

# Quantum dynamics for a fractal spectrum

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## Mandelbrot's definition of a fractal(1986)

*A fractal is a shape made of parts similar to the whole in some way*

## More mathematical formulation (1982)

*A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension*

In our case : objects with a scaling symmetry.

### Example

Sierpinsky gasket

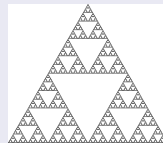


Figure:



Scaling property and log-periodic oscillations :

$$f(x) = \frac{1}{b} f(ax)$$

$$f(x) = x^{\ln(b)/\ln(a)} g\left(\frac{\ln(x)}{\ln(a)}\right)$$

$g$  periodic of period 1.

Another example : triadic Cantor set

"zoom in" by a factor 3 and get the same object, with total mass or number of states rescaled by factor  $\frac{1}{2}$



Figure: Triadic Cantor set

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## Fibonacci cavity

Spacially varying potential yields modification of the density of states of the eigenmodes ;  
quasiperiodic potential  $\rightarrow$  ???  
Longitudinal cavity of  
quasiperiodically varying width  
according to Fibonacci order  $\rightarrow$   
Cantor like spectrum.

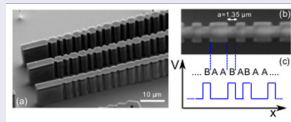


Figure: Fibonacci cavity. Source : [1]

Previous numerical and experimental results<sup>1</sup> show log-periodic behaviour of some quantities...

1. D. Tanese, E. Gurevich, A. Lemaitre, E. Galopin, I. Sagnes, A. Amo, J. Bloch, E. Akkermans, arxiv :1311.3453v1

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Hamiltonian  $H$ , spectrum = triadic Cantor set.

- Dynamics of the wave packet ?
- Which physical quantities highlight the fractal feature ?
- Analytical expression ?
- Conversely : determine the spectrum from these quantities ?

Define an appropriate measure :

$$\int_C f(x) d\mu(x) = \lim_{n \rightarrow +\infty} \left(\frac{3}{2}\right)^n \sum_{P_n} \int_{a_j}^{a_j+3^{-n}} f(x) dx$$

with  $P_n = \{a_j\}$  the set of the left edges of the remaining segments after  $n$  steps. Satisfies :

- $\int_C d\mu(x) = 1$
- $\int_C f(x) d\mu(x) = \frac{1}{2} \int_C f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_C f\left(\frac{x+2}{3}\right) d\mu(x)$

*Note : informally, one can think of  $d\mu$  as  $d\mu(\epsilon) \equiv \rho(\epsilon) d\epsilon$*

Integrated density of states (IDOS) :

$$\begin{aligned}
 N(\epsilon) &= \int_{\text{spectrum}} \theta(\epsilon - \epsilon') d\mu(\epsilon') \\
 &= \frac{1}{2} \int_{\text{spectre}} \theta\left(\epsilon - \frac{\epsilon'}{3}\right) d\mu(\epsilon') + \frac{1}{2} \int_{\text{spectre}} \theta\left(\epsilon - \frac{\epsilon' + 2}{3}\right) d\mu(\epsilon') \\
 &= \frac{1}{2} N(3\epsilon) + g(\epsilon)
 \end{aligned}$$

## Correlation function or return probability

$$C(t) = \frac{1}{t} \int_0^t | \langle \psi(0) | \psi(t') \rangle |^2 dt'$$

Results for the Fibonacci cavity : log-periodic oscillations.  
Likely to be the same for Cantor.

$|\psi(t)\rangle$  > wave function.



Decomposition over the eigenfunction (orthonormal basis) of the Hamiltonian  $H$

$$|\psi(t)\rangle = \int_C g(\epsilon) e^{-i\epsilon t} |\phi(\epsilon)\rangle d\mu(\epsilon)$$

Normalization :

$$\begin{aligned} \langle \psi(t) | \psi(t) \rangle &= \int_C |g(\epsilon)|^2 d\mu(\epsilon) \\ &= 1 \end{aligned}$$

Define :  $d\tilde{\mu}(\epsilon) = |g(\epsilon)|^2 d\mu(\epsilon)$

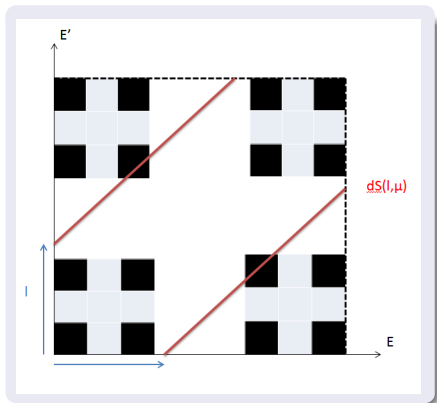
$$\begin{aligned}
C(t) &= \frac{1}{t} \int_0^t | \langle \psi(0) | \psi(t') \rangle |^2 dt' \\
&= \frac{1}{t} \int_0^t \left| \int_C |g(\epsilon)|^2 e^{-i\epsilon t'} d\mu(\epsilon) \right|^2 dt' \\
&= \frac{1}{t} \int_0^t \iint_{C \times C} e^{i(\epsilon' - \epsilon)t'} d\tilde{\mu}(\epsilon) d\tilde{\mu}(\epsilon') \\
&= \frac{1}{t} \iint_{C \times C} \frac{e^{i(\epsilon' - \epsilon)t} - 1}{i(\epsilon' - \epsilon)} d\tilde{\mu}(\epsilon) d\tilde{\mu}(\epsilon') \\
&= \iint_{C \times C} \text{sinc}((\epsilon' - \epsilon)t) d\tilde{\mu}(\epsilon) d\tilde{\mu}(\epsilon') + i \underbrace{\iint_{C \times C} \frac{\sin^2((\epsilon' - \epsilon)t/2)}{(\epsilon' - \epsilon)t}}_{\text{nul by symmetry}}
\end{aligned}$$

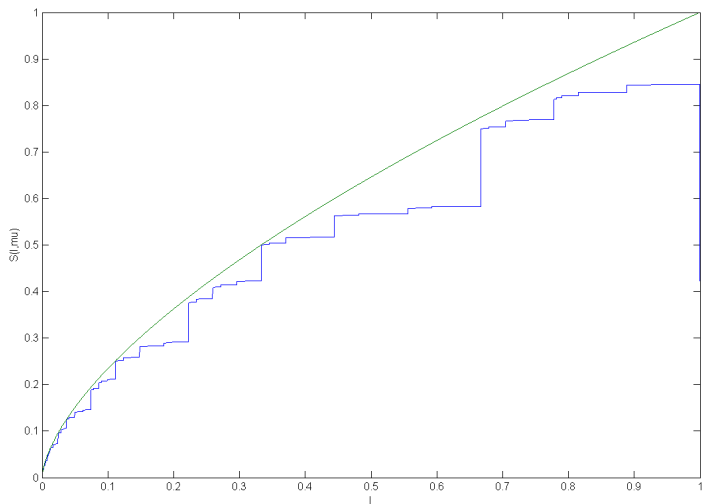
$$C(t) = \iint_{C \times C} \text{sinc}((\epsilon' - \epsilon)t) d\tilde{\mu}(\epsilon) d\tilde{\mu}(\epsilon')$$

Change of variable :  $l = |\epsilon' - \epsilon|$   
 $dS(l, \mu)$  = area for measure  $\mu$  of  
 the Cantor carpet in the strips

Correlation function becomes

$$C(t) = \int_0^1 \text{sinc}(lt) dS(l, \mu)$$





Scaling property + Mellin to calculate  $dS(l, \mu)$ .

## Mellin transform

- $M_f(s) = \int_0^\infty f(x)x^{s-1}dx$
- $M_\phi^{-1}(x) = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \phi(s)x^{-s}ds$

$$\begin{aligned}
 \zeta(\alpha) &= \int_0^1 l^{-\alpha} dS(l, \mu) \\
 &= \iint_{C \times C} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \\
 &= \frac{1}{2} \iint_{C \times C} \frac{d\mu(x)d\mu(y)}{3^{-\alpha}|x-y|^\alpha} + \iint_{C \times C} \frac{d\mu(x)d\mu(y)}{3^{-\alpha}|x-y+2|^\alpha} \\
 &= \frac{3^\alpha}{2} \zeta(\alpha) + \epsilon(\alpha)
 \end{aligned}$$

Therefore :  $\zeta(\alpha) = \frac{\epsilon(\alpha)}{1 - \frac{3^\alpha}{2}}$  and

$$\begin{aligned} dS(l, \mu) &= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \zeta(1-\alpha) l^{-\alpha} d\alpha \\ &= \sum_n \left[ \frac{\alpha - (1 - \alpha_n)}{1 - \frac{3^{1-\alpha}}{2}} l^{-\alpha} \epsilon(\alpha) \right]_{\alpha=\alpha_n} \end{aligned}$$

$$1 - \alpha_n = \frac{\ln(2)}{\ln(3)}, \quad d = \frac{\ln(2)}{\ln(3)}$$

$$dS(l, \mu) = \frac{l^{d-1}}{\ln(3)} \sum_n l^{\frac{2i\pi n}{\ln(3)}} \epsilon \left( d + \frac{2i\pi n}{\ln(3)} \right) = \frac{l^{d-1}}{\ln(3)} \sum_n l^{\frac{2i\pi n}{\ln(3)}} \epsilon_n$$

## Correlation function when the spectrum is a Cantor set

$$C(t) = \frac{1}{\ln(3)} \sum_n \epsilon_n t^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_n - 1)\right) \Gamma(d_n - 1)$$

$$C(t) = \left(\frac{E}{e.u.}\right)^2 \left(\frac{Et}{\hbar}\right)^{-d} \sum_n \frac{\epsilon_n}{\ln(3)} \left(\frac{Et}{\hbar}\right)^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_n - 1)\right) \Gamma(d_n - 1)$$

$$d_n = d + \frac{2i\pi n}{\ln(3)}$$

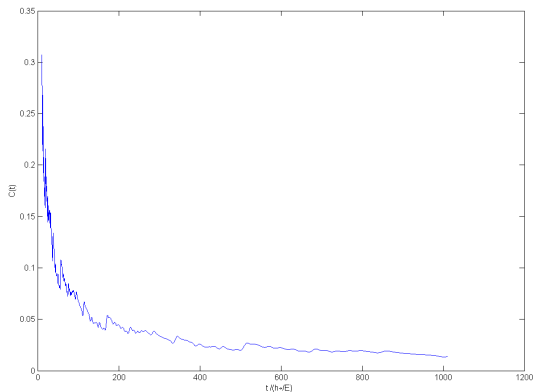
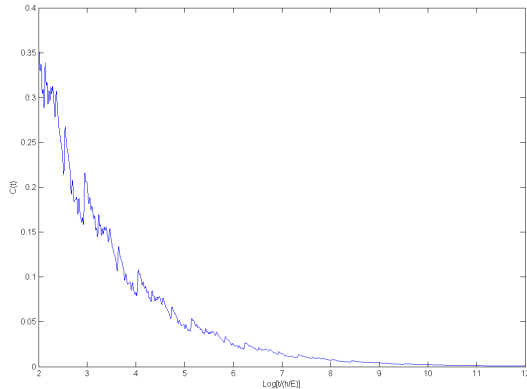


Figure: Correlation function  $C(t)$  for a triadic Cantor set embedded in  $[0, E]$





**Figure:** Correlation function  $C(t)$  for a triadic Cantor set embedded in  $[0,E]$ , log scale

Asymptotic behaviour :

$$|\epsilon_n \sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)| = O\left(\frac{1}{n^{5/2-d}}\right)$$

## Participation ratio

$$P_r(t) = \frac{1}{t} \int_0^t \frac{dt'}{|\langle x | \psi(t') \rangle|^4}$$

Again, numerics for the Fibonacci cavity gave log-periodic oscillations ; so it is likely to be the same for Cantor.

Hamiltonian  $H$ , wave function  $|\psi(t)\rangle$ , orthonormal eigenfunctions basis  $\{|\phi(\epsilon)\rangle\}$

### Green function

$$G(x, x', t) = \langle x | e^{-iHt} | x' \rangle = \int_{\text{spectrum}} \phi(x, \epsilon)^* \phi(x', \epsilon) e^{-i\epsilon t} d\mu(\epsilon)$$

$$G^{A,R}(x, x', t) = \pm i \theta(\mp t) \int_{\text{spectrum}} \phi(x, \epsilon)^* \phi(x', \epsilon) e^{-i\epsilon t} d\mu(\epsilon)$$

$$\text{Im}(G^R)(x, x', t) = \frac{G^A(x, x', t) - G^R(x, x', t)}{i}$$

### Fourier transform

$$\begin{aligned} G^{A,R}(x, x', \epsilon) &= \int_{-\infty}^{+\infty} G^{A,R}(x, x', t) e^{i\epsilon t} dt \\ &= \lim_{\eta \rightarrow 0^+} \int_{\text{spectrum}} \frac{\phi(x, \epsilon')^* \phi(x', \epsilon')}{\epsilon - \epsilon' \mp i\eta} d\mu(\epsilon') \end{aligned}$$

Using :  $\frac{1}{x+i0} = PP\left(\frac{1}{x}\right) - i\pi\delta(x)$  one has

Relation between the Green function and the local density of states

$$\text{Im}(G^R)(x, x', \epsilon) = 2\pi\phi(x, \epsilon)^*\phi(x', \epsilon)d\mu(\epsilon)$$

Density of states, local density of states

$$"\rho(\epsilon)d\epsilon" = d\mu(\epsilon)$$

$$"\rho(\epsilon, x)d\epsilon" = |\phi(x, \epsilon)|^2d\mu(\epsilon)$$

$$"\rho(\epsilon, x, x')d\epsilon" = \phi(x, \epsilon)^*\phi(x', \epsilon)d\mu(\epsilon)$$

*These definition are generic ; in some cases,  $\rho$  is not defined (like for fractals)-> quotation marks.*

$$\begin{aligned}
|\psi(x, t)| &= |\langle x | \psi(t) \rangle| \\
&= \left| \langle x | e^{-iHt} | \psi(0) \rangle \right| \\
&= \left| \int_{\text{space}} \langle x | e^{-iHt} | x' \rangle \psi(x', 0) dx' \right| \\
&= \left| \int_{\text{space}} \int_{\text{spectrum}} e^{-i\epsilon t} \text{Im} G(x, x', \epsilon) \psi(0, t) dx' d\mu(\epsilon) \right| \\
&= \left| \int_{\text{space}} \int_{\text{spectrum}} e^{-i\epsilon t} \phi(x, \epsilon)^* \phi(x', \epsilon) \psi(x', 0) dx' d\mu(\epsilon) \right|
\end{aligned}$$

What can we say about  $\phi(x, \epsilon)^* \phi(x', \epsilon) d\mu(\epsilon)$  ?

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Ideas for continuing the study of the participation ratio :

- ① Assume  $\overline{\phi(0, \epsilon)^* \phi(x, \epsilon)} = f\left(\frac{x}{l_n}\right)$  ?
- ② Find/assume dispersion relation and use Fourier theory
- ③ Prove that  $\int_{band} \phi(x, \epsilon)^* \phi(x', \epsilon) d\mu(\epsilon) < a \|x - x'\|^{-b}$

*Note* (1) is reasonable for the eigenfunctions of the Laplacian ; try numerics on a system with (fractal ?) shape in space in order to have a fractal spectrum for the Laplacian.

(2) Related to definition and convergence of Fourier series on fractals.

(3) Such a result holds for diagonal terms of the resolvent <sup>a</sup> :  
 $G(x, x, t)$

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a. E. Akkermans, O. Benichou, G. V. Dunne, A. Teplyaev, and Raphael Voituriez, Physical Review E 86, 061125 (2012)

One could apply the same kind of calculation to study spontaneous emission and superradiance, and thermodynamics on fractal structures.