

Deterministic Strategies in Concurrent Games and their Relation with Semantics and Logic

M1 Internship Defense

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How did it go?

Remote internship under the supervision of Prof. Glynn Winskel (University of Strathclyde) and Dr. Simon Castellan (INRIA Rennes-Bretagne-Atlantique).

Remote work has quite a lot of drawbacks such as schedule, motivation and communication.

Worked on concurrent interactive processes represented as strategies.

Approximative timeline:

Feb.-Mar. Bibliographic work

Apr.-May Equivalence between Berry-stable maps and deterministic strategies

June Equivalence between dialectica maps and winning deterministic strategies in dialectica games

July-Aug. Report and defense

Concurrent games are ubiquitous.

Aim: providing equivalences between
concurrent games and some known objects in
theoretical CS.

From sequential games to concurrent games

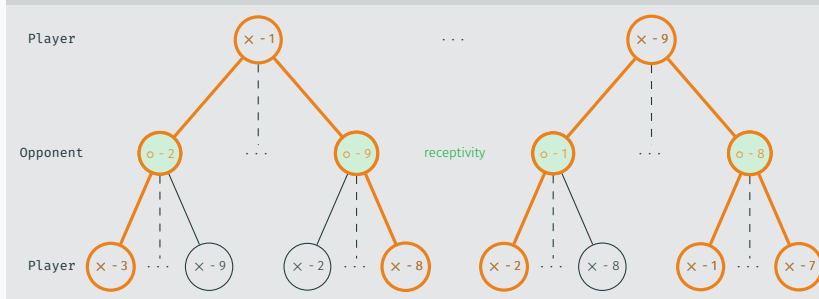
Usual games

Games are often represented **sequentially**.

Games are represented by a **forest** and strategies are **sub-forests**.

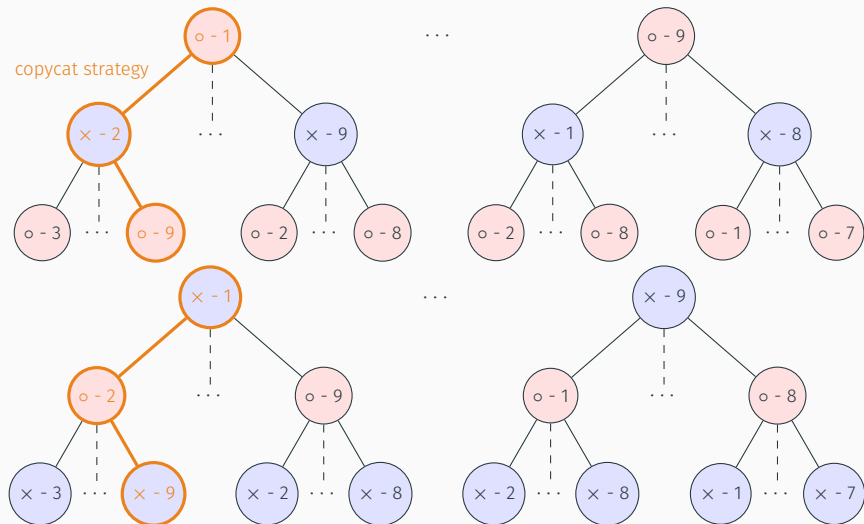
Nodes of **even height** have **same polarity** (e.g. **Player**) and those of **odd height** have the **opposite polarity** (e.g. **Opponent**).

Tic Tac Toe



Parallel games

TicTacToe[⊥] || TicTacToe where **Player** plays x and **Opponent** o.



A bunch of drawbacks

Trace representation of a play

$$\times - 1 \cdot \circ - 2 \cdot \times - 3 \dots$$

which leads to the representation

$$\text{TicTacToe} = \{\varepsilon, \times - 1, \dots, \times - 9, \times - 1 \cdot \circ - 2, \dots, \times - 1 \cdot \circ - 9, \dots\}$$

paths of the tree.

Drawbacks

- **Combinatorial explosion:** $\text{TicTacToe}^\perp \parallel \text{TicTacToe}$ has exponentially more states than TicTacToe
- **Causal dependencies**
- **Consistency**
- **Very rigid:** moves alternate, etc.

A solution: event structures

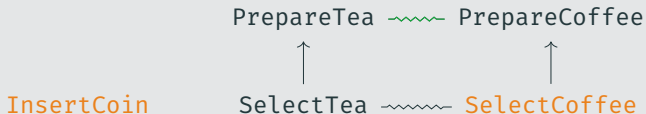
A way to describe **events with causality**: concurrent representation of trees.

Event structure

Tuple $\langle E, \leq_E, \text{Con}_E \rangle$ with E **set of events**, \leq_E **causal dependency** relation on E and $\text{Con}_E \subseteq 2^E$ set of **consistent subsets** of E .

Such that (roughly) \leq_E is well-founded and **conflicts are inherited**.

Vending machine



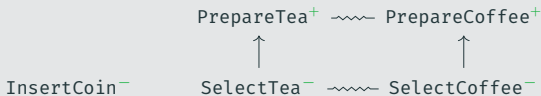
Concurrent games

Main ingredient: event structures.

Polarity to account for the two players: $\text{pol}: E \rightarrow \{+, -\}$.

A play of the game is a **configuration** of E : **downward-closed** and **consistent**.

Vending machine



A possible play:

$\{\text{InsertCoin}^-, \text{SelectTea}^-, \text{PrepareTea}^+\}$

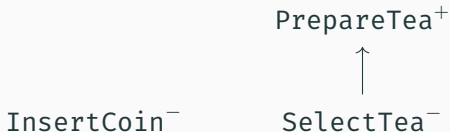
impossible plays:

$\{\text{InsertCoin}^-, \text{SelectTea}^-, \text{SelectCoffee}^-\}$ $\{\text{InsertCoin}^-, \text{PrepareTea}^+\}.$

Concurrent strategies

A first idea

A sub-event structure, like for trees.



Drawbacks

- Might want additional dependencies, e.g. hardcode that user must pay



- Conjunctive causality.

Disjunctive vs conjunctive causality

A vending machine with which we can pay cash or card.



We need to be able to add **additionnal events** to account for **disjunctive causality**.

Intuition: a strategy is a “sub-event structure” with **additional causality** and **possibly duplicated events**.

Concurrent strategy

A strategy on E is the data of an **event structure** S and a **labeling** function $\sigma: S \rightarrow E$ such that:

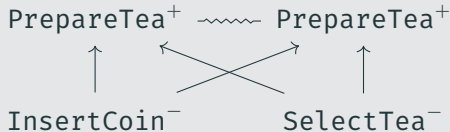
- σ preserves polarity,
- σ preserves configurations,
- σ is injective over configurations.

In addition, we add the following constraints:

- **receptivity**: σ disallows no **Opponent** moves,
- **innocence**: we can add dependencies $\ominus \leq \oplus$ but not $\oplus \leq \ominus$.

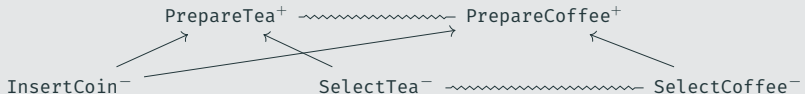
Example strategy

Vending machine 1



- The above **is not** a strategy on the vending machine: it is not receptive since **Opponent** cannot select coffee.
- It is a strategy on vending machines that only sell tea.

Vending machine 2



Determinism

A strategy is deterministic when **consistency depends only on Opponent's moves**.

Deterministic strategy

A strategy $\sigma: S \rightarrow E$ is deterministic iff for every $X \subseteq S$, $X \in \text{Cons}_S$ iff the set $\text{Neg}[X] := \{s' \in S \mid \text{pol}(s) = - \wedge \exists s \in S \cdot s' \leq s\}$ of **Opponent's moves on which moves in X depend** is consistent.

Examples

- Not deterministic:

- Deterministic:

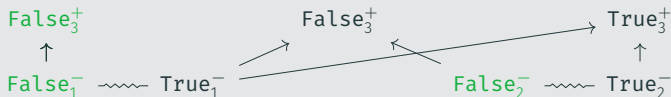
The framework of purely positive games

From strategies to functions

A, B purely +-ve and $\sigma: S \rightarrow A^\perp \parallel B$ deterministic strat. from A to B .

Lazy left-first "and" evaluation σ_{left}

Event structure Bool of booleans: $\text{False}^+ \rightsquigarrow \text{True}^+$.



where $A = \text{Bool} \parallel \text{Bool}$ and $B = \text{Bool}$.

An input $x \in \mathcal{C}^\infty(\text{Bool} \parallel \text{Bool})$ induces a unique +-max configuration $Z \in S$. (existence: receptivity; uniqueness: determinism)

$\sigma[z] = \bar{x} \parallel y$ for some $y \in \mathcal{C}^\infty(\text{Bool})$, thus we can consider y to be the output $\tilde{\sigma}(x)$ of σ on x .

$$\tilde{\sigma}(\text{False}_1 \parallel \text{False}_2) = \text{False}_3$$

The converse transformation?

Turns out that not all functions are can be transformed into a strat.

Berry-stable map

A **Berry-stable map** $f: \langle \mathcal{C}^\infty(A), \subseteq \rangle \rightarrow \langle \mathcal{C}^\infty(B), \subseteq \rangle$ is a map s.t.:

- f is Scott-continuous,
- $f(x \cap y) = f(x) \cap f(y)$ if $x, y \subseteq z \in \mathcal{C}^\infty(A)$.

(Almost) represent **sequential programs**.

The domain of booleans $\mathcal{C}^\infty(\text{Bool})$

Three "and" evaluation strategies:

$$\wedge_{\text{left}} : \begin{cases} \perp, b \mapsto \perp \\ \text{False}, b \mapsto \text{False} \\ \text{True}, b \mapsto b \end{cases} \quad \wedge_{\text{right}} : \begin{cases} b, \perp \mapsto \perp \\ b, \text{False} \mapsto \text{False} \\ b, \text{True} \mapsto b \end{cases} \quad \wedge_{\text{par}} : \begin{cases} \perp, \perp \mapsto \perp \\ \text{False}, b \mapsto \text{False} \\ b, \text{False} \mapsto \text{False} \\ \text{True}, \text{True} \mapsto \text{True} \end{cases}$$

but \wedge_{par} isn't stable: take $x := \{\text{False}\} \parallel \perp$ and $y := \perp \parallel \{\text{False}\}$.

The function associated with σ_{left} is \wedge_{left} .

From stable maps to strategies

The thorny part of the work...

$f: \mathcal{C}^\infty(A) \rightarrow \mathcal{C}^\infty(B)$ **Berry-stable**.

Aim Finding σ whose **+max configurations** are of the form $\bar{x} \parallel f(x)$.

Solution Take the configurations of S to be

$$\mathcal{F} := \{\bar{x} \parallel y \in \mathcal{C}^\infty(A^\perp \parallel B) \mid y \subseteq_B f(x)\}.$$

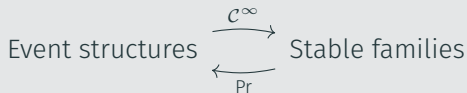
Main issue Build an event structure with prescribed configurations.

Sketch of proof

Stable families and functor Pr

\mathcal{C}^∞ takes an event structure to a **domain of configuration**.

Stable family \mathcal{F} = right notion of domain of configuration.



We also have a map $\text{top}_{\mathcal{F}}: \mathcal{C}^\infty(\text{Pr}(\mathcal{F})) \rightarrow \mathcal{F}$.

We take $\tilde{f}: \text{Pr}(\mathcal{F}) \rightarrow A^\perp \parallel B$ to be $\text{top}_{\mathcal{F}}$.

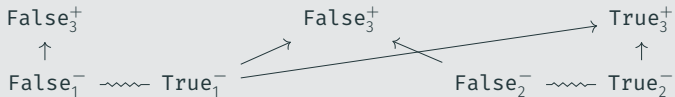
Remark

This construction solves the pb. of **disjunctive causal dependency**.
Vending machine with cash or card gives two events **PrepareTea**,
 $\{\text{PrepareTea}^+, \text{InsertCoin}^-, \text{SelectTea}^-\}$ and
 $\{\text{PrepareTea}^+, \text{InsertCard}^-, \text{SelectTea}^-\}$.

Example

Lazy left-first "and" \wedge_{left}

1. Compute the (complicated) stable family $\mathcal{F}_{\text{left}}$,
2. Compute $\Pr(\mathcal{F})$,
3. Get the associated strategy:



This example illustrates that the correspondence is **one-to-one**.

Conclusion

We have shown that **Berry-stable maps**, which are central in **semantics**, can be emulated in concurrent games.

Continuation? Two games in parallel $A_p \parallel A_n$ with A_p purely +-ve and A_n purely --ve. A little trickier.

These games are connected to **logic**, and more specifically to **dialectica maps**.

Questions?

Winning conditions

Aim: defining what it means for a player to **win**.

Winning conditions and winning strategies

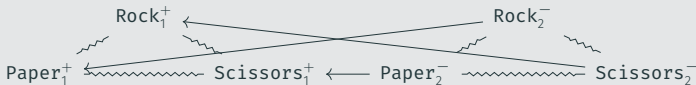
A game with winning condition is basically a game A with **winning conditions (configurations)** $W_A \subseteq \mathcal{C}^\infty(A)$.

Similarly, **losing conditions** L_A , by default $L_A = \mathcal{C}^\infty(A) \setminus W_A$.

- Winning in A^\perp is for the other player to lose in A .
- Winning in $A \parallel B$ is winning either in A or B .

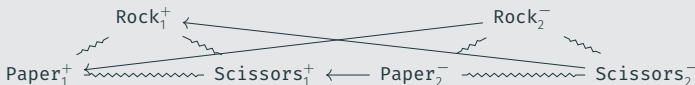
A **winning strategy** ensures that **Player** wins whenever he plays **optimally**.

Winning strategy (cheaty) in Rock-Paper-Scissors



Imperfect information

Cheaty strategy in Rock-Paper-Scissors



A kind of **one-way mirror** for games.

Assign to all events a **level** in $l: E \rightarrow \mathbb{N}$. An event e' is **visible** by e (w.r.t. causal dependency) iff $l(e') \leq l(e)$.

Strategies cannot allow dependencies on maske moves:

$$\forall s, s' \in S \cdot s \leq_s s' \implies l(\sigma(s)) \leq l(\sigma(s')).$$

How to fix Rock-Paper-Scissors?

+ -ve events have level 1 and - -ve have level 2.

- $\ominus \leq \oplus$ dependencies disabled by imperfect info.
- $\oplus \leq \ominus$ dependencies disallowed by innocence.

Dialectica games

Dialectica game = a game $A_p \parallel A_n$ with winning conditions and imperfect info s.t. **Opponent** can see **Player** but not the opposite.

Dialectica maps

A and B dialectica games. A dialectica map is a pair of Berry-stable maps $f: \mathcal{C}^\infty(A_p) \rightarrow \mathcal{C}^\infty(B_p)$ and $g: \mathcal{C}^\infty(A_p \parallel B_n) \rightarrow \mathcal{C}^\infty(A_n)$ s.t.

$$x \parallel g(x \parallel y) \in W_A \implies f(x) \parallel y \in W_B.$$

Wait, where does this come from?

Gödel's dialectica interpretation

Coined by Kurt Gödel in 1958.

<i>Computational world</i>	<i>Dialectica world</i>
λ -calculus	T -theory
Curry-Howard isomorphism	Dialectica interpretation
Cartesian closed category	Dialectica category

1991, Valeria de Paiva and Martin Hyland showed connexion between **Girard's linear logic** and **Dialectica Categories**. **A kind of Lambek's correspondence.**

The category of dialectica games is the **dialectica category** associated with the category of games with **imperfect information** and **winning conditions**.

The equivalence

Dialectica maps are exactly deterministic winning strategies
between dialectica games.

► *From dialectica maps to strategies $\langle f, g \rangle$*

$$f': \left\{ \begin{array}{ccc} \mathcal{C}^\infty(A_p \parallel B_n) & \longrightarrow & \mathcal{C}^\infty(B_p) \\ x \parallel y & \longmapsto & f(x) \end{array} \right. \quad h: \left\{ \begin{array}{ccc} \mathcal{C}^\infty(A_p \parallel B_n) & \longrightarrow & \mathcal{C}^\infty(A_n) \times \mathcal{C}^\infty(B_p) \\ x \parallel y & \longmapsto & (f(x), g(x \parallel y)) \end{array} \right.$$

h is (almost) a Berry-stable map $h: \mathcal{C}^\infty(A_p \parallel B_n^\perp) \rightarrow \mathcal{C}^\infty(A_n^\perp \parallel B_p)$ so it induces a strategy.

► *From strategies to dialectica maps $\sigma: S \rightarrow A^\perp \parallel B$*

σ induces a Berry-stable map $\tilde{\sigma}: \mathcal{C}^\infty(A_p \parallel B_n^\perp) \rightarrow \mathcal{C}^\infty(A_n^\perp \parallel B_p)$ which we can project (up to iso.) into $f: \mathcal{C}^\infty(A_p \parallel B_n) \rightarrow \mathcal{C}^\infty(A_n)$ and $g: \mathcal{C}^\infty(A_p \parallel B_n^\perp) \rightarrow \mathcal{C}^\infty(B_p)$. By innocence and imperfect info, f doesn't depend on its second argument so $\langle f, g \rangle$ suits.

Here again, the correspondence is one-to-one.