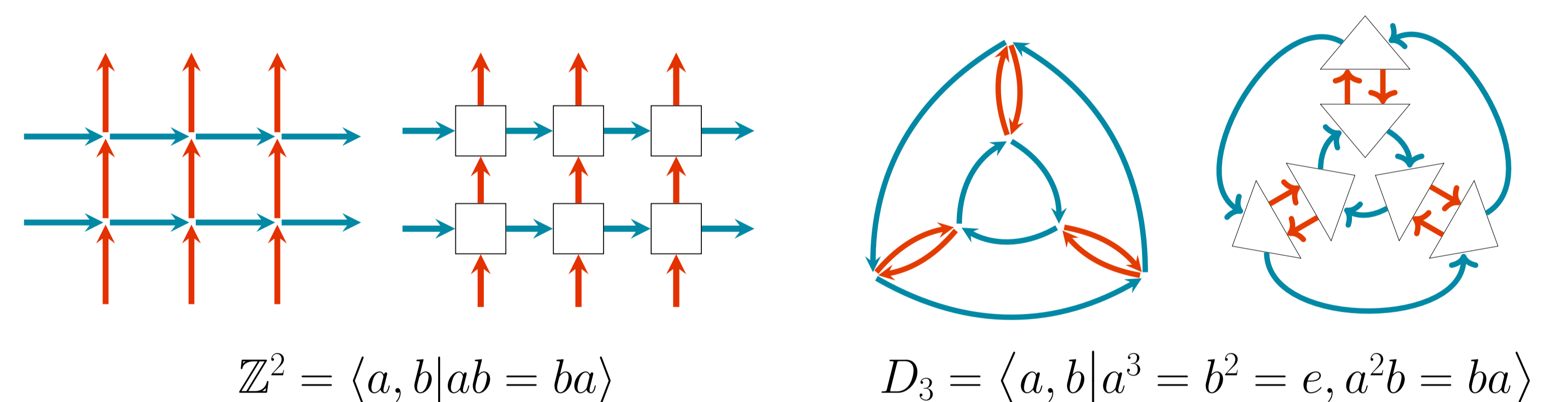
Whereas  $\tau = \{ \text{[Red][Blue]}, \text{[Green][Blue]}, \text{[Red][Yellow]}, \text{[Green][Yellow]} \}$  tiles the line,  $|\tau\rangle$  doesn't:  $T^2 = 0$ .

Interferences destruct the blocks of two dominos!

## Tilings on Cayley Graphs

Cayley graphs are a tool to represent mathematical groups.

With these graphs and our tensors, we can generalize the tilings on groups.



## Conclusion and future work

### Quantum objectives:

- Describing states of infinite systems
- New way of thinking about condensed matter?

### Symbolic dynamics objectives:

- Characterization of other definitions of subshifts (SFT, effective...)
- Periodicity property of tilings

### Graphical language objectives:

- Find the correct categorical framework for our diagrams
- Write a complete equivalence rewriting set of rules between diagrams

## References

- [1] D. Lind and B. Marcus. An Introduction to Symbolic Dynamics and Coding. *Cambridge University Press*, 1995.
- [2] T. Carette and E. Moutot. Aperiodicity in Quantum Wang Tilings. 2023.
- [3] E. Moutot. Around the Domino Problem - Combinatorial Structures and Algebraic Tools. *PhD Thesis*, 2020.