

# 1 Path Integrals in Quantum Mechanics

## 1.1 Path integrals

You have seen in the Quantum Theory course that transition amplitudes in non-relativistic quantum mechanics can be computed by path integrals. Consider a simple particle of mass  $m$  in a potential  $V(q)$ ,  $q$  being the position. The classical action  $S$  and the classical Lagrangian  $L$  are

$$S[q] = \int dt L(q(t), \dot{q}(t)) \quad L(q, \dot{q}) = \frac{m}{2} \dot{q}^2 - V(q) \quad (1.1)$$

and the classical equation of motion, satisfied by a classical trajectory  $q_{cl}(t)$  are obtained by extremizing the action with respect to variations of the trajectory (Euler-Lagrange equations)

$$q(t) = q_{cl}(t) + \delta q(t) \quad \frac{\delta S[q]}{\delta q(t)} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = -m \ddot{q}(t) - \frac{\partial}{\partial q} V(q) = 0 \quad (1.2)$$

the end-point of the trajectory being fixed (between initial time  $t_i$  and final time  $t_f$ )

$$\delta q(t_i) = \delta q(t_f) = 0 \quad (1.3)$$

The quantum probability amplitude  $K$  for propagation from  $q_i$  at time  $t_i$  to  $q_f$  at time  $t_f$ , is the matrix element of the evolution operator  $U$  (sometimes called the propagator, or the propagation kernel)

$$K(q_f, t_f; q_i, t_i) = \langle q_f | U(t_f - t_i) | q_i \rangle = \langle q_f, t_f | q_i, t_i \rangle \quad U(t) = \exp\left(\frac{t}{i\hbar} H\right) \quad (1.4)$$

(the first notation refers to the Schrödinger picture, the second one to the Heisenberg picture).  $H$  is the quantum Hamiltonian

$$H = \frac{1}{2m} P^2 + V(Q) \quad (1.5)$$

$K$  can be written as a sum of histories  $q(t)$

$$K(q_f, t_f; q_i, t_i) = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right) \quad (1.6)$$

The precise derivation of this formula, as well as its proper mathematical definition, is obtained by decomposing the evolution of the system in a large number  $N$  of evolutions during elementary time step  $\Delta t = \epsilon = t/N$ , at arbitrary intermediate positions  $q(t_n = n\epsilon)$ ,  $n \in \{1, \dots, N-1\}$ , using the superposition principle. One then uses the explicit formula for the propagation kernel at small time (the potential  $V(q)$  may be considered as constant locally)

$$K(q_f, \epsilon; q_i, 0) \simeq \left(\frac{2i\pi\hbar\epsilon}{m}\right)^{-1/2} \exp\left(\frac{i}{\hbar} \left(\frac{m}{2} \frac{(q_f - q_i)^2}{\epsilon} - \epsilon V\left(\frac{q_f + q_i}{2}\right)\right)\right) \quad (1.7)$$

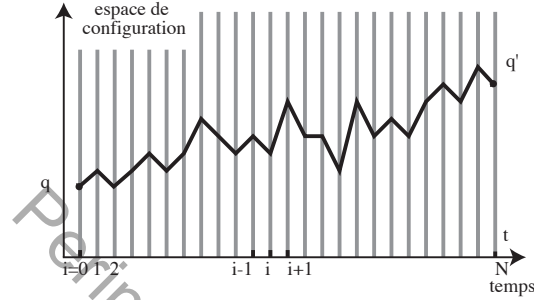


Figure 1: Path integral: time discretization

and one then takes the continuous time limit  $\epsilon \rightarrow 0$ . The precise definition of the measure over histories or paths is (from the prefactor)

$$\mathcal{D}[q] = \prod_{n=1}^{N-1} \left( dq(t_n) \left( \frac{2i\pi\hbar\epsilon}{m} \right)^{-1/2} \right) \tag{1.8}$$

### 1.2 Euclidean time

Before discussing the meaning of path integrals, let us generalize the evolution operator  $U(t)$  to "complex time". Let us consider the case of a particle in a potential well  $V(q)$  with a minimum, and growing fast enough at infinity (for instance an harmonic oscillator), so that the spectrum  $E_k$  of the hamiltonian  $H$  is discrete, bounded from below (there is a ground state), and unbounded from above. The energy levels are labelled as  $E_0 < E_1 < E_2 < \dots$  etc. and  $E_n \rightarrow +\infty$  with  $n$ . Then the evolution operator

$$U(t) = \exp\left(\frac{t}{i\hbar}H\right) = \sum_{i=0}^{\infty} \exp\left(\frac{t}{i\hbar}E_i\right) |i\rangle\langle i| \tag{1.9}$$

(considered as an element of the algebra  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  of the bounded <sup>2</sup> operators over the Hilbert space  $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$  of the system), exists not only for real  $t$  (where it is unitary) but for  $t$  complex provided that  $t$  is in the lower half plane

$$Im(t) \leq 0 \implies \|U(t)\|_{\infty} < \infty \implies U(t) \in \mathcal{B}(\mathcal{H}) \tag{1.10}$$

<sup>2</sup>The norm  $\|A\|_{\infty}$  of an operator is defined as  $\|A\|_{\infty}^2 = \sup_{\psi \in \mathcal{H}} \frac{\langle \psi | AA^{\dagger} | \psi \rangle}{\langle \psi | \psi \rangle}$  and corresponds for matrices to the modulus of the largest eigenvalue of  $A$

Not only  $U(t)$  exists, but it is an analytic function of  $t$  (namely all matrix elements  $\langle \psi | U(t) | \psi' \rangle$  are finite and are analytic functions of  $t$ ).

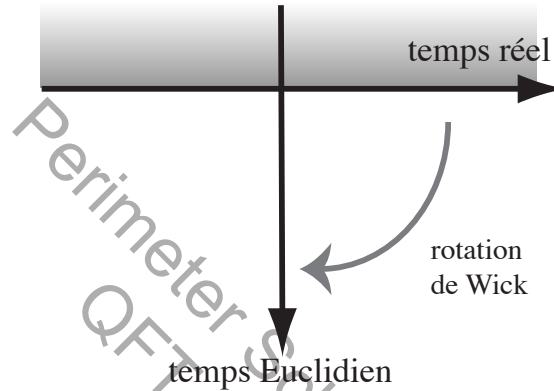


Figure 2: Complex time and Wick rotation

Now consider  $U(t)$  for purely imaginary times  $t = -i\tau$ . This rotation by  $-\pi/2$  in the complex time plane is called a Wick rotation. We shall call  $\tau = it$  "Euclidean time" for reasons that will become clear in the next lecture. Obviously the evolution operator becomes (the "E" stands for "Euclidean")

$$U_E(\tau) = U(-i\tau) = \exp\left(-\frac{\tau}{\hbar}H\right) \quad (1.11)$$

$U_E$  is proportional to the density matrix  $\rho_\beta$  of the same quantum system at finite temperature  $T$ , if we identify

$$\frac{\tau}{\hbar} = \beta = \frac{1}{k_B T} \quad (1.12)$$

since the density matrix is

$$\rho_\beta = \frac{1}{Z(\beta)} \exp(-\beta H) = \frac{1}{\text{tr}[U_E(\tau)]} U_E(\tau) \quad (1.13)$$

The partition function is

$$Z(\beta) = \text{tr}[U_E(\tau)] \quad (1.14)$$

Let us thus compute this partition function, using path integral methods. We now have to compute the evolution operator at imaginary time  $\sigma = it$ , starting from  $q_i$  at  $\sigma_i = 0$ , up to  $q_f$  at  $\sigma_f = \tau = \hbar/(k_B T)$ , and take the trace

$$\text{tr}[U_E(\tau)] = \int dq \langle q | U_E(\tau) | q \rangle \quad \text{i.e.} \quad q_i = q_f = q \quad (1.15)$$

We can easily repeat the path integral calculation, decomposing the evolution at Euclidean time into a large number of small steps. Everything amounts in fact to replace real times  $s$  by imaginary times  $\sigma = is$ . The action  $S[q(s)]$  becomes

$$S[q(s)] = \int ds \left( \frac{m}{2} \frac{dq(s)^2}{ds} - V(q(s)) \right) \tag{1.16}$$

$$= \int (-i)d\sigma \left( -\frac{m}{2} \frac{dq(\sigma)^2}{d\sigma} - V(q(\sigma)) \right) \tag{1.17}$$

$$= i \int d\sigma \left( \frac{m}{2} \frac{dq(\sigma)^2}{d\sigma} + V(q(\sigma)) \right) = iS_E[q(\sigma)] \tag{1.18}$$

Computing the trace amounts to sum over periodic histories such that  $q(0) = q(\tau)$  (since  $q_i = q_f$ ), but integrating also over all possible initial positions at  $\sigma = 0$ . One obtains a path integral over periodic Euclidean histories (with period  $\tau = \hbar/(k_B T)$ )

$$Z(\beta) = \int_{q(0)=q(\tau)} \mathcal{D}q \exp \left( -\frac{1}{\hbar} S_E[q] \right) \tag{1.19}$$

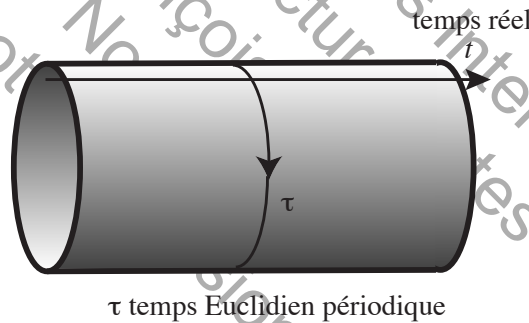


Figure 3: Periodic Euclidean time

But now each history  $q = [q(\sigma)]$  is weighted by a real positive number  $W[q]$ , similar to a Boltzmann weight for the Gibbs distribution in statistical mechanics

$$W[q] = \exp \left( -\frac{1}{\hbar} S_E[q] \right) \simeq W_{\text{Gibbs}} = \exp \left( -\frac{\text{Energy}}{k_B \text{Temperature}} \right) \tag{1.20}$$

The Euclidean action

$$S_E[q] = \int d\sigma \left( \frac{m}{2} \frac{dq(\sigma)^2}{d\sigma} + V(q(\sigma)) \right) \tag{1.21}$$

plays the rôle of the energy of a configuration of a statistical system, and the (reduced) Planck constant  $\hbar$  plays the role of the temperature for this statistical system. Note the + sign in front of the potential energy term in the Euclidean action, to be compared with the - sign in the ordinary action at real time. In our example the Euclidean action amounts to the integrated Hamiltonian over the time... but this is not always so simple.

### 1.3 Dictionary Quantum Mechanics ↔ Statistical Mechanics

One thus has a mathematical analogy between a quantum mechanical system at Euclidean time (hence at finite temperature  $T_Q$  (for instance the 1D harmonic oscillator with a single degree of freedom  $q$ , its position); and an extended 1 dimensional statistical system at finite temperature  $T_S$  (a chain of linearly coupled oscillators along a closed circle with total length  $L$ , the local order parameter of the oscillator at position  $\sigma$  being the variable  $q(\sigma)$ ). The dictionary is

Quantum system	Statistical 1D system
Euclidean time $\sigma$	$x$ position along the chain
degree of freedom $q(\sigma)$	$q(x)$ local order parameter
Euclidean action $S_E[q]$	$\mathcal{E}[q]$ energy of a microstate
Planck's constant $\hbar$	$T_S$ temperature
temperature $T_Q$	$L^{-1}$ $L =$ size of the system

As we shall see, this analogy extends to Euclidean Relativistic Quantum Field Theories ↔ Extended Statistical Systems  $D > 1$ .

Remember the fundamental relation between the temperature of a quantum system and the periodicity in Euclidean time of the corresponding path integral for this system.

$$k_B \text{ Temperature} = \frac{\hbar}{\text{Period in Euclidean Time}} \tag{1.22}$$

This mathematical relation allows for instance to derive the Beckenstein-Hawking temperature of a black hole, or the Unruh temperature of the vacuum in an accelerated frame in a very elegant way.

### 1.4 Semiclassical expansion and quantum fluctuations

Path integrals allow to grasp what is happening for simple systems (with a few degrees of freedom) in the regime when quantum fluctuations are small. Beware! Small quantum

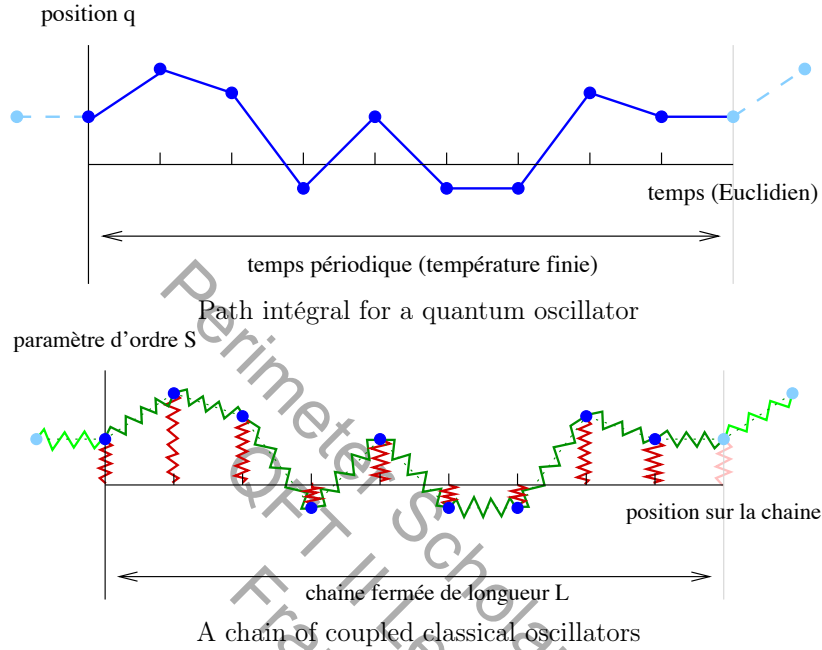


Figure 4: Equivalence between a single quantum oscillator and a classical chain of oscillators

fluctuations do not necessarily mean small quantum effects! Let us consider  $\hbar$  as a parameter in the path integral, that we are free to tune to small values. When  $\hbar \rightarrow 0$ , in fact only trajectories  $q(t)$  “close to” classical trajectories  $q_{classical}(t)$  (solutions of the equation of motion) contribute.

**1.4.1 Gaussian integrals**

As a toy model, let us consider the Gaussian integral (A “no-time” quantum mechanical model). The real Gaussian integral is

$$\int_{-\infty}^{+\infty} dx e^{-(x-x_0)^2/2\sigma} = (2\pi\sigma)^{1/2} \tag{1.23}$$

The integrand  $e^{-(x-x_0)^2/2\sigma}$  is depicted on Fig. 5. It has a maximum at  $x = x_0$ , and is of order  $\mathcal{O}(1)$  for  $|x - x_0| \sim \sqrt{\sigma}$  (the variance). The term  $e^{-x_0^2/2\sigma}$  is the contribution of the maximum, and the coefficient  $(2\pi\sigma)^{1/2}$  the contribution of the fluctuations around the maximum. This is the zero dimensional analog of the path integral at imaginary time.

Now consider the imaginary Gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{i(x-x_0)^2/2\sigma} = (2i\pi\sigma)^{1/2} \tag{1.24}$$

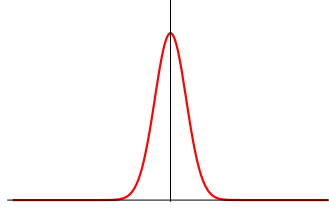


Figure 5: The real Gaussian function

The real and imaginary part of the distribution  $e^{i(x-x_0)^2/2\sigma}$  are depicted on Fig. 6. It is a pure phase, but is slowly varying around the stationary phase point  $x = x_0$ , in a domain of width  $\sqrt{\sigma}$ . away from this domain, the phase is oscillating very quickly, and the contributions cancel out. The term  $e^{ix_0^2/2\sigma}$  is the contribution of the stationary phase point  $x = x_0$ , the coefficient  $(2i\pi\sigma)^{1/2}$  the contribution of the fluctuations around  $x_0$ .

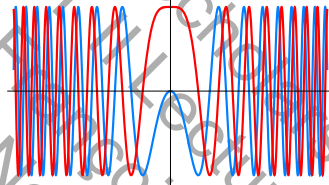


Figure 6: The imaginary Gaussian function: the real part and the imaginary part are blue and red.

#### 1.4.2 Semiclassical limit: real time

This simple picture extends to quantum mechanics and path integrals (with non trivial mathematics!). In the limit  $\hbar \rightarrow 0$  the path integral

$$K(q_f, q_i; t) = \int_{\substack{q(0)=q_i \\ q(t)=q_f}} \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right) \quad (1.25)$$

is dominated by the stationary phase trajectory  $q_0(s)$  which extremises the action  $S[q]$

$$\left. \frac{\delta S[q]}{\delta q(s)} \right|_{q_c} = 0 \quad \Longrightarrow \quad m\ddot{q}_0 = -V'(q_0) \quad (1.26)$$

that is the classical trajectory going from  $q_i$  to  $q_f$  in a time  $t$ . This gives a leading classical phase  $\exp\left(\frac{i}{\hbar} S[q_0]\right)$ . Only the non-classical trajectories  $q = q_0 + \delta q$  “close enough” to  $q_0$  contributes to the path integral. They give the multiplicative factor in front of the phase,

which can be obtained from the “determinant” of the quadratic form  $S_0''$  given by the second functional derivative of the action  $S[q]$ ; this quadratic form is a symmetric function<sup>3</sup>  $H_0(u, v)$  of two time variables  $u$  and  $v$ , called the Hessian of the action  $S$ .

$$S_0'' = H_0(u, v) = \frac{\delta^2 S[q]}{\delta q(u)\delta q(v)} \Big|_{q_0} \tag{1.27}$$

One obtains

$$K(q_f, q_i; t) = \left( \det \left[ \frac{i}{\hbar} \frac{S_0''}{2\pi} \right] \right)^{-1/2} \exp \left( \frac{i}{\hbar} S[q_0] \right) (1 + \mathcal{O}(\hbar)) \tag{1.28}$$

The determinant  $\det \left[ \frac{i}{\hbar} \frac{S_0''}{2\pi} \right]$  is the determinant of a differential operator (an infinite dimensional matrix) and requires some care to be calculated (one must go back to the precise discretization in time which defines the path integral). The subleading  $\mathcal{O}(\hbar)$  terms to the right are subdominant terms coming from the fact that the action is in general not quadratic in  $q$ . They can be calculated by different methods.

The strongly non-classical trajectories, far away from the classical ones, do not contribute to the path integral when  $\hbar$  is small because of the strong destructive interferences between such trajectories.

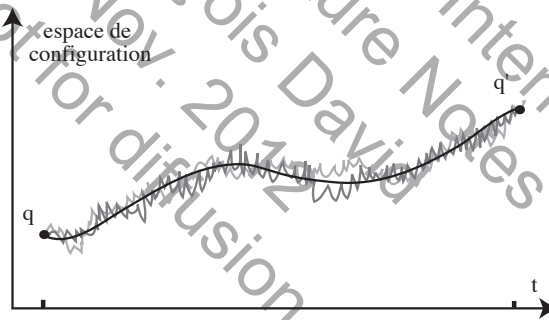


Figure 7: Path integral is the semi-classical regime: the smooth classical trajectory dominates, the irregular non-classical trajectories near the classical one give the main contribution.

In fact the determinant can be computed more easily using the WKB approximation (thus going back to the Schrödinger equation). There is a remarkable formula, thanks to Van Vleck

<sup>3</sup>In fact it is a distribution, giving the integral kernel of a linear operator acting on functions  $\psi(t)$ .



and Morette, which gives this determinant in terms of the second derivative of the classical action  $S_0$ , considered as a function of the initial and final conditions<sup>4</sup>, with respect to  $q_i$  and  $q_f$

$$\det \left[ \frac{i}{\hbar} \frac{S_0''}{2\pi} \right] = \left( -\frac{1}{2i\pi\hbar} \frac{\partial^2 S_0(q_f, q_i; t)}{\partial q_i \cdot \partial q_f} \right)^{-1} \quad (1.29)$$

### 1.4.3 Semiclassical limit: imaginary time

The same arguments are valid for the Euclidean path integral

$$K_E(q_f, q_i; \tau) = \int_{\substack{q(0)=q_i \\ q(\tau)=q_f}} \mathcal{D}[q] \exp \left( -\frac{1}{\hbar} S_E[q] \right) \quad (1.30)$$

It is dominated by the saddle point trajectory  $q_0(\sigma)$  which minimizes the Euclidean action  $S_E[q]$

$$\left. \frac{\delta E[q]}{\delta q(\sigma)} \right|_{q_0} = 0 \implies m\ddot{q}_0 = V'(q_0) \quad (1.31)$$

Note the change of sign in front of the potential  $V$ . One gets

$$K_E(q_f, q_i; \tau) = \left( \det \left[ \frac{1}{\hbar} \frac{S_{E0}''}{2\pi} \right] \right)^{-1/2} \exp \left( -\frac{1}{\hbar} S_E[q_0] \right) (1 + \mathcal{O}(\hbar)) \quad (1.32)$$

### 1.4.4 Multiple trajectories, quantum interferences

As you have seen in the Quantum Theory course, this is valid when several different classical trajectories can contribute to the path integral. In these cases, one has simply to add the contributions of the different classical contributions, each with its quantum phase (given by the classical action), and its normalization (given by the quantum fluctuations). In many cases the gaussian approximation (neglecting the  $\mathcal{O}(\hbar)$  corrections) are sufficient. One recovers easily the classical results for phenomena involving quantum interferences (two slits experiments, the Aharonov-Bohm effect).

$$\text{Transition Amplitude} \simeq \sum_{\text{classical trajectories}} \det[S''[q_a]]^{-1/2} e^{\frac{i}{\hbar} S[q_a]} \quad (1.33)$$

Another impressive application of semiclassical calculations is the estimation of the density of states of complicated quantum systems which may even be classically chaotic. The density of states  $\rho(E)$  is given by a "trace formula" (Gutzwiller formula) involving the classical density of states  $\rho_{\text{classical}}(E)$ , plus a quantum contribution given by a sum over all the

<sup>4</sup>As in the derivation of the Hamilton-Jacobi equations in classical mechanics.

periodic orbits  $q_a$  of energy  $E$  (now the energy  $E$  is fixed, but the period  $T_a$  is arbitrary), of the form

$$\rho(E) = \rho_{\text{classical}}(E) + \frac{1}{\pi\hbar} \sum_{\substack{\text{periodic orbits} \\ \text{of energy } E}} \text{Re} \left[ \Delta_a(E) e^{\frac{i}{\hbar}(S[q_a] + T_a E)} \right] \quad (1.34)$$

the coefficients  $\Delta_a(E)$  are given by the contribution of the fluctuations around each orbit in the path integral, and contains a topological phase factor related to the stability of the trajectories (Maslov index). I do not give more details here. See also the works by Balian and Bloch.

### 1.5 Operators and correlation functions

The path integral formalism allows to compute matrix elements of operators (observables). For instance, inserting a  $q(t_1)$  in the path integral amounts to insert a position operator  $Q$  at time  $t_1$  (in the Heisenberg picture, between the initial and final states, as depicted in Fig. 8

$$\langle q', t | Q(t_1) | q, 0 \rangle = \int D[q] \exp \left[ \frac{i}{\hbar} S[q] \right] q(t_1) \quad (1.35)$$

Inserting a function  $A(q(t))$  amounts to insert the operator  $A(Q)(t)$ .

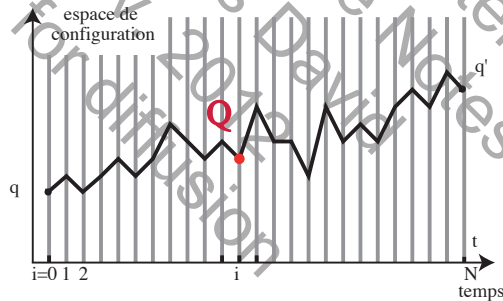


Figure 8: One operator e.v.

Similarly, inserting two  $q$ 's at two different times  $t_1$  and  $t_2$  amounts to insert the *time ordered product*  $T[Q(t_1)Q(t_2)]$

$$T[Q(t_1)Q(t_2)] = \begin{cases} Q(t_2)Q(t_1) & \text{if } t_1 < t_2, \\ Q(t_1)Q(t_2) & \text{if } t_1 > t_2. \end{cases} \quad (1.36)$$

Namely

$$\langle q', t | T[Q(t_1) Q(t_2)] | q, 0 \rangle = \int D[q] \exp \left[ \frac{i}{\hbar} S[q] \right] q(t_1) q(t_2) \quad (1.37)$$

As depicted on Fig. 9, the path integral automatically performs the time ordering of operators. This is obviously valid for an arbitrary number of operators  $A_i(t_i)$ ,  $i = 1, 2, 3, \dots$

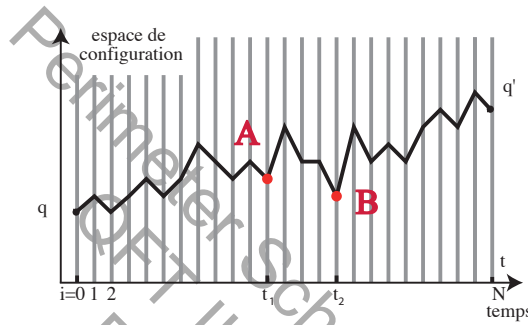


Figure 9: Two operators e.v.

This works also when computing the expectation value of operators on thermal state. For computing the e.v. of an operator  $A$  (which is independent of time since thermal states are stationary states), since it is given by a ratio of two traces

$$\langle \mathbf{A}(t) \rangle_{\beta} = \langle \mathbf{A} \rangle_{\beta} = \frac{\text{tr} (\mathbf{A} e^{-\beta \mathbf{H}})}{\text{tr} (e^{-\beta \mathbf{H}})} \quad (1.38)$$

it is given by the ratio of two path integrals at imaginary periodic time, with period  $\tau = \hbar/k_B T$ .

$$\langle \mathbf{A} \rangle_{\beta} = \frac{\int_{q(0)=q(\tau)} \mathcal{D}[q] \exp \left[ -\frac{1}{\hbar} S_E[q] \right] A[q(\tau_0)]}{\int_{q(0)=q(\tau)} \mathcal{D}[q] \exp \left[ -\frac{1}{\hbar} S_E[q] \right]} \quad (1.39)$$

The numerator being depicted of Fig. 10

Real time correlators of two operators  $A$  and  $B$  on a thermal state are easily obtained. Start from the e.v. of these operators at imaginary times  $\tau_1 < \tau_2$ . It is defined as

$$\begin{aligned} \langle B(\tau_2) A(\tau_1) \rangle_{\beta} &= \frac{1}{Z_{\beta}} \text{tr} [e^{-\beta H} B(\tau_2) A(\tau_1)] \\ &= \frac{1}{Z_{\beta}} \text{tr} [e^{-(\tau-\tau_2)H/\hbar} B e^{-(\tau_2-\tau_1)H/\hbar} A e^{-\tau_1 H/\hbar}] \end{aligned} \quad (1.40)$$

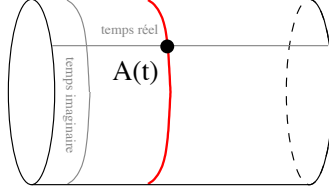


Figure 10: Periodic imaginary time contour for evaluating  $\langle \mathbf{A}(t) \rangle_\beta$  at finite temperature.

and given by the path integral

$$\text{tr} [e^{-\beta H} B(\tau_2) A(\tau_1)] = \int_{\mathcal{E}} \mathcal{D}[q] e^{-\frac{1}{\hbar} S_E[q]} A(\tau_1) B(\tau_2) \quad (1.41)$$

depicted on Fig. 11

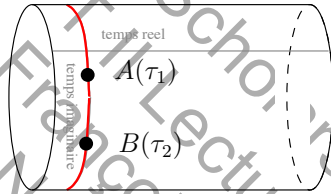


Figure 11: Imaginary times contour  $\mathcal{T}$  for the correlator  $\langle A(\tau_1) B(\tau_2) \rangle_\beta$  at finite temperature.

Now perform the Wick rotation

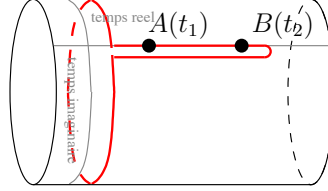
$$\tau_1 = it_1, \quad \tau_2 = it_2 \quad (1.42)$$

to obtain the e.v. of the time ordered product *at real times*

$$\langle T[B(t_2)A(t_1)] \rangle_\beta = \frac{1}{Z_\beta} \text{tr} (e^{-\beta H} B(t_2)A(t_1)) \quad t_1 < t_2 \quad (1.43)$$

It is given by a path integral mixing evolutions at real times and imaginary times, depicted on Fig. 12.

Notice that it is necessary to include both a “time-forward contour” (to represent the evolution operators  $U(t)$ ) and a “time-backward contour” (to represent the evolution operators  $U(-t) = U^\dagger(t)$ ). This is a simple example of the so-called two-times representation at the basis of the Schwinger-Keldisch formalism used to study off-equilibrium quantum systems in condensed matter, nuclear and high energy physics, etc. Please remember that there is nothing mysterious here. This simply reflects that in quantum physics the physical probabilities are given by probability amplitudes  $\times$  their complex conjugates, and path integrals construct probability amplitudes, not probabilities!

Figure 12: Complex times contour  $\mathcal{T}$  for the correlator  $\langle T[A(t_1)B(t_2)] \rangle_\beta$  at finite temperature.

## 1.6 Projection on the ground state

One last simple but important point. Starting from a thermal state at temperature  $T$ , taking the zero temperature limit  $T \rightarrow 0$  (i.e.  $\beta = 1/k_B T \rightarrow \infty$ ) amounts to project on the lowest energy state, that is the ground state  $|0\rangle$ . Indeed, for a single operator

$$\lim_{\beta \rightarrow +\infty} \langle A(\tau_0) \rangle_\beta = \lim_{\beta \rightarrow +\infty} \frac{\sum_n \langle n|A|n \rangle e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = \langle 0|A(\tau_0)|0 \rangle \quad (1.44)$$

and for a product of operators

$$\lim_{\beta \rightarrow +\infty} \langle A_k(\tau_k) \cdots A_1(\tau_1) \rangle_\beta = \langle 0|A_k(\tau_k) \cdots A_1(\tau_1)|0 \rangle \quad (1.45)$$

Thus in the imaginary time path integral, taking the limit of infinite period

$$\tau = \hbar/k_B T \rightarrow \infty \quad (1.46)$$

amounts to compute vacuum expectation values of observables.

## 1.7 Green functions

We can then in the path integral formalism “forget about” the evolution over imaginary time, and consider only the real time evolution from initial time  $t_{in} = -\infty$  to final time  $t_{out} = +\infty$ . The corresponding “in” and “out” vacuum are simply related to the vacuum (in the Heisenberg picture) by a phase

$$|0_{in}\rangle = |0; t_{in}\rangle = e^{i \frac{t_{in} E_0}{\hbar}} |0\rangle \quad \text{with } t_{in} \rightarrow -\infty \quad (1.47)$$

$$|0_{out}\rangle = |0; t_{out}\rangle = e^{i \frac{t_{out} E_0}{\hbar}} |0\rangle \quad \text{with } t_{out} \rightarrow +\infty \quad (1.48)$$

The vacuum expectation value of time ordered products of operators are called the correlation functions or the Green functions. They are given by the ratio of IN|OUT matrix elements

$$\langle 0|T[A_1(t_1) \cdots A_K(t_K)]|0\rangle = \frac{\langle 0_{out}|T[A_1(t_1) \cdots A_K(t_K)]|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} \quad (1.49)$$

The numerator is given by the path integral over the red “time forward” contour of Fig. 13, the denominator by the blue “time backward” contour. This seems a bit formal and trivial, but keep it in mind for the next lectures. These points are important in quantum field theory, in particular when comparing the path integral formalism and the canonical quantization formalism.

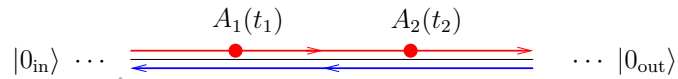


Figure 13: Path integral for vacuum expectation values of

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