

Ordinary differential equations, transport theory and Sobolev spaces

R.J. DiPerna¹ and P.L. Lions²

¹ Department of Mathematics, University of Berkeley, Berkeley, CA 94720, USA

² Ceremade, Université de Paris-Dauphine, Place de Lattre de Tassigny, F-75775 Paris Cedex 16, France

Summary. We obtain some new existence, uniqueness and stability results for ordinary differential equations with coefficients in Sobolev spaces. These results are deduced from corresponding results on linear transport equations which are analyzed by the method of renormalized solutions.

Contents

- I. Introduction 511
- II. Linear transport equations 514
 - II.1 Existence and regularization 514
 - II.2 Uniqueness 517
 - II.3 Existence of renormalized solutions and stability 520
 - II.4 Duality 529
 - II.5 Stability and time compactness 530
- III. Applications to ordinary differential equations 532
 - III.1 The divergence free autonomous case 532
 - III.2 The general autonomous case 537
 - III.3 Time-dependent theory 539
- IV. Counterexamples and remarks 540
 - IV.1 $W^{1,p}$ vector-fields with unbounded divergence 540
 - IV.2 Divergence free vector-fields without integrable first derivatives 541
 - IV.3 Small noise approximations 543
 - IV.4 Remarks 545

I. Introduction

The famous Cauchy-Lipschitz theorem (in its global version) provides global solutions of ordinary differential equations

$$\dot{X} = b(X) \quad \text{for } t \in \mathbb{R}, X(0) = x \in \mathbb{R}^N \tag{1}$$

where b , say, is Lipschitz on \mathbb{R}^N ($N \geq 1$). To simplify matters in this introduction, we restrict temporarily our attention to such autonomous cases. In fact, the

Cauchy-Lipschitz theorem provides much more information than the mere existence and uniqueness of a solution of (1) since it provides a unique continuous flow $X(t, x)$ i.e. a unique continuous function X on $\mathbb{R} \times \mathbb{R}^N$ satisfying (1)—in integral form—and

$$X(t + s, \cdot) = X(t, X(s, \cdot)) \quad \text{on } \mathbb{R}^N, \text{ for all } t, s \in \mathbb{R}. \tag{2}$$

And the continuity in x of X reflects the continuity of the solution upon initial conditions, which in fact can be strengthened to

$$|X(t, x_1) - X(t, x_2)| \leq e^{C_0|t|}|x_1 - x_2| \quad \text{for } t \in \mathbb{R}, x_1, x_2 \in \mathbb{R}^N \tag{3}$$

where C_0 is the Lipschitz constant of b . The stability of X with respect to perturbations on initial conditions can be also modified to take into account stability with respect to perturbations on b : for instance, if b_n converges uniformly on compact sets to b , X_n solves (1) with b replaced by b_n and X_n is bounded on compact sets of $\mathbb{R} \times \mathbb{R}^N$ uniformly in n , then $X_n(t, x)$ converges to $X(t, x)$ uniformly on compact sets of $\mathbb{R} \times \mathbb{R}^N$ —notice that b_n does not need to be Lipschitz. In all these standard results, measure theory plays no role. However, since our goal is to extend all this theory to vector-fields lying in Sobolev spaces instead of being Lipschitz, it is then natural to add the following (easy but no so standard) information also derived from the Cauchy-Lipschitz theorem:

$$e^{-C_1 t} \lambda \leq \lambda \circ X(t) \leq e^{C_1 t} \lambda \quad \text{for all } t \geq 0 \tag{4}$$

for some $C_1 \geq 0$, where λ is the Lebesgue measure on \mathbb{R}^N and $\lambda \circ X(t)$ denotes the image measure of λ by the map $X(t)$ from \mathbb{R}^N into \mathbb{R}^N i.e.

$$\int_{\mathbb{R}^N} \phi d(\lambda \circ X(t)) = \int_{\mathbb{R}^N} \phi(X(t, x)) dx, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^N).$$

Several proofs of (4) are possible: the simplest—but the wrong one—uses (2) and (3) to deduce

$$|X(t, x_1) - X(t, x_2)| \geq e^{-C_0 t}|x_1 - x_2| \quad \text{for all } t \geq 0, x_1, x_2 \in \mathbb{R}^N \tag{5}$$

and thus $X(t)$ is a Lipschitz homeomorphism from \mathbb{R}^N onto \mathbb{R}^N satisfying (4) with $C_0 = C_1$. A better proof—better since it yields a sharper estimate and the correct explanation of (4)—is based upon the following (standard) observation: let $\bar{\lambda}(t)$ denote $\lambda \circ X(t)$, then one can show that $\bar{\lambda}(t)$ satisfies in the sense of distributions

$$\frac{\partial \bar{\lambda}}{\partial t} - \text{div}(b \bar{\lambda}) = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^N, \quad \bar{\lambda}|_{t=0} = \lambda$$

and $\bar{\lambda}$ admits a density r with respect to λ which satisfies

$$\frac{\partial r}{\partial t} - \text{div}(br) = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^N, \quad r|_{t=0} \equiv 1 \text{ on } \mathbb{R}^N. \tag{6}$$

And one deduces easily

$$e^{-C_1 t} \leq r(t, x) \leq e^{C_1 t} \quad \text{on } (0, \infty) \times \mathbb{R}^N \tag{7}$$

where

$$C_1 = \|\operatorname{div} b\|_{L^\infty} . \tag{8}$$

Roughly speaking, the divergence of b governs the exponential rate of compression or dilation of Lebesgue’s measure transported by the flow.

It has been a permanent question to extend any part of this elementary theory to less regular vector fields b —question pertinent to a wide variety of applications ranging from Fluid Mechanics to Control Theory. Various (somewhat limited) extensions have been proposed but seemed to be of restricted applicability in view of standard examples.

It is our goal here to provide a quite general (and natural) extension to vector-fields b having bounded divergence and some Sobolev type regularity. Our motivation stems from kinetic theory and fluid mechanics (see for instance [5], [6]) where such questions are fundamental to understand the “characteristics” of the physical system and where only limited Sobolev regularity seems to be available. More precisely, we will show that if $b \in W_{loc}^{1,1}(\mathbb{R}^N)$, $\operatorname{div} b \in L^\infty(\mathbb{R}^N)$ and

$$b = b_1 + b_2, \quad b_1 \in L^p(\mathbb{R}^N), \quad b_2(1 + |x|)^{-1} \in L^\infty(\mathbb{R}^N) \quad (\text{for some } 1 \leq p \leq \infty)$$

(we will in fact cover even more general situations) then there exists a unique “flow” $X \in C(\mathbb{R}; L_{loc}^p(\mathbb{R}^N))$ solving (1), satisfying (4) and (2) a.e.. In addition, $X \in L_{loc}^p(\mathbb{R}^N; C([-T, T]))$ (for all $T \in (0, \infty)$). Finally, we will also obtain stability results under perturbations of b (and, in particular, convergence of the flows obtained by smoothing b). The corresponding time-dependent theory will also be considered assuming an L^1 -time dependence in all the conditions above. All these results will be obtained in section III below.

We will also present in section IV below examples showing the sharpness of these results: two different types of counterexamples will be presented, the first class is taken from A. Beck [1] and provides for any $p \in (1, \infty)$ a vector-field $b \in C_b(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ with two (in fact infinitely many) distinct continuous flows, showing thus the relevance of the bound on $\operatorname{div} b$. The second class includes an example of a vector-field $b \in W_{loc}^{s,1}(\mathbb{R}^2)$ for any $s < 1$ satisfying $\operatorname{div} b = 0$, with two distinct measure preserving L^1 -flows, showing the sharpness of the $W_{loc}^{1,1}$ regularity.

It is worth emphasizing a striking aspect of our method of attack: all these results on ODE’s will be deduced from the analysis of the associated PDE namely the following transport equation

$$\frac{\partial u}{\partial t} - b \cdot \nabla u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N . \tag{10}$$

In some sense, the Lagrangian formulation will be deduced from the Eulerian one. This analysis will be based upon the use of renormalized solutions (introduced by the authors in the context of kinetic models—see [2], [3], [4], [5]), and a regularization argument. It will lead to existence, uniqueness and stability results which are presented and proved in section II.

Let us conclude this Introduction by mentioning several forthcoming applications of our results to kinetic Vlasov-type models ([5]), fluid mechanics including

the evidence of singular phenomena in 3-D Euler equations ([7]) or existence results for density-dependent models ([6]).

II. Linear transport equations

II.1 Existence and regularization

We begin with a simple existence result for the following linear transport equation

$$\frac{\partial u}{\partial t} - b \cdot \nabla_x u + cu = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{11}$$

where $T > 0$ is given and we will always assume that b, c satisfy at least

$$b \in L^1(0, T; (L^1_{loc}(\mathbb{R}^N))^N), \quad c \in L^1(0, T; L^1_{loc}(\mathbb{R}^N)). \tag{12}$$

Given an initial condition u^0 in $L^p(\mathbb{R}^N)$ for some $p \in [1, \infty]$, we wish to build a solution of (11) in $L^\infty(0, T; L^p(\mathbb{R}^N))$. Of course, the equation will be understood in distributions sense that is (for instance)

$$-\int_0^T dt \int_{\mathbb{R}^N} dx u \frac{\partial \phi}{\partial t} - \int_{\mathbb{R}^N} u^0 \phi(0, x) dx + \int_0^T dt \int_{\mathbb{R}^N} dx u \{ \text{div}(b\phi) + c\phi \} = 0 \tag{13}$$

for all test functions $\phi \in C^\infty([0, T] \times \mathbb{R}^N)$ with compact support in $[0, T) \times \mathbb{R}^N$ — we will denote this space by $\mathcal{D}([0, T) \times \mathbb{R}^N)$.

Observe however that this definition makes sense provided we assume

$$c + \text{div } b \in L^1(0, T; L^q_{loc}(\mathbb{R}^N)), \quad b \in L^1(0, T; (L^q_{loc}(\mathbb{R}^N))^N) \tag{14}$$

where q is the conjugate exponent of p ($\frac{1}{q} + \frac{1}{p} = 1$).

With these notations, we have the

Proposition II.1. *Let $p \in [1, \infty]$, $u^0 \in L^p(\mathbb{R}^N)$, assume (12), (14) and*

$$\begin{cases} c + \frac{1}{p} \text{div } b \in L^1(0, T; L^\infty(\mathbb{R}^N)) & \text{if } p > 1 \\ c, \text{div } b \in L^1(0, T; L^\infty(\mathbb{R}^N)) & \text{if } p = 1. \end{cases} \tag{15}$$

Then, there exists a solution u of (11) in $L^\infty(0, T; L^p(\mathbb{R}^N))$ corresponding to the initial condition u^0 .

Remark. The same result holds if we replace 0 in the right-hand side of (11) by $f \in L^1(0, T; L^p(\mathbb{R}^N))$.

Proof. The proof consists only in a justification by approximation and regularization of the following formal estimates. First of all, if $p = \infty$, one has formally by standard arguments

$$\|u(t)\|_\infty \leq \|u^0\|_\infty + \int_0^t \|cu\|_\infty ds$$

hence in view of (12)

$$\|u(t)\|_\infty \leq C_0 \|u^0\|_\infty \quad \text{a.e. on } (0, T) \tag{16}$$

where C_0 depends only on the norm of c in $L^1(0, T; L^\infty(\mathbb{R}^N))$. Next, if $p < \infty$, one observes that formally

$$\frac{\partial}{\partial t} |u|^p - b \cdot \nabla_x |u|^p + pc|u|^p = 0$$

and thus integrating this equation over \mathbb{R}^N , one deduces

$$\frac{d}{dt} \int_{\mathbb{R}^N} |u|^p dx \leq \left(\int_{\mathbb{R}^N} |u|^p dx \right) \{ \|pc + \operatorname{div} b\|_\infty(t) \}.$$

Therefore, using (15)

$$\|u(t)\|_p \leq C_0 \|u^0\|_p \quad \text{a.e. on } (0, T) \tag{17}$$

where C_0 depends only on the norm of $c + \frac{1}{p} \operatorname{div} b$ in $L^1(0, T; L^\infty(\mathbb{R}^N))$.

Now, to prove existence, we regularize b, c, u^0 by convolution in x i.e. we consider $b_\varepsilon = b * \rho_\varepsilon$, $c_\varepsilon = c * \rho_\varepsilon$, $u_\varepsilon^0 = u^0 * \rho_\varepsilon$ where $\rho_\varepsilon = \frac{1}{\varepsilon^N} \rho\left(\frac{\cdot}{\varepsilon}\right)$, $\rho \in \mathcal{D}_+(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho dx = 1$. Since we assumed only L^1_{loc} integrability in (12), a further approximation by truncation is necessary that we leave to the reader and we thus assume that $b_\varepsilon \in L^1(0, T; C^1_b(\mathbb{R}^N))$, $c_\varepsilon \in L^1(0, T; C^1_b(\mathbb{R}^N))$. Then, by standard considerations, there exists a unique solution $u_\varepsilon \in C([0, T]; C^1_b(\mathbb{R}^N))$ of

$$\frac{\partial u_\varepsilon}{\partial t} - b_\varepsilon \cdot \nabla_x u_\varepsilon + c_\varepsilon u_\varepsilon = 0 \quad \text{in } (0, T) \times \mathbb{R}^N, \quad u_\varepsilon|_{t=0} = u_\varepsilon^0 \quad \text{in } \mathbb{R}^N.$$

Then, in view of (16) and (17)—estimates which can now be proved rigorously— u_ε is bounded in $L^\infty(0, T; L^p(\mathbb{R}^N))$ uniformly in ε . Extracting subsequences if necessary, we may assume when $p > 1$ that u_ε converges weakly in $L^\infty(0, T; L^p(\mathbb{R}^N))$ and weakly $*$ if $p = \infty$ to some u . Checking that (13) holds is now a simple exercise that we skip: remark only that

$$c_\varepsilon + \operatorname{div} b_\varepsilon, \quad b_\varepsilon \rightarrow c + \operatorname{div} b, \quad b \quad \text{in } L^1(0, T; L^q_{\text{loc}}(\mathbb{R}^N)).$$

When $p = 1$, the same proof applies provided we show that u_ε is weakly relatively compact in $L^\infty(0, T; L^1_{\text{loc}}(\mathbb{R}^N))$. In order to do so, we consider $u_n^0 \in \mathcal{D}(\mathbb{R}^N)$ converging in $L^1(\mathbb{R}^N)$ to u^0 and we denote by $u_{n,\varepsilon}$ the corresponding approximated solutions as above.

By the preceding arguments, using (15), we see that

$$\|u_{n,\varepsilon}\|_{L^\infty(0, T; L^p(\mathbb{R}^N))} \leq C(n, p) \text{ (ind' of } \varepsilon) \quad \text{for all } p > 1$$

while

$$\|u_\varepsilon - u_{n,\varepsilon}\|_{L^\infty(0, T; L^1(\mathbb{R}^N))} \leq C_0 \|u^0 - u_{n,\varepsilon}^0\|_1 \leq C_0 \|u^0 - u_n^0\|_1.$$

And this yields the desired weak compactness. △

We now turn to the main result of this section: this result will show that, under appropriate conditions on b , (weak) solutions of (11) can be approximated by smooth (in x) solutions of (11) with small error terms. This result will be one of the fundamental technical tools required throughout the paper. Let ρ_ε be a regularizing kernel i.e.

$$\rho_\varepsilon = \frac{1}{\varepsilon^N} \rho\left(\frac{\cdot}{\varepsilon}\right) \quad \text{with} \quad \rho \in \mathcal{D}_+(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \rho \, dx = 1, \quad \varepsilon > 0.$$

Theorem II.1. *Let $1 \leq p \leq \infty$, let $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ be a solution of (11) and assume that b, c satisfy*

$$b \in L^1(0, T; W_{loc}^{1,\alpha}(\mathbb{R}^N)), \quad c \in L^1(0, T; L_{loc}^\alpha(\mathbb{R}^N)) \quad \text{for some } \alpha \geq q. \quad (18)$$

Then, if we denote by $u_\varepsilon = u * \rho_\varepsilon$, u_ε satisfies

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon + c u_\varepsilon = r_\varepsilon \quad (19)$$

where r_ε converges to 0 as ε goes to 0 in $L^1(0, T; L_{loc}^\beta(\mathbb{R}^N))$ and β is given by: $\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}$ if α or $p < \infty$, $\beta < \infty$ is arbitrary if $\alpha = p = \infty$.

Remarks. 1) The same results hold if we replace 0 in the right-hand side of (11) by $f \in L^1(0, T; L_{loc}^\beta(\mathbb{R}^N))$.

2) The same results hold if we replace the equality in (11) by an inequality; then, of course, (19) becomes the corresponding inequality.

3) The same results hold if we replace b in (19) by $b_\varepsilon = b * \rho_\varepsilon$.

4) Analogous results hold if we modify the time integrability of u and b, c . For instance, if $b \in L^\gamma(0, T; W_{loc}^{1,\alpha}(\mathbb{R}^N))$, $c \in L^\gamma(0, T; L_{loc}^\alpha(\mathbb{R}^N))$ where $1 \leq \gamma < \infty$ (to simplify), then $r_\varepsilon \rightarrow 0$ in $L^\gamma(0, T; L_{loc}^\beta(\mathbb{R}^N))$.

5) The above result still holds with $\beta = 1$ if we take u continuous in $(t, x) \in [0, T] \times \mathbb{R}^N$, $b, c \in L^1(0, T; L_{loc}^1)$ and

$$\frac{\partial}{\partial x_i} b_j(t, x) \text{ is a bounded measure on } [0, T] \times \mathbb{R}^N$$

for all compact sets $K \subset \mathbb{R}^N$, $1 \leq i, j \leq N$. △

The proof of Theorem II.1 is a trivial consequence of the following

Lemma II.1. i) *Let $B \in (W_{loc}^{1,\alpha}(\mathbb{R}^N))^N$, $w \in L_{loc}^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$, $\alpha \geq q$. Then*

$$(B \cdot \nabla w) * \rho_\varepsilon - B \cdot \nabla(w * \rho_\varepsilon) \rightarrow 0 \quad \text{in } L_{loc}^\beta(\mathbb{R}^N)$$

where β is given in Theorem II.1.

ii) *Let $B \in L^1(0, T; (W_{loc}^{1,\alpha}(\mathbb{R}^N))^N)$, $w \in L^\infty(0, T; L_{loc}^p(\mathbb{R}^N))$; then*

$$(B \cdot \nabla w) * \rho_\varepsilon - B \cdot \nabla(w * \rho_\varepsilon) \rightarrow 0 \quad \text{in } L^1(0, T; L_{loc}^\beta(\mathbb{R}^N)).$$

Proof. Part i) of Lemma II.1 seems to belong to the folklore of real analysis and thus we will present a rather sketchy proof of it. And we will entirely skip the proof

of ii) since it requires only to reproduce carefully the proof of i), keeping track of the time dependence. In order to prove i), we first observe that

$$\begin{aligned} (B \cdot \nabla w) * \rho_\varepsilon - B \cdot \nabla (w * \rho_\varepsilon) &= - \int w(y) [\operatorname{div}_y \{B(y) \rho_\varepsilon(x - y)\} + B(x) \cdot \nabla \rho_\varepsilon(x - y)] dy \\ &= \int w(y) \{(B(y) - B(x)) \cdot \nabla \rho_\varepsilon(x - y)\} dy - (w \operatorname{div} B) * \rho_\varepsilon. \end{aligned}$$

By standard results on convolutions, the second term converges in L^p_{loc} as ε goes to 0 to $w \operatorname{div} B$.

Next, we estimate the first term as follows for ε small enough

$$\begin{aligned} &\left\| \int w(y) \{(B(y) - B(x)) \cdot \nabla \rho_\varepsilon(x - y)\} dy \right\|_{L^p(B_R)} \leq C \|w\|_{L^p(B_{R+1})} \\ &\left\{ \int_{B_{R+1}} dx \int_{|x-y| \leq C\varepsilon} \left\{ \frac{|B(y) - B(x)|}{\varepsilon} \right\}^\alpha dy \right\}^{1/\alpha} \end{aligned}$$

where B_M denotes the ball of radius M , R is fixed, and C denote various constants independent of ε , R , w , B . Then, we remark that

$$\begin{aligned} &\left\{ \int_{B_{R+1}} dx \int_{|x-y| \leq C\varepsilon} \left\{ \frac{|B(y) - B(x)|}{\varepsilon} \right\}^\alpha dy \right\}^{1/\alpha} = \\ &\left\{ \int_{B_{R+1}} dx \int_{|z| \leq C} dz \left(\int_0^1 dt |\nabla B(x + t\varepsilon z)| \right)^\alpha \right\}^{1/\alpha} \\ &\leq C \|\nabla B\|_{L^\alpha(B_{R+1+c})}. \end{aligned}$$

In order to conclude, we just need to observe that it is now enough to show that

$$\int w(y) \{B(y) - B(x)\} \cdot \nabla \rho_\varepsilon(x - y) dy \xrightarrow{\varepsilon} w \operatorname{div} B \quad \text{in } L^p_{\text{loc}}$$

when w and B are smooth. Indeed, the general case follows by density using the above bounds. But, this convergence is clear if w and B are smooth since

$$\int w(y) \{B(y) - B(x)\} \cdot \nabla \rho_\varepsilon(x - y) dy \xrightarrow{\varepsilon} -w(x) \sum_{i,j=1}^N \frac{\partial}{\partial x_i} B_j(x) \cdot \int z_i \frac{\partial}{\partial z_j} \rho(z) dz$$

and

$$- \sum_{i,j=1}^N \frac{\partial}{\partial x_i} B_j(x) \cdot \int z_i \frac{\partial}{\partial z_j} \rho(z) dz = \operatorname{div} B. \quad \triangle$$

II.2 Uniqueness

Theorem II.2. *Let $1 \leq p \leq \infty$, let $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ be a solution of (11) for the initial condition $u^0 \equiv 0$ (i.e. u satisfies (13) with $u^0 \equiv 0$). We assume that $c, \operatorname{div} b \in L^1(0, T; L^\infty(\mathbb{R}^N))$, $b \in L^1(0, T; W^{1,q}_{\text{loc}}(\mathbb{R}^N))$ and*

$$\frac{b}{1 + |x|} \in L^1(0, T; L^1(\mathbb{R}^N)) + L^1(0, T; L^\infty(\mathbb{R}^N)). \quad (20)$$

Then, $u \equiv 0$.

Remark. It will be clear from the proof below that (20) may be somewhat relaxed . . .

Combining Proposition II.1 and Theorem II.2, we immediately deduce the

Corollary II.1. *Let $1 \leq p \leq \infty$, let $u^0 \in L^p(\mathbb{R}^N)$. We assume that $c, \operatorname{div} b \in L^1(0, T; L^\infty(\mathbb{R}^N)), b \in L^1(0, T; W_{\text{loc}}^{1,q}(\mathbb{R}^N))$ and (20). Then, there exists a unique solution u of (11) in $L^\infty(0, T; L^p(\mathbb{R}^N))$ corresponding to the initial condition u^0 .*

Proof of Theorem II.2. We first apply Theorem II.1 and deduce

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon + cu_\varepsilon = r_\varepsilon \xrightarrow{\varepsilon} 0 \text{ in } L^1(0, T; L_{\text{loc}}^1(\mathbb{R}^N)).$$

From this we deduce that if $\beta \in C^1(\mathbb{R}), \beta'$ is bounded on \mathbb{R} then

$$\frac{\partial}{\partial t} \beta(u_\varepsilon) - b \cdot \nabla \beta(u_\varepsilon) + cu_\varepsilon \beta'(u_\varepsilon) = r_\varepsilon \beta'(u_\varepsilon).$$

And letting ε go to 0, we obtain

$$\frac{\partial}{\partial t} \beta(u) - b \cdot \nabla \beta(u) + cu \beta'(u) = 0 \text{ in } (0, T) \times \mathbb{R}^N. \tag{21}$$

Next, we consider some smooth cut-off functions $\phi_R = \phi\left(\frac{\cdot}{R}\right)$ for $R \geq 1$ where $\phi \in \mathcal{D}_+(\mathbb{R}^N), \operatorname{Supp} \phi \subset B_2, \phi \equiv 1$ on B_1 . Then, we multiply (21) by ϕ_R and we find

$$\frac{d}{dt} \int \beta(u) \phi_R dx + \int \{cu \beta'(u) + \operatorname{div} b \beta(u)\} \phi_R = - \int \beta(u) b \cdot \nabla \phi_R. \tag{22}$$

Let $M \in (0, \infty)$, we would like to choose $\beta(t) = (|t| \wedge M)^p$ which is Lipschitz on \mathbb{R} but not C^1 : this point may be overcome by tedious approximation arguments that we skip and we deduce from (22)

$$\frac{d}{dt} \int (|u| \wedge M)^p \phi_R dx \leq C \int (|u| \wedge M)^p \phi_R dx + \frac{C}{R} \int_{R \leq |x| \leq 2R} (|u| \wedge M)^p |b(t, x)| dx.$$

Next, we observe that $(|u| \wedge M)^p \in L^\infty(0, T; L^1 \cap L^\infty)$ while

$$\frac{|b(t, x)|}{R} \mathbf{1}_{R \leq |x| \leq 2R} \leq \frac{|b(t, x)|}{1 + |x|} \mathbf{1}_{R \leq |x|}.$$

Therefore, we deduce from (20)

$$\frac{d}{dt} \int (|u| \wedge M)^p \phi_R \leq C \int (|u| \wedge M)^p \phi_R + m(t) \int_{|x| \geq R} (|u| \wedge M)^p + CM^p \int_{|x| \geq R} \frac{|b_1(t, x)|}{1 + |x|}$$

where $b = b_1 + b_2, m = \|b_2/(1 + |x|)\|_\infty, b_1/(1 + |x|) \in L^1(0, T; L^1(\mathbb{R}^N))$ and $b_2/(1 + |x|) \in L^1(0, T; L^\infty(\mathbb{R}^N))$. Letting R go to ∞ , this yields

$$\frac{d}{dt} \int (|u| \wedge M)^p \leq C \int (|u| \wedge M)^p.$$

Therefore, $|u| \wedge M \equiv 0$, and we conclude when $p < \infty$ letting M go to ∞ .

When $p = \infty$, some further arguments seem to be necessary. First of all, if $u \in L^\infty(0, T; L^\infty \cap L^1)$ the proof above applies and yields the uniqueness. In the general case, we will use a duality argument that we only sketch below (since we will deal with much more general duality results later on): let $\phi \in \mathcal{D}((0, T) \times \mathbb{R}^N)$, it is enough to show that

$$\int_0^T \int_{\mathbb{R}^N} u \phi dx dt = 0 .$$

In order to do so, one considers the solution of the following backwards problem

$$\frac{\partial \Phi}{\partial t} - b \cdot \nabla \Phi - (c + \operatorname{div} b) \Phi = \phi \quad \text{in } (0, T) \times \mathbb{R}^N, \quad \Phi|_{t=T} = 0 \quad \text{on } \mathbb{R}^N .$$

By Proposition II.1, such a solution Φ exists and is in fact unique by the above proof. Furthermore, $\Phi \in L^\infty(0, T; L^1 \cap L^\infty)$.

Next, we invoke the regularization result Theorem II.1 to deduce

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon + c u_\varepsilon = r_\varepsilon \quad \text{in } (0, T) \times \mathbb{R}^N, \quad u_\varepsilon|_{t=0} = 0 \quad \text{on } \mathbb{R}^N$$

$$\frac{\partial \Phi_\varepsilon}{\partial t} - b \cdot \nabla \Phi_\varepsilon - (c + \operatorname{div} b) \Phi_\varepsilon = \phi + \psi_\varepsilon \quad \text{in } (0, T) \times \mathbb{R}^N, \quad \Phi_\varepsilon|_{t=T} = 0 \quad \text{on } \mathbb{R}^N$$

where $r_\varepsilon, \psi_\varepsilon \rightarrow 0$ in $L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^N))$. Multiplying the first equation by $\Phi_\varepsilon \phi_R$, integrating by parts and using the second equation we find

$$-\int_0^T \int_{\mathbb{R}^N} u_\varepsilon (\phi + \psi_\varepsilon) \phi_R dx dt + r_\varepsilon \int_0^T \int_{\mathbb{R}^N} \Phi_\varepsilon \phi_R dx dt + \int_0^T \int_{\mathbb{R}^N} u_\varepsilon \Phi_\varepsilon b \cdot \nabla \phi_R dx dt = 0 .$$

Letting ε go to 0, we deduce

$$\left| \int_0^T \int_{\mathbb{R}^N} u \phi dx dt \right| \leq \int_0^T \int_{\mathbb{R}^N} |u| |\Phi| \frac{|b|}{1 + |x|} 1_{R \leq |x| \leq 2R} dx dt .$$

And we conclude easily since $|u| |\Phi| \in L^\infty(0, T; L^1 \cap L^\infty)$. △

In fact, the above proof also shows the

Corollary II.2. *Under the assumptions of Corollary II.1, u satisfies*

$$u \in C([0, T]; L^p(\mathbb{R}^N)) \quad \text{if } p < \infty . \tag{23}$$

$$\frac{\partial}{\partial t} \beta(u) - b \cdot \nabla \beta(u) + c u \beta'(u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{24}$$

for all functions $\beta \in C^1(\mathbb{R})$ such that

$$|\beta'(t)| \leq C(1 + |t|^r) \tag{25}$$

with $r = p - 1$ if $q > N$, $r < p - 1$ if $q = N$, $r = \frac{p}{N}$ if $q < N$.

Proof. (24) is an easy consequence of (21) observing that $\beta(u) \in L^\infty(0, T; L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^N))$ and using Sobolev inequalities to deduce that $b \cdot \beta(u) \in L^1(0, T; L^1_{loc}(\mathbb{R}^N))$ if (25) holds. Next, the proof of Theorem II.2 shows that

$$\frac{d}{dt} \int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} \{pc + \operatorname{div} b\} |u|^p dx = 0 \quad \text{a.e. on } (0, T). \tag{26}$$

Therefore, $\|u(t)\|_p \in C([0, T])$ and this implies easily, in view of (11), that $u \in C([0, T]; L^p(\mathbb{R}^N))$ if $p > 1$. The case $p = 1$ is slightly more delicate: first of all, approximating u^0 by $u_n^0 \in L^1 \cap L^p$ (for some $p > 1$) and using (26) to deduce that the corresponding solution u_n of (11) converges to u in $L^1(\mathbb{R}^N)$ uniformly on $[0, T]$, we already obtain that $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$ and that

$$\sup_{t \in [0, T]} \operatorname{ess} \int_{\mathbb{R}^N} |u(t)| 1_{|u(t)| \geq M} dx \rightarrow 0 \quad \text{as } M \rightarrow \infty. \tag{27}$$

Next, we consider $\zeta \in C^\infty(\mathbb{R}^N)$, $0 \leq \zeta \leq 1$, $\zeta \equiv 0$ on $B_{1/2}$, $\zeta \equiv 1$ if $|x| \geq 1$ and we introduce $\zeta_R = \zeta(Rx)$ for $R \geq 1$. Then, copying the proof of Theorem II.1 we find for all $M \geq 0$

$$\frac{d}{dt} \int_{\mathbb{R}^N} |u| \wedge M \zeta_R dx \leq C \int_{\mathbb{R}^N} |u| \wedge M \zeta_R dx + C \int_{R/2 \leq |x| \leq R} |u| \wedge M \frac{|b|}{1 + |x|}.$$

This yields

$$\sup_{t \in [0, T]} \operatorname{ess} \int_{\mathbb{R}^N} |u| \wedge M \zeta_R dx \rightarrow 0 \quad \text{as } R \rightarrow \infty, \text{ for all } M > 0. \tag{28}$$

And we conclude combining (27), (28) and the fact that $u \in C([0, T]; L^1_{loc})$. \triangle

II.3 Existence of renormalized solutions and stability

In this section, we extend the range of the existence and uniqueness results proven in the preceding sections by requiring less integrability conditions on the derivatives of b and the initial conditions and we prove a fundamental stability result. In order to state precisely our results, we need to introduce a few notions and notations.

First of all, the conditions on b, c we will assume throughout this section (and the following ones) are

$$\begin{cases} b \in L^1(0, T; W^{1,1}_{loc}(\mathbb{R}^N)), \quad \operatorname{div} b \in L^1(0, T; L^\infty(\mathbb{R}^N)), \\ c \in L^1(0, T; L^\infty(\mathbb{R}^N)), \end{cases} \tag{*}$$

$$\frac{|b(t, x)|}{1 + |x|} \in L^1(0, T; L^1(\mathbb{R}^N)) + L^1(0, T; L^\infty(\mathbb{R}^N)). \tag{**}$$

Next, we need to introduce a set of functions that we will denote by L^0 : L^0 is the set of all measurable functions u on \mathbb{R}^N with values in $\overline{\mathbb{R}}$ such that

$$\operatorname{meas} \{ |u| > \lambda \} < \infty, \quad \text{for all } \lambda > 0. \tag{29}$$

Observe that whenever $\beta \in C(\mathbb{R})$ is bounded and vanishes near 0 then $\beta(u) \in L^1 \cap L^\infty(\mathbb{R}^N)$. We will say that $u^n \rightarrow u$ in L^0 if $\beta(u^n) \rightarrow \beta(u)$ in L^1 for all such β and that u^n is bounded in L^0 if $\beta(u^n)$ is bounded in L^1 for all such β . In this way, the sets $L^\infty(0, T; L^0)$, $C([0, T]; L^0)$ are well-defined. Finally, L^0_{loc} will stand for the corresponding local versions (in fact L^0_{loc} is nothing but the set of all measurable functions from \mathbb{R}^N into \mathbb{R}).

We now turn to the notion of renormalized solutions of (11). We will say that $u \in L^\infty(0, T; L^0)$ is a renormalized solution of (11) if the following holds

$$\frac{\partial}{\partial t} \beta(u) - b \cdot \nabla \beta(u) + cu\beta'(u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{30}$$

for all $\beta \in C^1(\mathbb{R})$, β and $\beta'(1 + |t|)^{-1}$ are bounded on \mathbb{R} and β vanishes near 0. We will call such functions β admissible functions. Observe that these conditions imply that

$$\beta(u) \quad \text{and} \quad u\beta'(u) \in L^\infty(0, T; L^1(\mathbb{R}^N)).$$

And, of course, $u \in L^\infty(0, T; L^0)$ will be a renormalized solution of (11) corresponding to the initial condition u^0 (given) in L^0 if $\beta(u)$ solves (30) with $\beta(u^0)$ as initial condition for all β as above. We may now state our main results.

Theorem II.3. *We assume (*) and (**).*

1) (Consistency). *Let $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ and let $b \in L^1(0, T; L^p(\mathbb{R}^N))$ with $1 \leq p \leq \infty$. If u is a renormalized solution of (11), then u is a solution of (11). If u is a solution of (11) and $b \in L^1(0, T; W^{1,q}_{\text{loc}}(\mathbb{R}^N))$, then u is a renormalized solution.*

2) (Existence and uniqueness). *Let $u^0 \in L^0(\mathbb{R}^N)$, then there exists a unique renormalized solution u of (11) in $L^\infty(0, T; L^0(\mathbb{R}^N))$ corresponding to the initial condition u^0 . Furthermore, $u \in C([0, T]; L^0(\mathbb{R}^N))$; $u \in C([0, T]; L^p(\mathbb{R}^N))$ if $u^0 \in L^p(\mathbb{R}^N)$ for some $1 \leq p \leq \infty$ and*

$$u \in L^\infty(0, T; L^\infty(\mathbb{R}^N)) \cap C([0, T]; L^p_{\text{loc}}(\mathbb{R}^N)) \quad (\forall p < \infty) \text{ if } u^0 \in L^\infty(\mathbb{R}^N).$$

Finally, the following identity holds for all $\beta \in C(\mathbb{R})$ bounded and vanishing near 0

$$\frac{d}{dt} \int_{\mathbb{R}^N} \beta(u) dx + \int_{\mathbb{R}^N} cu\beta'(u) + \text{div } b\beta(u) dx = 0, \quad \text{a.e. on } (0, T). \tag{31}$$

The next result is a stability result which corresponds to the case when $c = 0$. We will indicate briefly after the proof of all these results how stability results may be obtained in the general case by a simple trick (reducing the general case to the case when $c = 0$).

Theorem II.4. (Stability). *Let $b_n, c_n \in L^1(0, T; L^1_{\text{loc}})$ be such that $\text{div } b_n \in L^1(0, T; L^1_{\text{loc}})$ and $b_n, c_n, \text{div } b_n$ converges as n goes to ∞ to $b, 0, \text{div } b$ (respectively) in $L^1(0, T; L^1_{\text{loc}})$ where b satisfies (*) and (**) (with $c = 0$). Let u^n be a bounded sequence in $L^\infty(0, T; L^0)$ such that u^n is a renormalized solution of (11) with (b, c) replaced by (b_n, c_n) corresponding to an initial condition $u^n_0 \in L^0$. Assume that u^n_0 converges in L^0_{loc} as n goes to ∞ to some $u^0 \in L^0$.*

1) (Local convergence). Then, u_n converges as n goes to ∞ in $C([0, T]; L^0_{loc})$ to the renormalized solution u of (11) (with $c = 0$) corresponding to the initial condition u^0 . In addition, we assume now that u_n^0 converges to u^0 in L^p_{loc} for some $p \in [1, \infty)$, that u^n is bounded in $L^\infty(0, T; L^p_{loc})$, that $b_n, c_n, \operatorname{div} b_n$ are bounded in $L^1(0, T; L^\infty_{loc})$ or $\{|u^n(t)|^p/t \in [0, T], n \geq 1\}$ is relatively weakly compact in L^1_{loc} . Then, u_n converges to u in $C([0, T]; L^p_{loc})$.

2) (Global convergence). Assume that c_n converges to 0 in $L^1(0, T; L^1 + L^r)$ (for some $r < \infty$), that $\operatorname{div} b_n = \beta_n^1 + \beta_n^2$ where β_n^2 is bounded in $L^1(0, T; L^\infty)$ and β_n^1 converges in $L^1(0, T; L^1)$, that u_n^0 converges to u^0 in L^0 and that u^n satisfies (31) with (b, c) replaced by (b_n, c_n) . Then, u_n converges to u in $C([0, T]; L^0)$. In addition, we assume now that u_n^0 converges to u^0 in L^p for some $1 \leq p < \infty$, that u^n is bounded in $L^\infty(0, T; L^p)$ or $c_n = 0$, that $c_n, \operatorname{div} b_n$ are bounded in $L^1(0, T; L^\infty)$ or $\{|u^n(t)|^p/t \in [0, T], n \geq 1\}$ is relatively weakly compact in L^1 . Then, u_n converges to u in $C([0, T]; L^p)$.

Remarks. 1) Notice that we are not assuming in the stability result that $b_n \rightarrow b$ in $L^1(0, T; W^{1,1}_{loc})$.

2) Similar results hold for equations with a right-hand side. △

We will prove Theorems II.3 and Theorem II.4 in several steps: first of all, we prove part 1) of Theorem II.3 and the uniqueness statement of part 2) in the case when $c = 0$. Then, we will prove Theorem II.4 in two steps. Next, we prove the existence statement of part 2) in Theorem II.3. Finally, we will explain how to recover the general case from the case when $c = 0$.

Step 1. In order to prove part 1) of Theorem II.3, we first recall that solutions of (11) (in distributions sense) are renormalized solutions of (11) when $b \in L^1(0, T; W^{1,q}_{loc})$, a fact which has been shown already (see Corollary II.2 and (24) in particular). Next, if u is a renormalized solution of (11) and $u \in L^\infty(0, T; L^p)$ then u is a solution of (11): indeed, one just needs to choose a sequence of admissible functions β_n such that

$$|\beta_n(t)| \leq |t| \quad \text{and} \quad \beta_n \rightarrow t \quad \text{uniformly on compact sets of } \mathbb{R} .$$

Then, (11) follows from (30) by easy measure theory considerations.

Next, the uniqueness assertion in part 2) of Theorem II.3 also follows from Theorem II.2 when $c \equiv 0$ since $\beta(u)$ is then a solution of (11) in $L^\infty(0, T; L^1 \cap L^\infty)$. Therefore, $\beta(u)$ is unique and since this holds for all admissible β we deduce easily that

$$u \mathbf{1}_{\infty > |u| > 0} = v \mathbf{1}_{\infty > |v| > 0} \text{ a.e. , } \quad \mathbf{1}_{u=0} = \mathbf{1}_{v=0} \text{ a.e. ,}$$

$$\mathbf{1}_{u = \pm \infty} = \mathbf{1}_{v = \pm \infty} \text{ a.e. ,}$$

i.e. $u \equiv v$ a.e., if u, v are two renormalized solutions. Observe also that all the continuity in time statements and the identity (31) contained in part 2) of Theorem II.3 follow in the same way from Corollaries II.1 and II.2.

Step 2. Pointwise stability

We now wish to show, under the assumptions of part 1) of Theorem II.4, that u_n converges a.e. on $(0, T) \times \mathbb{R}^N$ to u or that for any admissible β , $\beta(u_n)$ converges a.e. on $(0, T) \times \mathbb{R}^N$ to $\beta(u)$. We thus fix such a β and denote by $v_n = \beta(u_n)$. Observe that v_n is bounded in $L^\infty(0, T; L^1 \cap L^\infty)$ and solves

$$\frac{\partial v_n}{\partial t} - b_n \cdot \nabla_x v_n + c_n u_n \beta'(u_n) = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{32}$$

while $v_n|_{t=0} = v_n^0 = \beta(u_n^0)$. Remarking that β^2 is still admissible, using the definition of renormalized solutions, we see that $w_n = v_n^2 \in L^\infty(0, T; L^1 \cap L^\infty)$ solves

$$\frac{\partial w_n}{\partial t} - b_n \cdot \nabla_x w_n + 2c_n u_n v_n \beta'(u_n) = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{33}$$

and $w_n|_{t=0} = (v_n^0)^2$.

Without loss of generality, we may assume that v_n and w_n converge weakly (say in $L^\infty((0, T) \times \mathbb{R}^N) - *$) to v and $w \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^N))$ which are solutions of (11) (in distributions sense) in view of the assumed convergences of $b^n, c^n, \text{div } b^n$. In addition, v and w correspond respectively to the initial conditions $\beta(u^0)$ and $\beta(u^0)^2$ since u_n^0 converges to u^0 in L^0_{loc} .

Then, in view of part 1) of Theorem II.3, v is a renormalized solution of (11) and thus v^2 is a solution of (11) corresponding to the initial condition $(v^0)^2 = \beta(u^0)^2$: indeed, recall that $c = 0$ here and that one just has to let the admissible nonlinearity in (30) go to t^2 . Therefore, by the uniqueness result Theorem II.2, $v^2 \equiv w$. But this means that

$$v_n^2 \rightarrow v^2 \quad \text{weakly in } L^\infty((0, T) \times \mathbb{R}^N) - *$$

hence v_n converges in $L^2(0, T; L^2_{\text{loc}})$ to v , therefore in measure. Recalling that $v_n = \beta(u_n)$ and β is an arbitrary admissible function, we see easily by varying β among a countable collection of such admissible functions β_k such that

$$\begin{cases} \beta_k = 0 & \text{if } |t| \leq \frac{1}{k}, & 0 < \beta'_k(t) & \text{if } |t| > \frac{1}{k} \\ \beta'_k(t)(1 + |t|)^{-1}, & & \beta_k & \text{are bounded on } \mathbb{R} \end{cases}$$

that u_n has to converge in measure to some u . But then v_n has to converge to $\beta(u)$. Hence $v = \beta(u)$ and u is a renormalized solution of (11) corresponding to the initial condition u^0 (and thus is unique).

Step 3. Conclusion of the proof of Theorem II.4

There only remains to show that convergences are uniform in t and global when the data converge globally.

The uniform convergences in t follows from Ascoli type arguments. Indeed, if we first fix an admissible function β , we know by step 2 that $\beta(u_n)$ and $\gamma(u_n)$

converge respectively to $\beta(u)$ and $\gamma(u)$ in $L^p(0, T; L^p_{loc})$ for all $1 \leq p < \infty$ where $\gamma = \beta^2$. Furthermore, choosing ϕ_R as in the proof of Theorem II.2, we see that

$$\frac{d}{dt} \int_{\mathbb{R}^N} \gamma(u_n) \phi_R dx + \int_{\mathbb{R}^N} \{c_n u_n \gamma'(u_n) + \operatorname{div} b_n \gamma(u_n)\} \phi_R + \gamma(u_n) b_n \cdot \nabla \phi_R dx = 0.$$

And one deduces easily that

$$\frac{d}{dt} \int_{\mathbb{R}^N} \gamma(u_n) \phi_R dx \xrightarrow{n} \int_{\mathbb{R}^N} \{\operatorname{div} b \phi_R + b \cdot \nabla \phi_R\} \gamma(u) dx \quad \text{in } L^1(0, T).$$

Therefore, since u is a renormalized solution of (11),

$$\int_{\mathbb{R}^N} \beta(u_n)^2 \phi_R dx \xrightarrow{n} \int_{\mathbb{R}^N} \beta(u)^2 \phi_R dx, \quad \text{uniformly in } [0, T]$$

and

$$\int_{\mathbb{R}^N} \beta(u_n(t_n))^2 \phi_R dx \xrightarrow{n} \int_{\mathbb{R}^N} \beta(u(t))^2 \phi_R dx, \quad \text{if } t_n \xrightarrow{n} t \text{ in } [0, T]. \tag{34}$$

On the other hand, for any bounded ball B_R , one checks easily using (30) that $\beta(u_n)$ is relatively compact in $C([0, T]; H^{-s}(B_R))$ for some large $s > 0$ (independent of n). Therefore, if $t_n \xrightarrow{n} t$ in $[0, T]$, $\beta(u_n(t_n)) \xrightarrow{n} \beta(u(t))$ in $H^{-s}(B_R)$ for all $R < \infty$, and thus weakly in $L^2(B_R)$. Then, in view of (34), $\beta(u_n(t_n)) \xrightarrow{n} \beta(u(t))$ in $L^2(B_R)$. Since $\beta(u) \in C([0, T]; L^p)$ ($\forall 1 \leq p < \infty$), this implies that $\beta(u_n) \xrightarrow{n} \beta(u)$ in $C([0, T]; L^2_{loc})$.

One proves in a similar fashion the remaining assertions of Theorem II.4 concerning the L^p convergences at least when $p > 1$ replacing $\beta(u_n)^2$ by $|u_n|^p$, or the global convergences (again when $p > 1$). One just has to notice that when $c_n = 0$, L^p bounds may be obtained using (31) to deduce

$$\frac{d}{dt} \int_{\mathbb{R}^N} \{(|u_n| - \lambda)^+ \wedge M\}^p dx \leq C \int_{\mathbb{R}^N} \{(|u_n| - \lambda)^+ \wedge M\}^p dx$$

which yields a $L^\infty(0, T; L^p)$ bound upon letting λ go to 0 and M go to ∞ .

There just remains to show the local and global convergences in L^1 . In both cases, observing that u_n^+, u_n^- are also renormalized solutions, we deduce that we may assume without loss of generality that u_n, u are nonnegative. Next, by the convergences proved above, we see that in the local situation

$$(u_n - \lambda)^+ \wedge M \xrightarrow{n} (u - \lambda)^+ \wedge M \quad \text{in } C([0, T]; L^1_{loc}) \tag{35}$$

for all $\lambda > 0, M < \infty$. But then the assumption on weak L^1_{loc} compactness implies that, for all $R < \infty$, we have

$$\sup_{t, n} \int_{B_R} |u_n| 1_{|u_n| \geq M} dx \rightarrow 0 \quad \text{as } M \rightarrow +\infty. \tag{36}$$

Combining (35) and (36), it is easy to deduce the convergence in $C([0, T]; L^1_{loc})$.

In the global situation, we want to show that, for any sequence t_n in $[0, T]$ converging to some t , $u_n(t_n)$ converges in L^1 to $u(t)$. This is clearly enough since $u \in C([0, T]; L^1)$. Since u_n converges to u in $C([0, T]; L^0)$, we already know that

$u_n(t_n)$ converges to $u(t)$ in measure or to simplify the presentation almost everywhere (extract a subsequence if necessary). On the other hand, because of (31), we have

$$\int_{\mathbb{R}^N} e^{-a(t_n)} u_n(t_n) dx + \int_0^{t_n} \int_{\mathbb{R}^N} \{c_n + \operatorname{div} b_n + a'(t)\} e^{-a(t)} u_n dt dx = 0 \tag{37}$$

$$\int_{\mathbb{R}^N} e^{-a(t)} u(t) dx + \int_0^t \int_{\mathbb{R}^N} \{\operatorname{div} b + a'\} e^{-a(t)} u dt dx = 0 \tag{38}$$

for any function $a \in W^{1,1}(0, T)$. And we choose a such that

$$a(0) = 0, a'(t) + c_n + \operatorname{div} b_n \geq 0 \quad \text{a.e. on } (0, T) \times \mathbb{R}^N.$$

Then, by Fatou's lemma (recalling that $u_n, u \geq 0$), we deduce that

$$\int_{\mathbb{R}^N} u_n(t_n) dx \xrightarrow{n} \int_{\mathbb{R}^N} u(t) dx. \tag{39}$$

And recalling a standard exercise in measure theory we conclude:

$$\int_{\mathbb{R}^N} |u_n(t_n) - u(t)| dx = \int_{\mathbb{R}^N} u_n(t_n) - u(t) dx + 2 \int_{\mathbb{R}^N} (u_n(t_n) - u(t))^- dx$$

and $(u_n(t_n) - u(t))^- \rightarrow 0$ in L^1 by Lebesgue's lemma.

Step 4. Existence when $c = 0$

We just approximate b by $b_\epsilon = \int_{\mathbb{R}^N} b(t, y) \rho_\epsilon(x - y) dy$ where $\rho_\epsilon = \frac{1}{\epsilon^N} \rho\left(\frac{\cdot}{\epsilon}\right)$, $\rho \in \mathcal{D}_+(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho dx = 1$. Then, $\frac{b_\epsilon}{(1 + |x|^2)^{1/2}} \in L^1(0, T; W^{k, \infty}(\mathbb{R}^N))$ for all $k \geq 1$ because of (*) and (**).

Next, consider β_k as in Step 2 above: we will impose in addition that $\beta_{k'} = \gamma_{k', k} \circ \beta_k$ for some $\gamma_{k', k} \in C^1(\mathbb{R})$, for all $k' \geq k \geq 1$. Then, we denote by $u_k^0 = \beta_k(u^0)$, $u_{k, \delta}^0 = \beta_k(u^0) * \rho_\delta$ for $\delta \geq 0$ (with the convention $\rho_0 = \delta_0$). By standard results, there exists a unique solution $u_{k, \epsilon}^\delta$ in $L^\infty(0, T; L^\infty(\mathbb{R}^N))$ of the following problem

$$\frac{\partial u_{k, \epsilon}^\delta}{\partial t} - b_\epsilon \cdot \nabla u_{k, \epsilon}^\delta = 0 \quad \text{in } (0, T) \times \mathbb{R}^N, u_{k, \epsilon}^\delta|_{t=0} = u_{k, \delta}^0 \quad \text{on } \mathbb{R}^N \tag{40}$$

and $u_{k, \epsilon}^\delta \in W^{1, \infty}((0, T) \times B_R)$ ($\forall R < \infty$) for $\delta > 0$, $u_{k, \epsilon}^\delta$ converges to $u_{k, \epsilon}^0$ to 0 (say in $C([0, T]; L^1_{loc})$) and we denote by $u_{k, \epsilon} = u_{k, \epsilon}^0$. Clearly, $\tilde{u}_{k', \epsilon}^\delta = \gamma_{k', k}(u_{k, \epsilon}^\delta)$ solves

$$\frac{\partial}{\partial t}(\tilde{u}_{k', \epsilon}^\delta) - b_\epsilon \cdot \nabla \tilde{u}_{k', \epsilon}^\delta = 0 \quad \text{in } (0, T) \times \mathbb{R}^N, \tilde{u}_{k', \epsilon}^\delta = \gamma_{k', k}(u_{k, \delta}^0) \quad \text{on } \mathbb{R}^N$$

for all $k' \geq k, \delta > 0$. Therefore, letting δ go to 0_+ and comparing with (40) we see that

$$u_{k', \epsilon} = \gamma_{k', k}(u_{k, \epsilon}) \quad \text{on } (0, T) \times \mathbb{R}^N, \quad \text{for all } k' \geq k \geq 1.$$

From this, we deduce that there exists $u_\varepsilon \in L^\infty(0, T; L^0)$ such that $u_{k,\varepsilon} = \beta_k(u_\varepsilon)$ and thus u_ε is a renormalized solution of (11) with (b, c) replaced by $(b_\varepsilon, 0)$ corresponding to the initial condition u_0 . In order to conclude the existence proof one just needs to check that u_ε is bounded in $L^\infty(0, T; L^0)$ and to use the stability result letting ε go to 0.

Step 5. Reduction to the case when $c = 0$

Let $a \in L^1(0, T; L^\infty(\mathbb{R}^N))$. In view of (*) and (**), there exists a unique solution

$$\frac{\partial \Phi}{\partial t} - b \cdot \nabla \Phi = a \quad \text{on } (0, T) \times \mathbb{R}^N, \quad \Phi|_{t=0} = 0 \quad \text{on } \mathbb{R}^N \quad (41)$$

(see Corollary II.1 and II.2).

Then, the reduction of the general case to the case when $c = 0$ follows immediately from the following.

Lemma II.2. *Let (b, c) satisfy (*) and (**). Let $u^0 \in L^0$. Then, $u \in L^\infty(0, T; L^0)$ is a renormalized solution of (11) for the initial condition u^0 if and only if $e^{-\Phi}u$ is a renormalized solution of (11) with (b, c) replaced by $(b, a + c)$ for the initial condition u_0 .*

Proof. Formally, this is nothing but the chain rule and we have to justify the obvious formal manipulations. Of course, by symmetry, it is enough to show one direction of the above equivalence. Hence, let $u \in L^\infty(0, T; L^0)$ be a renormalized solution of (11) for the initial condition u^0 and let γ, β be admissible functions. We want to show that $\gamma(e^{-\Phi}\beta(u)) = \omega$ solves

$$\begin{aligned} \frac{\partial \omega}{\partial t} - b \cdot \nabla \omega + \gamma'(e^{-\Phi}\beta(u))e^{-\Phi}\{c\beta'(u)u + a\beta(u)\} &= 0 \\ \text{on } (0, T) \times \mathbb{R}^N, \quad \omega|_{t=0} &= \gamma \circ \beta(u^0) \quad \text{on } \mathbb{R}^N. \end{aligned} \quad (42)$$

To this end, we apply the regularizing result Theorem II.1 and we find that

$$\begin{cases} \frac{\partial \Phi_\varepsilon}{\partial t} - b \cdot \nabla \Phi_\varepsilon = a + \psi_\varepsilon & \text{in } (0, T) \times \mathbb{R}^N, \quad \Phi_\varepsilon|_{t=0} = 0 \quad \text{on } \mathbb{R}^N \\ \frac{\partial v_\varepsilon}{\partial t} - b \cdot \nabla v_\varepsilon + c\beta'(u)u = r_\varepsilon & \text{in } (0, T) \times \mathbb{R}^N, \quad v_\varepsilon|_{t=0} = v_\varepsilon^0 \quad \text{on } \mathbb{R}^N \end{cases} \quad (43)$$

where $v = \beta(u)$, $v^0 = \beta(u^0)$, $\Phi_\varepsilon = \Phi * \rho_\varepsilon$, $v_\varepsilon = v * \rho_\varepsilon$, $v_\varepsilon^0 = v^0 * \rho_\varepsilon$, and $\psi_\varepsilon, r_\varepsilon \rightarrow 0$ in $L^1(0, T; L^1_{loc})$.

We next set $\omega^\varepsilon = \gamma(e^{-\Phi_\varepsilon}v_\varepsilon)$ and we may now use the chain rule to deduce from (43)

$$\begin{aligned} \frac{\partial \omega_\varepsilon}{\partial t} - b \cdot \nabla \omega^\varepsilon &= \gamma'(e^{-\Phi_\varepsilon}v_\varepsilon)e^{-\Phi_\varepsilon}\{r_\varepsilon - c\beta'(u)u - a\beta(u) - \psi_\varepsilon\beta(u)\} \\ &\text{in } (0, T) \times \mathbb{R}^N, \quad \omega^\varepsilon|_{t=0} = \gamma(v_\varepsilon^0) \quad \text{on } \mathbb{R}^N. \end{aligned}$$

And (42) follows upon letting ε go to 0 in the preceding equation, concluding thus the proof of Theorems II.3 and II.4. \triangle

Remarks. 1) Theorem II.4 with Lemma II.2 yields corresponding stability results for general equations (without assuming $c = 0$).

2) It is possible to extend some of the results above to more general initial conditions namely $u^0 \in \bar{L} = \{v \text{ measurable from } \mathbb{R}^N \text{ into } \bar{\mathbb{R}}\} (= L^0_{loc})$. Then, we may define admissible functions β as follows: $\beta \in C^1(\mathbb{R})$, β and $\beta'(1 + |t|)$ are bounded. And we then use the same definition as before for renormalized solutions observing that if $u \in L^\infty(0, T; \bar{L})$ then $\beta(u) \in L^\infty((0, T) \times \mathbb{R}^N)$. Let us remark at this point that if $u^0 \in L^0$ then both definitions are easily shown to be equivalent.

Then, the proofs above show that there exists a unique renormalized solution u of (11) in $L^\infty(0, T; \bar{L})$ for any initial condition $u^0 \in \bar{L}$, and $u \in C([0, T]; \bar{L})$. Part 1) (concerning the $L^0_{loc} = \bar{L}$ convergence) is then still true.

Furthermore, if $|u^0| < \infty$ a.e., and u^n_0 converges in measure locally to u^0 , then checking that

$$\sup_{t \in [0, T]} |\{|u| \geq M\} \cap B_R| \rightarrow 0 \quad \text{as } M \rightarrow \infty \text{ (for all } R < \infty)$$

and deducing from that for all $R < \infty$ and $\varepsilon > 0$, there exists M large enough such that for all n large

$$\sup_{t \in [0, T]} |\{|u_n| \geq M\} \cap B_R| \leq \varepsilon$$

we see that u^n converges to u locally in measure uniformly in t .

3) Let us observe also that the method of proof used in the proof of Lemma II.2 may also be used to show that if $u^0, v^0 \in L^0$ (or \bar{L} , see 2) above) and $|u^0|, |v^0| < \infty$ a.e. so that $u^0 + v^0$ makes sense and belongs to L^0 , the renormalized solution w of (11) corresponding to $u^0 + v^0$ is $u + v$ where u, v are respectively the renormalized solutions of (11) corresponding to u^0, v^0 . (Notice that, because of (31), $|u|, |v| < \infty$ a.e.).

4) Using the regularizing result, one may check that if u_1, \dots, u_m (for some $m \geq 1$) are renormalized solutions of (11) say in $L^\infty(0, T; \bar{L})$ with $|u_i| < \infty$ a.e., $c \equiv 0$ and F is a continuous map from \mathbb{R}^m into \mathbb{R} , then $u = F(u_1, \dots, u_m)$ is still a renormalized solution of (11) (with $c \equiv 0$). \triangle

Let us indicate another possible strategy for proving stability results (of a slightly different kind). We first state the type of results one can obtain with this strategy.

Theorem II.5. *Let $b_n \in L^1(0, T; L^1_{loc})$ be such that $\text{div } b_n$ is bounded in $L^1(0, T; L^\infty)$ and b_n converges, as n goes to ∞ , to b in $L^1(0, T; L^1_{loc})$ where b satisfies (*) and (**) (with $c = 0$). Let u^n be a bounded sequence in $L^\infty(0, T; L^0)$ such that u^n is a renormalized solution of (11) satisfying (31) with (b, c) replaced by $(b_n, 0)$, corresponding to an initial condition $u^n_0 \in L^0$. Assume that u^n_0 converges to u^0 in L^0 (resp. in L^p for some $1 \leq p < \infty$), then u^n converges in $C([0, T]; L^0)$ (resp. in $C([0, T]; L^p)$) to the renormalized solution u of (11) (with $c = 0$) corresponding to the initial condition u^0 .*

Since a large part of the proof of this result is analogous to the proof of Theorem II.4, we sketch it and consider only the case of an initial condition u_n^0 which is bounded in $L^1 \cap L^\infty$ and converges in L^1 , therefore u^n is bounded in $L^\infty(0, T; L^1 \cap L^\infty)$. Next, because of (*) and (**), with the notations of the proofs above, we deduce the existence of u_ε^R solution of

$$\frac{\partial u_R^\varepsilon}{\partial t} - b \cdot \nabla u_R^\varepsilon = f_R^\varepsilon \quad \text{in } (0, T) \times \mathbb{R}^N$$

with $u_R^\varepsilon|_{t=0} = \phi_R u_\varepsilon^0$, where u_R^ε is smooth in x (uniformly in t), compactly supported in x (uniformly in t) and $f_R^\varepsilon \xrightarrow{L^1} 0$, $u_R^\varepsilon \xrightarrow{C(L^1)} u$ as $R \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$. In fact, u_R^ε is nothing but $\phi_R u_\varepsilon$ where u_ε is obtained using the regularizing result Theorem II.1. Next, we write

$$\frac{\partial}{\partial t}(u^n - u_R^\varepsilon) - b_n \cdot \nabla(u^n - u_R^\varepsilon) = (b - b_n)\nabla u_R^\varepsilon + f_R^\varepsilon.$$

Since u^n is a renormalized solution satisfying (31), one deduces from the above equation and a tedious approximation argument

$$\frac{d}{dt} \int_{\mathbb{R}^N} |u^n - u_R^\varepsilon| dx \leq \int_{\mathbb{R}^N} |b - b_n| |\nabla u_R^\varepsilon| dx + \int_{\mathbb{R}^N} |f_R^\varepsilon| dx + \|\operatorname{div} b_n(t)\|_{L^\infty} \int_{\mathbb{R}^N} |u^n - u_R^\varepsilon| dx.$$

Hence, setting $A_n(t) = \int_0^t \|\operatorname{div} b_n(s)\|_{L^\infty} ds$, we deduce

$$\begin{aligned} \sup_{[0, T]} \left\{ \left(\int_{\mathbb{R}^N} |u^n - u_R^\varepsilon|(t) dx \right) e^{-A_n(t)} \right\} &\leq \int_{\mathbb{R}^N} |u_n^0 - \phi_R u_\varepsilon^0| dx \\ &+ \int_0^T e^{-A_n(t)} dt \int_{\mathbb{R}^N} |b - b_n| |\nabla u_R^\varepsilon| + |f_R^\varepsilon| dx \end{aligned}$$

or, in view of the bounds on $(\operatorname{div} b_n)_n$,

$$\begin{aligned} \sup_{[0, T]} \left(\int_{\mathbb{R}^N} |u^n - u_R^\varepsilon| dx \right) (t) &\leq C_0 \int_{\mathbb{R}^N} |u_n^0 - \phi_R u_\varepsilon^0| dx \\ &+ C_0 \int_0^T \int_{\mathbb{R}^N} dt dx \{ |b - b_n| |\nabla u_R^\varepsilon| + |f_R^\varepsilon| \} \end{aligned}$$

for some constant C_0 independent of n, ε, R .

Hence, we have

$$\begin{aligned} \overline{\lim}_n \sup_{[0, T]} \left(\int_{\mathbb{R}^N} |u^n - u| dx \right) (t) &\leq C_0 \sup_{[0, T]} \left(\int_{\mathbb{R}^N} |u - u_R^\varepsilon| dx \right) (t) \\ &+ C_0 \int_0^T \int_{\mathbb{R}^N} dt dx |f_R^\varepsilon|, \end{aligned}$$

and we conclude letting first R go to $+\infty$ and then ε go to 0. △

II.4 Duality

Theorem II.6. *Let (b, c) satisfies $(*)$ and $(**)$, let $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$, $v \in L^\infty(0, T; L^q(\mathbb{R}^N))$ with $1 \leq p \leq \infty$, $q = \frac{p}{p-1}$, be respectively renormalized solutions of (11) and*

$$\frac{\partial v}{\partial t} - b \cdot \nabla v - (c + \operatorname{div} b)v = f \quad \text{in } (0, T) \times \mathbb{R}^N \quad (44)$$

where $f \in L^1(0, T; L^q(\mathbb{R}^N))$. Then, we have the following duality formula

$$\int_{\mathbb{R}^N} u(T)v(T) dx - \int_{\mathbb{R}^N} u(0)v(0) dx = \int_0^T \int_{\mathbb{R}^N} f u dt dx . \quad (45)$$

Remarks. 1) Recall that u (and similarly for v) belongs to $C([0, T]; L^p)$ (if $p < \infty$, $C([0, T]; L^p_{\text{loc}})$ for all $r < \infty$ if $p = \infty$).

2) It is of course possible to define solutions of (11) for an initial condition $u^0 \in L^p$ by imposing that the above formula (45) holds for all v solving (in renormalized sense, or in distributions sense) (44) with $f \in \mathcal{D}'((0, T) \times \mathbb{R}^N)$. Then, under conditions $(*)$ and $(**)$, the above result shows that u is the unique solution of (11) in that (dual) sense. \triangle

Proof. Formally, the above result is nothing but an integration by parts and so we need to justify once more these formal manipulations. To this end, we take an admissible function β and use the regularizing result Theorem II.1 to deduce

$$\frac{\partial}{\partial t} \tilde{u}_\varepsilon - b \cdot \nabla \tilde{u}_\varepsilon + c \tilde{u} \beta'(\tilde{u}) = r_\varepsilon \quad \text{in } (0, T) \times \mathbb{R}^N , \quad (46)$$

$$\frac{\partial}{\partial t} \tilde{v}_\varepsilon - b \cdot \nabla \tilde{v}_\varepsilon - (c + \operatorname{div} b) v \beta'(v) = s_\varepsilon + f \quad \text{in } (0, T) \times \mathbb{R}^N \quad (47)$$

where $\tilde{u} = \beta(u)$, $\tilde{v} = \beta(v)$, $\tilde{u}_\varepsilon = \tilde{u} * \rho_\varepsilon$, $\tilde{v}_\varepsilon = \tilde{v} * \rho_\varepsilon$ and $r_\varepsilon, s_\varepsilon \rightarrow 0$ in $L^1(0, T; L^1_{\text{loc}})$. Introducing ϕ_R as in the proof of Theorem II.2, we may now multiply (46) by $\tilde{v}_\varepsilon \phi_R$, integrate by parts and use (47) to deduce

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{u}_\varepsilon(T) \tilde{v}_\varepsilon(T) \phi_R dx - \int_{\mathbb{R}^N} \tilde{u}_\varepsilon(0) \tilde{v}_\varepsilon(0) \phi_R dx &= \int_0^T \int_{\mathbb{R}^N} (f + s_\varepsilon) \tilde{u}_\varepsilon \phi_R dt dx + \\ &+ \int_0^T \int_{\mathbb{R}^N} r_\varepsilon \tilde{v}_\varepsilon \phi_R - b \nabla \phi_R \tilde{u}_\varepsilon \tilde{v}_\varepsilon + (c + \operatorname{div} b) v \beta'(v) \tilde{u}_\varepsilon \phi_R dt dx + \\ &- \int_0^T \int_{\mathbb{R}^N} c \tilde{u} \beta'(\tilde{u}) \tilde{v}_\varepsilon \phi_R + \operatorname{div} b \tilde{u}_\varepsilon \tilde{v}_\varepsilon \phi_R dt dx . \end{aligned}$$

Then, we may let ε go to 0 and then argue as in the proof of Theorem II.2 letting R go to ∞ to obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{u}(T) \tilde{v}(T) dx - \int_{\mathbb{R}^N} \tilde{u}(0) \tilde{v}(0) dx &= \int_0^T \int_{\mathbb{R}^N} f \tilde{u} + (c + \operatorname{div} b) v \beta'(v) \tilde{u} + \\ &- \int_0^T \int_{\mathbb{R}^N} c \tilde{u} \beta'(\tilde{u}) \tilde{v} + \operatorname{div} b \tilde{u} \tilde{v} dt dx . \end{aligned}$$

Finally, in order to conclude we just have to let β converge to $\beta_1(t) = t$ (imposing that $|\beta(t)| \leq t, |\beta'(t)| \leq 1$ on \mathbb{R}). △

We conclude this section with an example of the possible applications of the duality formula to weak convergence and stability results. It can also be applied to recover the “strong” stability results of the preceding section.

Corollary II.3. *Let b_n converge to b in $L^1(0, T; L^1_{loc})$ be such that $\text{div } b_n$ is bounded in $L^1(0, T; L^\infty)$. Let u_n be a renormalized solution of (11) with (b, c) replaced by $(b_n, 0)$ for an initial condition u_n^0 . We assume in addition that b_n satisfies (***) and that u_n^0 converges weakly in $L^p(\mathbb{R}^N)$ for some $p \in (1, +\infty]$ to some u^0 . Then, u_n converges weakly in $L^\infty(0, T; L^p)$ to the renormalized solution of (11) (with $c = 0$) for the initial condition u^0 .*

Proof. One first observes that, by the results of the preceding sections, u_n is bounded in $L^\infty(0, T; L^p(\mathbb{R}^N))$. Next, we consider the solution of

$$\frac{\partial \Phi_n}{\partial t} - b_n \cdot \nabla \Phi_n - \text{div } b_n \Phi_n = \phi \quad \text{in } (0, T) \times \mathbb{R}^N, \quad \Phi_n|_{t=T} = 0 \quad \text{on } \mathbb{R}^N$$

where ϕ is given in $\mathcal{D}((0, T) \times \mathbb{R}^N)$. By the stability result Theorem II.4 (and the remarks following its proof) Φ_n converges in $C([0, T]; L^r)$ for all $1 \leq r < \infty$ (and remains bounded in $L^\infty(0, T; L^\infty)$) to the solution Φ of

$$\frac{\partial \Phi}{\partial t} - b \cdot \nabla \Phi - \text{div } b \Phi = \phi \quad \text{in } (0, T) \times \mathbb{R}^N, \quad \Phi|_{t=T} = 0 \quad \text{on } \mathbb{R}^N.$$

Then, if u_n converges weakly in $L^\infty(0, T; L^p)$ to some u (extracting a subsequence if necessary), we may pass to the limit in (45) and we find

$$-\int_{\mathbb{R}^N} u_0 \Phi(0) dx = \int_0^T \int_{\mathbb{R}^N} \phi u dt dx.$$

Since ϕ is arbitrary, we may use the remark 2) following Theorem II.5 to conclude the proof. △

II.5 Stability and time compactness

The goal of this section is to present an extension of the stability results proven in the previous sections. The main difference lies in the time dependence of the coefficients. More precisely, we extend Theorem II.4 (for instance) as follows

Theorem II.7. *Theorem II.4 still holds if we replace the convergence of $(b_n, c_n, \text{div } b_n)$ to $(b, 0, \text{div } b)$ in $L^1(0, T; L^1_{loc})$ by the following assumptions: first, we assume that $(b_n, c_n, \text{div } b_n)$ converges to $(b, 0, \text{div } b)$ weakly in $L^1(0, T; L^1_{loc})$ and also that $\phi_n = b_n, c_n, \text{div } b_n$ satisfies*

$$\phi_n(t, x + h) \rightarrow \phi_n(t, x) \quad \text{as } h \rightarrow 0 \quad \text{in } L^1(0, T; L^1_{loc}), \text{ uniformly in } n. \quad (***)$$

Remarks. 1) Of course, if $b_n, c_n, \text{div } b_n$ do not depend on t , as it is well-known the combination of the weak convergence and of (***) is equivalent to the strong convergence. In this case, Theorem II.7 thus reduces to Theorem II.4.

2) In general, Theorem II.7 is more general than Theorem II.4 and, even if it seems a rather technical extension of Theorem II.4, the gain in generality will be quite important for applications. Observe in particular that (***) holds as soon as ϕ_n is bounded in $L^\alpha(0, T; X)$ where $\alpha > 1$, and X is any space with a compact embedding in L^1_{loc} —for instance any Sobolev space $W^{m,p}$ with $p \geq 1, m > 0$ has this property. \triangle

Proof of Theorem II.7. Since its proof is very much similar to the proof of Theorem II.4, we only explain the new ingredient: to this end, we take $c_n = 0$ and we consider v^n bounded in $L^\infty_{t,x}$ solution of

$$\frac{\partial v^n}{\partial t} - \operatorname{div}_x (b_n v^n) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N)$$

writing the equation in divergence form allows to simplify a bit the presentation, avoiding to keep track of the extra “divergence” term present in the case of the non-divergence form equation. Of course, we may assume that v^n converges weakly in L^*_* to some v and we want to prove that v satisfies

$$\frac{\partial v}{\partial t} - \operatorname{div}_x (bv) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N);$$

or in other words that $b_n v^n$ converges weakly (in \mathcal{D}' or in L^1) to bv .

To this end, we introduce a regularizing kernel as in the proof of Theorem II.1 and we observe that (***) yields immediately

$$\begin{aligned} b_n(v^n * \rho_\varepsilon) - (b_n v^n) * \rho_\varepsilon &\rightarrow 0 \quad \text{in } L^1(0, T; L^1_{loc}), \\ &\text{as } \varepsilon \rightarrow 0+, \text{ uniformly in } n. \end{aligned}$$

Since $\frac{\partial}{\partial t}(v^n * \rho_\varepsilon) = (\operatorname{div}_x (b_n v^n)) * \rho_\varepsilon$ is clearly uniformly (in n) bounded in $L^1(0, T; L^1_{loc})$, we deduce easily from the compactness of Sobolev embeddings that for each fixed $\varepsilon > 0$

$$v^n * \rho_\varepsilon \rightharpoonup_n v * \rho_\varepsilon \quad \text{a.e. in } (0, T) \times \mathbb{R}^N$$

(extracting a subsequence if necessary). This almost everywhere convergence combined with the uniform bounds on $v^n * \rho_\varepsilon$ and the weak* convergence of b_n in $L^1(0, T; L^1_{loc})$ yields the desired convergence namely

$$b_n(v^n * \rho_\varepsilon) \rightharpoonup_n b(v * \rho_\varepsilon) \quad \text{weakly in } L^1(0, T; L^1_{loc}), \text{ for all } \varepsilon > 0.$$

Indeed, collecting all the above convergences and the obvious one

$$b(v * \rho_\varepsilon) \rightarrow bv \quad \text{in } L^1(0, T; L^1_{loc}), \text{ as } \varepsilon \rightarrow 0+,$$

we deduce easily that we have

$$b_n v^n \rightharpoonup_n bv \quad \text{weakly in } L^1(0, T; L^1_{loc}). \quad \triangle$$

III. Applications to ordinary differential equations

III.1 The divergence free autonomous case

We consider in this section the case when b depends only on x and satisfies

$$b \in W_{loc}^{1,1}(\mathbb{R}^N), \quad \operatorname{div} b = 0 \quad \text{a.e. on } \mathbb{R}^N, \tag{48}$$

$$\frac{b}{1 + |x|} \in L^1 + L^\infty. \tag{49}$$

In some of the estimates below we will strengthen (49) as follows

$$b \in L^p + (1 + |x|)L^\infty \quad \text{for some } p \in [1, \infty]. \tag{50}$$

We are going to show the existence and uniqueness of solutions of (1). Stability results will be given in the next section in a more general situation. But, if we assume only (49), the solution map $X(t, x)$ we will obtain will not be in L_{loc}^1 (for a fixed t) so we have to define solutions of (1) in a manner similar to renormalized solutions of (1): we will show that $X(t) \in C(\mathbb{R}; L)^N$ where $L = \{ \phi \text{ measurable from } \mathbb{R}^N \text{ into } \mathbb{R} \text{ and } |\phi| < \infty \text{ a.e.} \}$ (for example) endowed with the distance

$$d(\phi, \psi) = \sum_{n \geq 1} \frac{1}{2^n} \| |\phi - \psi| \wedge 1 \|_{L^1(B_n)}$$

which corresponds to the convergence in measure on arbitrary balls. In addition, because of (48), X will satisfy

$$\lambda \circ X(t) = \lambda, \quad \text{for all } t \in \mathbb{R} \tag{51}$$

(recall that λ is the Lebesgue measure and that $\lambda \circ X(t)$ is the image measure of λ by $X(t)$ i.e.:

$$\int_{\mathbb{R}^N} \phi d(\lambda \circ X(t)) = \int_{\mathbb{R}^N} \phi(X(t)) dx.$$

Because of (51), $\phi \circ X(t)$ makes sense in L for all $\phi \in L$. The ODE (1) will hold in the following sense: for all $\beta \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ such that β and $|D\beta(z)|(1 + |z|)$ are bounded on \mathbb{R}^N , $\beta(X) \in L^\infty(\mathbb{R}; L_{loc}^1)$ and we have

$$\frac{\partial}{\partial t} \beta(X) = D\beta(X) \cdot b(X) \quad \text{on } \mathbb{R} \times \mathbb{R}^N, \quad \beta(X)|_{t=0} = \beta(x) \quad \text{on } \mathbb{R}^N \tag{52}$$

where the equation holds in distributions sense. We will also call admissible functions such functions β . Notice that because of (49) and (51),

$$\frac{b(X)}{1 + |X|} \in L^\infty(\mathbb{R}; L^1 + L^\infty).$$

Finally, the group property will now hold in the following sense

$$X(t + s, \cdot) = X(t, X(s, \cdot)) \quad \text{a.e. on } \mathbb{R}^N, \quad \text{for all } t, s \in \mathbb{R}. \tag{53}$$

We may now state our main existence and uniqueness result.

Theorem III.1. *We assume (48) and (49). Then, there exists a unique $X \in C(\mathbb{R}; L^N)$ satisfying (51), (52), (53). In addition, X satisfies*

$$\beta(X) \in L^1_{\text{loc}}(\mathbb{R}^N; C(\mathbb{R})), \quad \text{for all admissible } \beta \tag{54}$$

$$\text{for almost all } x \in \mathbb{R}^N, X \in C^1(\mathbb{R}), b(X) \in C(\mathbb{R}) \text{ and } \frac{\partial x}{\partial t} = b(X) \quad \text{on } \mathbb{R}. \tag{55}$$

Furthermore, if $u_0 \in L^0$ (or \bar{L}), $u(t, x) = u_0(X(t, x))$ is the unique renormalized solution in $C(\mathbb{R}; L^0)$ of (11) with $c \equiv 0$, for the initial condition u^0 (for all T). Finally, if b satisfies (50), then $X \in L^p_{\text{loc}}(\mathbb{R}^N; C(\mathbb{R}))$.

Proof. Step 1 (Existence)

We regularize b as usual: set $b_\epsilon = b * \rho_\epsilon$. By Cauchy-Lipschitz theorem, there exists a unique smooth map X_ϵ on $\mathbb{R} \times \mathbb{R}^N$ satisfying

$$\frac{\partial X_\epsilon}{\partial t} = b(X_\epsilon) \quad \text{on } \mathbb{R} \times \mathbb{R}^N, \quad X_\epsilon|_{t=0} = x \quad \text{on } \mathbb{R}^N. \tag{56}$$

In addition, (51) and (53) hold for X_ϵ (of course (53) holds now everywhere) and for each $u^0 \in L^0$ (or in \bar{L}), $u^0(X_\epsilon)$ is the unique (renormalized) solution of

$$\frac{\partial u_\epsilon}{\partial t} = b_\epsilon \cdot \nabla u_\epsilon \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \quad u_\epsilon|_{t=0} = u^0 \quad \text{on } \mathbb{R}^N. \tag{57}$$

In particular, X_ϵ solves

$$\frac{\partial X_\epsilon}{\partial t} = b_\epsilon \cdot \nabla X_\epsilon \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \quad X_\epsilon|_{t=0} = x \quad \text{on } \mathbb{R}^N. \tag{58}$$

Next, choosing $\beta_0(z) = z(1 + |z|^2)^{-1/2} \text{Log}(1 + |z|^2)$ for $z \in \mathbb{R}^N$, we deduce that

$$\frac{\partial}{\partial t} \beta_0(X_\epsilon) = b_\epsilon \cdot \nabla(\beta_0(X_\epsilon)) = \nabla \beta_0(X_\epsilon) \cdot b(X_\epsilon) \quad \text{in } \mathbb{R} \times \mathbb{R}^N. \tag{59}$$

Since $|\nabla \beta_0(z) \cdot b(z)| \leq C \frac{|b(z)|}{1 + |z|}$, we deduce from (49) and (51) that

$$\begin{aligned} \frac{\partial}{\partial t}(\beta_0(X_\epsilon)) \text{ is bounded in } L^\infty(\mathbb{R}; L^1 + L^\infty) \text{ and belongs} \\ \text{to a relatively compact set of } L^\infty(-T, T; L^1(B_R)) \quad (\forall R, T < \infty) \end{aligned} \tag{60}$$

In particular, $\beta_0(X_\epsilon)$ is bounded in $L^\infty(\mathbb{R}; L^1_{\text{loc}})$.

We may then use the stability results to deduce that X_ϵ converges as ϵ goes to 0, in $C([-T, T]; L^N)$ ($\forall T \in (0, \infty)$) to X which satisfies (52). In addition, choosing first u^0 in $\mathcal{D}(\mathbb{R}^N)$, using the stability results and then approximating general u^0 in L^0 , we see that for all $u^0 \in L^0$ (or \bar{L}), $u^0(X)$ is the unique renormalized solution of (11) with $c \equiv 0$ for the initial condition u^0 . In particular, we deduce that for all $u^0 \in \mathcal{D}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u^0(X(t, x)) dx = \int_{\mathbb{R}^N} u^0(x) dx, \quad \forall t \in \mathbb{R}$$

therefore (51) holds. The uniqueness of renormalized solutions also yields the group property (53).

Because of (51), (52) yields

$$\frac{\partial}{\partial t} \beta(X) \in L^\infty(\mathbb{R}; L^1 + L^\infty). \tag{61}$$

Since $L^\infty(\mathbb{R}; L^1 + L^\infty) \subset L^1_{loc}(\mathbb{R}^N; L^1_{loc}(\mathbb{R}))$, we deduce that

$$\beta(X) \in L^1_{loc}(\mathbb{R}^N; W^{1,1}_{loc}(\mathbb{R})) \subset L^1_{loc}(\mathbb{R}^N; C(\mathbb{R})).$$

Then, (60) also yields

$$\frac{\partial}{\partial t} \beta_0(X) \in L^\infty(\mathbb{R}; L^1 + L^\infty) \tag{62}$$

from which we deduce as above that $\beta_0(X) \in L^1_{loc}(\mathbb{R}^N; C(\mathbb{R}))$. In particular, for almost all $x \in \mathbb{R}^N$, $\beta_0(X) \in C(\mathbb{R})$ and since $\left(t \rightarrow \frac{t}{\sqrt{1+t^2}} \text{Log}(1+t^2) \right)$ is strictly increasing on $[0, \infty)$ we deduce that $X \in C(\mathbb{R})$.

Next, we show the claim contained in (55) about the time continuity of $b(X)$ for almost all $x \in \mathbb{R}^N$. In order to do so, we first choose $\psi \in C^1(\mathbb{R})$ such that $\psi > 0$ on \mathbb{R} , ψ is even and

$$\psi(|z|) |Db_\varepsilon(z)| \leq \beta_2(z) \in L^1_+(\mathbb{R}^N), \quad \text{for all } \varepsilon \in [0, 1] \tag{63}$$

$$|\psi'(|z|)| |b_\varepsilon| \text{Log}(1 + |b_\varepsilon|) \leq \beta_1(z) \in L^1_+(\mathbb{R}^N), \quad \text{for all } \varepsilon \in [0, 1]. \tag{64}$$

The existence of such a ψ is a simple consequence of the fact that $Db \in L^1_{loc}(\mathbb{R}^N)$ and thus $|b| \in L^{\frac{N}{N-1}}_{loc}(\mathbb{R}^N)$ by Sobolev embeddings therefore $|b| \text{Log}(1 + |b|) \in L^1_{loc}(\mathbb{R}^N)$. Then, we compute

$$\begin{aligned} \frac{\partial}{\partial t} \{ \psi(X_\varepsilon) \beta_0(b_\varepsilon(X_\varepsilon)) \} &= b_\varepsilon \cdot \nabla \psi(X_\varepsilon) \beta_0(b_\varepsilon(X_\varepsilon)) \\ &\quad + \psi(X_\varepsilon) \cdot \nabla \beta_0(b_\varepsilon(X_\varepsilon)) \cdot Db_\varepsilon(X_\varepsilon) \cdot b_\varepsilon(X_\varepsilon) \end{aligned}$$

therefore in view of (63) and (64)

$$\left| \frac{\partial}{\partial t} \{ \psi(X_\varepsilon) \beta_0(b_\varepsilon(X_\varepsilon)) \} \right| \leq \beta(X_\varepsilon)$$

where $\beta(z) = \beta_1 + \beta_2 \in L^1_+(\mathbb{R}^N)$. Since X_ε is measure preserving, this yields

$$\left\{ \begin{aligned} \frac{\partial}{\partial t} \{ \psi(X_\varepsilon) \beta_0(b_\varepsilon(X_\varepsilon)) \} &\text{ is bounded in } L^\infty(\mathbb{R}; L^1) \text{ and belongs} \\ &\text{to a relatively weakly compact set of } L^1(-T, T; L^1(B_R)) \quad (\forall R, T < \infty). \end{aligned} \right.$$

Then, letting ε go to 0, we deduce

$$\frac{\partial}{\partial t} \{ \psi(X) \beta_0(b(X)) \} \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^N)) \tag{65}$$

and exactly as before we deduce that, for almost all $x \in \mathbb{R}^N$, $\psi(X)\beta_0(b(X))$ is continuous on \mathbb{R} and thus, in view of the previous proofs, $b(X)$ is continuous on \mathbb{R} .

At this stage, proving that the ODE holds for almost all $x \in \mathbb{R}^N$ is easy: use (52) in integral form, let β go to the identity mapping, use the a.e. in x temporal continuity to deduce the integral form of the equation . . .

We conclude Step 1 by showing that $X \in L^p_{\text{loc}}(\mathbb{R}^N; C(\mathbb{R}))$ if b satisfies (50) i.e. $b = b_1 + b_2$ where $b_1 \in L^p(\mathbb{R}^N)$, $b_2(1 + |x|)^{-1} \in L^\infty(\mathbb{R}^N)$. Then, we have if $p < \infty$ (the case $p = \infty$ is easier)

$$\frac{\partial X_\varepsilon}{\partial t} = b_\varepsilon^1(X_\varepsilon) + b_\varepsilon^2(X_\varepsilon)$$

hence

$$\left| \frac{\partial |X_\varepsilon|}{\partial t} \right| \leq C \{1 + |X_\varepsilon|\} + |b_\varepsilon^1(X_\varepsilon)|$$

or setting $Y_\varepsilon = e^{-Ct}|X_\varepsilon|$

$$\left| \frac{\partial}{\partial t} Y_\varepsilon \right| \leq C e^{-Ct} + e^{-Ct}|b_\varepsilon^1(X_\varepsilon)|.$$

Letting ε go to 0, we find that

$$\left| \frac{\partial}{\partial t} \{e^{-Ct}|X|\} \right| \leq C e^{-Ct} + e^{-Ct}|b^1(X)|.$$

In particular $e^{-Ct}|X| \in L^p_{\text{loc}}(\mathbb{R}^N; W^{1,p}_{\text{loc}}(\mathbb{R})) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N; C(\mathbb{R}))$ and our claim is proven. Observe that the above proof also yields

$$X \in L^p_{\text{loc}}(\mathbb{R}^N; W^{1,p}_{\text{loc}}(\mathbb{R})). \tag{66}$$

Step 2. (Uniqueness)

In order to prove uniqueness, we just have to prove that, if X satisfies the conditions listed in the uniqueness statement and if $u_0 \in \mathcal{D}(\mathbb{R}^N)$, then $u_0(X(t, x))$ is the solution of (11) with $c = 0$ corresponding to the initial condition u_0 . Since u_0 is arbitrary, this yields of course the uniqueness. Hence, we set $u(t, x) = u_0(X(t, x))$ and we wish to show that u —which belongs to $C(\mathbb{R}; L^p_{\text{loc}}(\mathbb{R}^N))$ for all $1 \leq p < \infty$ and to $L^\infty(\mathbb{R}; L^p(\mathbb{R}^N))$ for all $1 \leq p \leq \infty$ —satisfies (11) in distributions sense.

In order to do so, we write for all $\psi \in \mathcal{D}(\mathbb{R}^N)$, $h > 0$, $t \in \mathbb{R}$

$$\begin{aligned} \Delta_h(t) &= \int_{\mathbb{R}^N} \frac{1}{h} \{u(t+h, x) - u(t, x)\} \psi(x) dx \\ &= \int_{\mathbb{R}^N} \frac{1}{h} \{u_0(X(t+h, x)) - u_0(X(t, x))\} \psi(x) dx \end{aligned}$$

and since X satisfies the group property, we deduce

$$\Delta_h(t) = \int_{\mathbb{R}^N} \frac{1}{h} \{u_0(X(t, X(h, x))) - u_0(X(t, x))\} \psi(x) dx .$$

And using the group property and the measure invariance of $X(h)$, this yields

$$\Delta_h(t) = \int_{\mathbb{R}^N} u(t, z) \{ \psi(X(-h, z)) - \psi(z) \} dz . \tag{67}$$

Next, we observe that $b(X) \cdot \nabla \psi(X) \in L^\infty(\mathbb{R}; L^1)$ and that for all admissible functions β

$$\frac{\partial}{\partial t} \psi(\beta(X)) = \nabla \psi(\beta(X)) \cdot D\beta(X) \cdot b(X) \quad \text{on } \mathbb{R} \times \mathbb{R}^N$$

and letting β converge to the identity map (as we did several times before) we deduce

$$\frac{\partial}{\partial t} \psi(X) = b(X) \cdot \nabla \psi(X) \quad \text{on } \mathbb{R} \times \mathbb{R}^N .$$

In particular, we have

$$\psi(X(-h, z)) - \psi(z) = - \int_0^h b(X(-\sigma, z)) \cdot \nabla \psi(X(-\sigma, z)) dz .$$

Inserting this expression in (67) and using once more the group property and the measure invariance of $X(\sigma)$ we finally obtain

$$\Delta_h(t) = - \int_{\mathbb{R}^N} \{ b(x) \cdot \nabla \psi(x) \} \cdot \left\{ \frac{1}{h} \int_0^h u(t + \sigma, x) - u(t, x) d\sigma \right\} dx .$$

Since $b \cdot \nabla \psi \in L^1$, u is bounded in $L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^N))$ and $u \in C(\mathbb{R}; L^p_{loc})$ (for $1 \leq p < \infty$), we deduce from this expression that

$$\Delta_h(t) \xrightarrow[n]{\mathbb{R}^N} - \int_{\mathbb{R}^N} b(x) \cdot \nabla \psi(x) u(t, x) dx \text{ uniformly for } t \text{ bounded} .$$

Since, on the other hand, we have obviously

$$\Delta_h(t) \xrightarrow[n]{\mathbb{R}^N} \frac{\partial}{\partial t} \int_{\mathbb{R}^N} u(t, x) \psi(x) dx \quad \text{in } \mathcal{D}'(\mathbb{R}) ,$$

we finally obtain the desired equation (11). △

Remarks. 1) In the uniqueness statement, it is possible to replace (52) by (55) and, in fact, one may show that (55) implies (52) (under the assumptions of the Theorem III.1). In the case when (50) holds, it is even possible to replace (55) by

$$\frac{\partial X}{\partial t} = b(X) \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^N) , \tag{68}$$

since in that case $X \in C(\mathbb{R}; L^1_{loc})$.

2) Under the assumptions of Theorem III.1, we do not know of any estimate on the dispersion $D_x X(t, x)$ except for the formal following one: differentiating the ODE with respect to x , we find formally

$$\frac{\partial}{\partial t} D_x X = Db(X) \cdot D_x X$$

hence

$$\frac{\partial}{\partial t} \{ \text{Log} |D_x X| \} \leq |Db(X)| .$$

Therefore, if $Db \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty]$, we deduce

$$\| \text{Log} |D_x X| \|_{L^p} \leq C|t|, \quad \text{for all } t \in \mathbb{R} . \tag{69}$$

Notice that when $p = \infty$ this yields the usual exponential rate for dispersion. \triangle

III.2 The general autonomous case

In this section, we replace the condition (48) by

$$b \in W_{\text{loc}}^{1,1}(\mathbb{R}^N), \quad \text{div } b \in L^\infty(\mathbb{R}^N) . \tag{70}$$

Of course, this will affect the property stated in (51) namely the invariance of λ by $X(t)$ and instead we will obtain for some $C_0 \in [0, \infty)$

$$e^{-C_0|t|} \lambda \leq \lambda \circ X(t) \leq e^{C_0|t|} \lambda, \quad \text{for all } t \in \mathbb{R} \tag{71}$$

or in other words, for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, $\phi \geq 0$ and for all $t \in \mathbb{R}$

$$e^{-C_0|t|} \int_{\mathbb{R}^N} \phi \, dx \leq \int_{\mathbb{R}^N} \phi(X(t, x)) \, dx \leq e^{C_0|t|} \int_{\mathbb{R}^N} \phi \, dx .$$

Then, we have the

Theorem III.2. *We assume (70) and (49). Then, the same conclusions as in Theorem III.1 hold provided condition (51) is replaced by (71).*

Remark. The unique solution $X(t, x)$ satisfies in fact (71) with $C_0 \leq \| \text{div } b \|_{L^\infty(\mathbb{R}^N)}$.

Proof of Theorem III.2. Step 1 of the proof of Theorem III.1 may be repeated without any changes; however, the uniqueness proof (step 2) has to be modified a bit. If we follow the proof given in step 2 (keeping the same notations) and use (71) instead of (51), we obtain for all $t \in \mathbb{R}$, $h > 0$, $\psi \in \mathcal{D}(\mathbb{R}^N)$

$$\begin{aligned} & \left| \Delta_h(t) - \left[- \int_{\mathbb{R}^N} u(t, x) \cdot \left\{ \frac{1}{h} \int_0^h b(X(-\sigma, z)) \cdot \nabla \psi(X(-\sigma, z)) \, d\sigma \right\} dx \right] \right| \\ & \leq \frac{1}{h} \left| \int_{\mathbb{R}^N} u(t, X(h, x)) \psi(x) \, dx - \int_{\mathbb{R}^N} u(t, x) \psi(X(-h, z)) \, dz \right| \\ & \leq \frac{C}{h} (e^{C_0 h} - 1) \| \psi \|_{L^1(\mathbb{R}^N)} \end{aligned}$$

and from this, we deduce letting h go to 0

$$\frac{\partial u}{\partial t} - \text{div}(bu) \in L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^N)) ;$$

(observe indeed that $\frac{1}{h} \int_0^h b(X(-\sigma, z)) \cdot \nabla \psi(X(-\sigma, z)) d\sigma$ is bounded in L^1 , uniformly integrable and converges in L^1_{loc} to $b(z) \cdot \nabla \psi(z)$). Therefore, if we set $F = \frac{\partial u}{\partial t} - b \cdot \nabla u$, we already know that $F \in L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^N))$ and we want to show that F vanishes.

We then use the regularizing result (Theorem II.1) to deduce that

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon = F + r_\varepsilon \quad \text{in } \mathbb{R} \times \mathbb{R}^N$$

where $r_\varepsilon \xrightarrow{\varepsilon} 0$ in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^N)$. Then, we introduce ϕ_R as in the proof of Theorem II.2 and we observe that

$$\frac{\partial}{\partial t}(u_\varepsilon \phi_R) - b \cdot \nabla(u_\varepsilon \phi_R) = F + r_\varepsilon - b \cdot \nabla \phi_R \quad \text{in } \mathbb{R} \times \mathbb{R}^N.$$

Using the regularity of u_ε (and (52)), it is now easy to integrate this equation “along the characteristics X ” in order to find

$$\begin{aligned} (\phi_R u_\varepsilon)(t, X(-t, x)) - (\phi_R u_\varepsilon)(x, 0) &= \int_0^t \{ \phi_R(F + r_\varepsilon) - b \cdot \nabla \phi_R u_\varepsilon \}(\sigma, X(-\sigma, x)) d\sigma \\ &\text{a.e. } x \in \mathbb{R}^N, \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

We then let ε go to 0, using (71), and we obtain

$$(\phi_R u)(t, X(-t, x)) - (\phi_R u)(x, 0) = \int_0^t \{ \phi_R F - b \cdot \nabla \phi_R u \}(\sigma, X(-\sigma, x)) d\sigma.$$

Then, letting R go to ∞ , using (71) and (49), this yields

$$u(t, X(-t, x)) - u(x, 0) = \int_0^t F(\sigma, X(-\sigma, x)) d\sigma \quad \text{a.e. } x \in \mathbb{R}^N, \quad \text{for all } t \in \mathbb{R}.$$

But the left-hand side vanishes, therefore we have

$$F(t, X(-t, x)) = 0 \quad \text{a.e. } x \in \mathbb{R}^N, \quad \text{for all } t \in \mathbb{R}.$$

And using once more (71), we finally obtain that F vanishes a.e. on $\mathbb{R} \times \mathbb{R}^N$, concluding thus the proof of Theorem III.2. △

Using the stability results proven in section II.3, we immediately deduce the

Corollary III.1. *Let $b_n \in L^1_{loc}$ be such that $\text{div } b_n \in L^1_{loc}$ and $b_n, \text{div } b_n$ converge as n goes to $b, \text{div } b$ in L^1_{loc} (respectively) where b satisfies (70) and (49). Assume that there exists $X_n \in C(\mathbb{R}; L^N)$ such that, for any $u_0 \in \mathcal{D}(\mathbb{R}^N)$, $u_0(X_n(t, x))$ is a renormalized solution of*

$$\frac{\partial u_n}{\partial t} - b_n \cdot \nabla u_n = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \quad u_n|_{t=0} = u_0 \text{ on } \mathbb{R}^N. \tag{72}$$

Then, for all $T \in (0, \infty)$, X_n converges in $C([-T, +T]; L^N)$ to the mapping $X \in C(\mathbb{R}, L^N)$ satisfying (71), (52), (53). In addition, X_n converges to X uniformly for t bounded, in measure for x bounded in \mathbb{R}^N .

Remark. Using Theorem II.5 instead of Theorem II.4, we see that we may assume that $\text{div } b_n$ is bounded in L^∞ instead of assuming its L^1_{loc} convergence.

III.3 Time-dependent theory

We now consider general vector fields $b = b(t, x)$ which satisfy (*) and (**) for all $T < \infty$. Then, we want to solve for all $t \geq 0, x \in \mathbb{R}^N$ the following ordinary differential equation

$$\frac{\partial X}{\partial s} = b(s, X) \quad \text{for } s \geq t, \quad X|_{s=t} = x \tag{73}$$

and thus X is a function of $(s, t, x): X = X(s, t, x)$. The mapping X will belong to $C(D; L)$ where $D = [0, \infty) \times [0, \infty)$.

Because of (*), we will find the following relation

$$\exp(-|A(t) - A(s)|\lambda) \leq \lambda \circ X \leq \exp(|A(t) - A(s)|\lambda), \quad \text{for all } t, s \geq 0 \tag{74}$$

where $A(t) \in W^{1,1}(0, R)$ ($\forall R < \infty$), $A(0) = 0, A'(t) \geq 0$ for $t \geq 0$. In fact, the solution we will build will satisfy (74) with

$$A(t) = \int_0^t \|\text{div}_x b\|_{L^\infty(\mathbb{R}^N)} ds. \tag{75}$$

Next, the group property we used in the autonomous case becomes

$$X(t_3, t_1, x) = X(t_3, t_2, X(t_2, t_1, x)) \quad \text{a.e. } x \in \mathbb{R}^N, \quad \text{for all } t_1, t_2, t_3 > 0. \tag{76}$$

In view of (74) and (**), $\frac{b(s, X)}{1 + |X|} \in L^1(0, T; L^1 + L^\infty)$ ($\forall T < \infty$) and thus we will define solutions of (73) in a similar way than in the preceding sections namely the following should hold for all admissible functions and for all $t \geq 0$

$$\frac{\partial}{\partial s} \beta(X) = D\beta(X) \cdot b(s, X) \quad \text{on } (0, \infty) \times \mathbb{R}^N, \quad \beta(X)|_{s=t} = \beta(x) \quad \text{on } \mathbb{R}^N, \tag{77}$$

where the equation holds in distributions sense.

We may now state our main existence and uniqueness result. Let us point out that we will not give stability results which are easily deduced from the stability results of section II.3 exactly as we did in Corollary III.1.

Theorem III.2. *We assume that b satisfies (*) and (**). Then, there exists a unique $X \in C(D; L)^N$ satisfying (74), (76) and (77). In addition, if $u^0 \in L^0$ (or \bar{L}), $u(s, t, x) = u^0(X(s(t, x)))$ is, for all $s \geq 0$, the unique renormalized solution in $C([0, \infty); L^0)$ of*

$$\frac{\partial u}{\partial t} + b \cdot \nabla_x u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N, \quad u|_{t=s} = u^0 \quad \text{on } \mathbb{R}^N. \tag{78}$$

Remarks. 1) The analogue of (54)–(55) is now

$$\beta(X) \in C([0 \leq t < \infty); L^1_{\text{loc}}(\mathbb{R}^N; C([0 \leq s < \infty)))) \quad \text{for all admissible } \beta \tag{79}$$

for all $t \geq 0$, for almost all $x \in \mathbb{R}^N$, $X(s) \in W^{1,1}$ and $\frac{\partial X}{\partial s} = b(s, X)$ on $(0, \infty)$

$$(80)$$

Finally, if b satisfies (for all $T < \infty$)

$$b \in L^1(0, T; L^p) + (1 + |x|)L^1(0, T; L^\infty) \text{ for some } 1 \leq p \leq \infty \quad (81)$$

then $X \in C([0 \leq t < \infty); L^p_{loc}(\mathbb{R}^N); C([0 \leq s < \infty)))$. In addition, in the last statement or in (79) one may permute s and t . △

We skip the proof of this result since it mimicks the proofs made in the preceding sections, keeping track carefully of the t -dependence (or s -dependence) using the stability result.

IV. Counterexamples and remarks

IV.1 $W^{1,p}$ vector-fields with unbounded divergence

In this section, we construct vector-fields b which are autonomous (i.e. b depends only on x) in two dimensions ($x \in \mathbb{R}^2$), belong to $W^{1,p}_{loc}(\mathbb{R}^2) \cap BUC(\mathbb{R}^2)$ for an arbitrary $p < \infty$ and yet yield infinitely many solutions of the ODE

$$\dot{X} = b(X), \quad X|_{t=0} = x \quad (82)$$

such that $X(t, x)$ satisfies the group property and X is continuous.

This construction follows in fact directly from the construction made by A. Beck in [1] that we recall now: let K be a Cantor set in $[0, 1]$ and let $g \in C^\infty(\mathbb{R})$ be such that $0 \leq g < 1$ on \mathbb{R} and $g(x) = 0$ if and only if $x \in K$. We denote by $f(x) = \int_0^x g(t) dt$ on \mathbb{R} . Next, we denote by M the set of atom-free, nonnegative, finite measures on K and for any measure m in M , we define a function f_m on \mathbb{R} by

$$f_m(x + m(K \cap [0, x])) = f(x), \quad \forall x \in \mathbb{R} . \quad (83)$$

Finally, we introduce

$$b(x) = (1, f'(f^{-1}(x^2))), \text{ for all } x = (x^1, x^2) \in \mathbb{R}^2 , \quad (84)$$

$$X_m(t, x) = (x^1 + t, f_m(t + f_m^{-1}(x^2))), \text{ for all } t \in \mathbb{R}, x = (x^1, x^2) \in \mathbb{R}^2 . \quad (85)$$

Notice that since f, f_m are strictly increasing, b, X_m are continuous in all their variables. Remark also that the group property is clearly satisfied.

We now claim that X_m is differentiable with respect to t or equivalently f_m is continuously differentiable and that

$$f'_m(t + f_m^{-1}(x_2)) = f'_m(f_m^{-1}(X_m^2)) = f'(f^{-1}(X_m^2)) \quad (86)$$

showing thus that X_m solves the ODE (82) for the choice of b given by (84).

Then, let $t \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that

$$x + m(K \cap [0, x]) = t .$$

If $x \notin K$ then for s close enough to t one has

$$(x + s - t) + m(K \cap [0, x + s - t]) = s$$

therefore f_m is differentiable at t and $f'_m(t) = f'(x)$ or in other words $f'_m(f_m^{-1}(f_m(t))) = f'(f^{-1}(x)) = f'(f^{-1}(f_m(t)))$. Next, if $x \in K$ and s is close to t , denoting by $x(s)$ the unique solution of

$$x(s) + m(K \cap [0, x(s)]) = t,$$

we observe first that

$$|x(s) - x| \leq |s - t|.$$

Then

$$|f_m(s) - f_m(t)| = |f(x(s)) - f(x)| \leq C|x(s) - x|^2 \leq C(s - t)^2$$

since $f'(x) = 0$. Hence, $f'_m(t) = 0 = f'(x)$ or in other words

$$f'_m(f_m^{-1}(f_m(t))) = f'(f^{-1}(f(x))) = f'(f^{-1}(f_m(t))).$$

As this proves our claim and (86) since t is arbitrary and f_m is strictly increasing.

There just remains to explain how g can be chosen in such a way that $b \in W_{loc}^{1,p}(\mathbb{R}^2)$ for an arbitrary $p < \infty$. Of course, we only have to check that $F = f'(f^{-1}(t)) \in L_{loc}^p(\mathbb{R}^2)$. To this end, we observe that

$$F' = g'(f^{-1}(t))g(f^{-1}(t))^{-1}$$

therefore, assuming for instance that $g(t)$ converges to 1 as $|t| \rightarrow \infty$ so that f maps \mathbb{R} onto \mathbb{R}

$$\begin{aligned} \int_{\mathbb{R}} |F'(t)|^p dt &= \int_{\mathbb{R}} |g'(f^{-1}(t))|^p |g(f^{-1}(t))|^{-p} dt \\ &= \int_{\mathbb{R}} |g(s)|^{-(p-1)} |g'(s)|^p ds. \end{aligned}$$

Hence, we only have to show the existence of a g making this last integral finite: in order to do so, we choose g_0 satisfying all the properties stated for g above and we set

$$g = g_0^m \quad \text{with} \quad m = 1 + \frac{p-1}{p};$$

so that $|g|^{-(p-1)}|g'|^p = m^p|g_0'|^p$. And we conclude requiring that $g_0' \in L^p(\mathbb{R})$.

Remarks. 1) Since $\text{div } b = g'(f^{-1}(x^2))g(f^{-1}(x^2))^{-1}$, $\text{div } b$ is clearly not bounded on \mathbb{R}^2 (even locally).

2) Notice also that $\lambda \circ X(t)$ is absolutely continuous with respect to λ and admits a density $p(t, x_2) \in L^\infty(\mathbb{R}; L^1(\mathbb{R}))$ and even in $L^\infty(\mathbb{R}; L_{loc}^q(\mathbb{R}))$ for some $q > 1$ provided $g^{-(q-1)} \in L_{loc}^1$ (in fact, if p above increases then q decreases!).

IV.2 Divergence free vector-fields without integrable first derivatives

In this section, we build an autonomous vector-field b on \mathbb{R}^2 such that

$$\text{div } b = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2), \quad b \in W_{loc}^{s,1}(\mathbb{R}^2) \quad \text{for all } s \in [0, 1) \tag{87}$$

$$b \in L^p(\mathbb{R}^2) + L^\infty(\mathbb{R}^2) \quad \text{for all } p \in [1, 2) \tag{88}$$

for which there exists two measure-preserving flows solving the associated ODE. Since we are in two dimensions, divergence-free vector-fields correspond to Hamiltonian systems and we will in fact build a singular Hamiltonian system as follows

$$\begin{cases} H(x) = -\frac{x_1}{|x_2|} & \text{if } |x_1| \leq |x_2|, \\ H(x) = -(x_1 - |x_2| + 1) & \text{if } x_1 > |x_2|, \\ H(x) = -(x_1 + |x_2| + 1) & \text{if } x_1 < -|x_2|, \end{cases} \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2 \quad (89)$$

then b will be given by

$$\begin{cases} b_2(x) = \frac{\partial H}{\partial x_1} = -\left\{ \frac{1}{|x_2|} 1_{|x_1| \leq |x_2|} + 1_{|x_1| > |x_2|} \right\} \\ b_1(x) = -\frac{\partial H}{\partial x_2} = -\text{sign}(x_2) \left\{ \frac{x_1}{|x_2|^2} 1_{|x_1| \leq |x_2|} + 1_{|x_1| > |x_2|} \right\} \end{cases} \quad (90)$$

And one checks easily that (87) and (88) hold: notice in fact that $\frac{\partial}{\partial x_i} b_j, \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$ ($\forall 1 \leq i, j \leq 2$) are bounded measures on $\mathbb{R}^2 - B_\delta$ for each $\delta > 0$ and the total variation of these measures on $\mathbb{R}^2 - B_\delta$ grows logarithmically as δ goes to 0.

Given an initial condition $x^0 = (x_1^0, x_2^0)$, we next wish to define two different flows $X^1, X^2 (= X^1, X^2(t, x^0))$. Since we are dealing with “ L^1 flows” (i.e. defined a.e.), we only need to define these flows on $I = \{x^0 \in \mathbb{R}^2 / x_1^0 \neq 0, x_2^0 \neq 0, |x_1^0| \neq |x_2^0|\}$. Then, by symmetry considerations, we only need to define X^1, X^2 on

$$Q = \{x^0 \in \mathbb{R}^2 / x_1^0 > 0, x_2^0 > 0, x_1^0 \neq x_2^0\}.$$

In the case when $x_1^0 > x_2^0$, we define X^1 and X^2 by

$$\begin{aligned} X_2^1 &= X_2^2 = x_2^0 - t; & X_1^1 &= X_1^2 = x_1^0 - t & \text{if } t \leq x_2^0, \\ X_1^1 &= X_1^2 = x_1^0 - 2x_2^0 + t & & & \text{if } t \geq x_2^0 \end{aligned} \quad \text{for all } x_1^0 > x_2^0 > 0. \quad (91)$$

In the case when $x_1^0 < x_2^0$, we define X^1, X^2 as follows

$$X_2^1 = |(x_2^0)^2 - 2t|^{1/2}, \quad X_1^1 = \frac{x_1^0}{x_2^0} |(x_2^0)^2 - 2t|^{1/2} \quad (92)$$

$$X_2^2 = \varepsilon |(x_2^0)^2 - 2t|^{1/2}, \quad X_1^2 = \frac{x_1^0}{x_2^0} \varepsilon |(x_2^0)^2 - 2t|^{1/2} \quad \text{where } \varepsilon = 1$$

$$\text{if } t \leq \frac{1}{2}(x_2^0)^2, \quad \varepsilon = -1 \quad \text{if } t \geq \frac{1}{2}(x_2^0)^2 \quad (93)$$

for all $t \in \mathbb{R}, x_2^0 > x_1^0 > 0$.

Notice that in both cases, X^1 and X^2 are continuous in t , belong to $W^{1,p}(-T, T)$ ($\forall T < \infty$) (for all $p < 2$), are smooth except for one t , solve the ODE with b for all t except for one value and such that $b(X)$ is continuous in t except for one value, $b(X(t)) \in L^p(-T, T)$ ($\forall T < \infty$) (for all $p < 2$). Furthermore, for $i = 1, 2$, we have

$$\sup_{-T \leq t \leq T} |X^i(t, x)| \leq C_T(1 + x) \quad \text{for all } T < \infty, t \in \mathbb{R}, x \in I \quad (94)$$

and

$$X^i \in L^\infty_{\text{loc}}(\mathbb{R}^2; C(\mathbb{R})) \cap C(\mathbb{R}; L^p_{\text{loc}}(\mathbb{R}^2)) \quad (\forall p < \infty). \tag{95}$$

One can also check that X^1 and X^2 are measure preserving i.e.

$$\lambda \circ X^i(t) = \lambda \quad \text{for all } t \in \mathbb{R}, \quad i = 1, 2$$

and satisfy the group property (53).

Finally, let us also remark that the proof of Theorem III.1 and the above properties of X^1, X^2 show that, for any $u^0 \in \mathcal{D}(\mathbb{R}^2)$ (or $L^p, L^0, L, \bar{L} \dots$), $u^i(t, x) = u^0(X^i(t, x))$ is, for each $i = 1, 2$, a renormalized solution (and thus a solution in distributions sense) in $C(\mathbb{R}; L^p(\mathbb{R}^2))$ ($\forall 1 \leq p < \infty$) of

$$\frac{\partial u^i}{\partial t} = b \cdot \nabla u^i \quad \text{on } \mathbb{R} \times \mathbb{R}^2, \quad u^i|_{t=0} = u^0 \quad \text{on } \mathbb{R}^2.$$

Notice that this also shows that: i) the regularizing result Theorem II.1 does not hold here—otherwise, we would deduce $u^1 \equiv u^2$, a contradiction—, ii) renormalized solutions cannot be compared with distributional solutions in the sense that the notions are not comparable. Indeed, we have seen in the preceding sections that renormalized solutions are more general than distributional solutions under some conditions on b . However, the above example shows that this is not always the case: indeed, if u^0 is an initial condition such that $u^1 \not\equiv u^2$ (i.e. $u^0 \not\equiv 0$!), then $v = u^1 - u^2$ solves the equation in distributions sense and satisfies: $v|_{t=0} = 0$ in \mathbb{R}^2 . Therefore, v cannot be a renormalized solution: indeed, if it were the case, we would easily deduce by a simple integration that

$$\int_{\mathbb{R}^2} \beta(v(t)) dx = 0, \quad \text{for each admissible } \beta, \text{ for all } t \in \mathbb{R}$$

i.e. $v \equiv 0$, contradicting the above choice.

IV.3 Small noise approximations

As it is well-known, it is possible to regularize ordinary differential equations by the addition of a “small” Brownian motion or equivalently to regularize the corresponding transport equation by the addition of a “small viscosity” term namely by considering

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \Delta u_\varepsilon - b \cdot \nabla u_\varepsilon = 0 \quad \text{in } (0, T) \times \mathbb{R}^N. \tag{96}$$

Even if it is possible to make a parallel theory of renormalized solutions for general parabolic equations including (96) and arbitrary initial conditions under the conditions (*) and (**) on b (for example), we will not do so here. Let us only remark that the arguments introduced in R. DiPerna and P.L. Lions [3] in the context of Fokker-Planck-Boltzmann equations may be used and in fact extended to cover much more general equations like (96).

Instead, we will concentrate here on the passage to the limit as ε goes to 0. One possible result in this direction is the following.

Theorem IV.1. *Assume that b satisfies (*) and (**). Let $u_\varepsilon \in L^\infty(0, T; L^1 \cap L^\infty)$ be a solution of (96) (in distributions sense) corresponding to an initial condition u_ε^0 . Assume that u_ε^0 converges in $L^p(\mathbb{R}^N)$ to u^0 for some $p \in (1, \infty)$. Then, u_ε converges in $C([0, T]; L^p(\mathbb{R}^N))$ to the renormalized solution of (11) (with $c = 0$) corresponding to the initial condition u^0 .*

Remarks. 1) In fact, the proof will show that $u_\varepsilon \in C([0, T]; L^p(\mathbb{R}^N))$.
 2) Similar results may be obtained in the case when $p = 1$.

Proof. The proof follows the argument introduced in the proof of Theorem II.4.

We first observe that we may still apply the regularizing result Theorem II.1 since its proof carries over without modifications. Then, this shows that $u_\varepsilon^\delta = u_\varepsilon * \rho_\delta$ satisfies

$$\frac{\partial u_\varepsilon^\delta}{\partial t} - \varepsilon \Delta u_\varepsilon^\delta - b \cdot \nabla u_\varepsilon^\delta = r_\varepsilon^\delta \xrightarrow{\delta} 0 \quad \text{in } L^1((0, T) \times B_R) \text{ (for all } R < \infty). \quad (97)$$

In particular, choosing cut-off functions ϕ_R as we did several times before and multiplying (97) by $\phi_R^2 |u_\varepsilon^\delta|^{\alpha-1} u_\varepsilon^\delta$, for some $\alpha > 0$, we find

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \frac{1}{\alpha+1} \int_{\mathbb{R}^N} \phi_R^2 |u_\varepsilon^\delta|^{\alpha+1} dx \right\} + \varepsilon \alpha \int_{\mathbb{R}^N} |\nabla u_\varepsilon^\delta|^2 |u_\varepsilon^\delta|^{\alpha-1} \phi_R^2 dx + \\ & + \int_{\mathbb{R}^N} \frac{\operatorname{div} b}{\alpha+1} |u_\varepsilon^\delta|^{\alpha+1} \phi_R^2 dx = \int_{\mathbb{R}^N} r_\varepsilon^\delta (u_\varepsilon^\delta)^\alpha \phi_R^2 dx - \int_{\mathbb{R}^N} 2b \cdot \nabla \phi_R (u_\varepsilon^\delta)^\alpha \phi_R dx + \\ & + 2\varepsilon \int_{\mathbb{R}^N} (\nabla \phi_R \cdot \nabla u_\varepsilon^\delta) \phi_R (u_\varepsilon^\delta)^\alpha dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, we deduce for all $v \in (0, \alpha)$

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \frac{1}{\alpha+1} \int_{\mathbb{R}^N} \phi_R^2 |u_\varepsilon^\delta|^{\alpha+1} dx \right\} + \varepsilon(\alpha - v) \int_{\mathbb{R}^N} |\nabla u_\varepsilon^\delta|^2 |u_\varepsilon^\delta|^{\alpha-1} \phi_R^2 dx + \\ & + \int_{\mathbb{R}^N} \frac{\operatorname{div} b}{\alpha+1} |u_\varepsilon^\delta|^{\alpha+1} \phi_R^2 dx \leq \int_{\mathbb{R}^N} |r_\varepsilon^\delta| |u_\varepsilon^\delta|^\alpha \phi_R^2 + |b \cdot \nabla \phi_R| |u_\varepsilon^\delta|^\alpha \phi_R^2 + \frac{\varepsilon}{v} |\nabla \phi_R|^2 |u_\varepsilon^\delta|^{\alpha+1} dx \end{aligned}$$

Letting δ go to 0, R go to ∞ , this yields: $|u_\varepsilon|^{(\alpha-1)/2} u_\varepsilon \in L^2(0, T; H^1(\mathbb{R}^N))$ for all $\alpha > 0$ and choosing $\alpha = p - 1$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^N} |u_\varepsilon|^p dx + \int_{\mathbb{R}^N} (\operatorname{div} b) |u_\varepsilon|^p dx \leq 0. \quad (98)$$

In particular, u_ε is bounded in $L^\infty(0, T; L^p(\mathbb{R}^N))$.

Next, we show that u_ε converges weakly in $L^\infty(0, T; L^p(\mathbb{R}^N))$ to the renormalized solution u of (11). To this end, we use a duality argument: we first observe that copying the proofs of Proposition II.1, Theorems II.1 and II.2, we obtain in particular the existence and uniqueness for every $\chi \in \mathcal{D}((0, T) \times \mathbb{R}^N)$ of a solution of

$$-\frac{\partial \psi_\varepsilon}{\partial t} - \varepsilon \Delta \psi_\varepsilon - \operatorname{div}(b \psi_\varepsilon) = \chi \quad \text{on } (0, T) \times \mathbb{R}^N, \quad \psi_\varepsilon|_{t=T} = 0 \quad \text{on } \mathbb{R}^N \quad (99)$$

Furthermore, $\psi_\varepsilon \in C([0, T]; L^q(\mathbb{R}^N)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^N))$ (for all $q \in [1, \infty)$) and we have

$$\int_0^T \int_{\mathbb{R}^N} u_\varepsilon \psi_\varepsilon dt dx - \int_{\mathbb{R}^N} u_\varepsilon^0 \psi_\varepsilon(0) dx = \int_0^T \int_{\mathbb{R}^N} u_\varepsilon \chi dt dx . \tag{100}$$

Next, the arguments given above show that ψ_ε is bounded in $L^\infty(0, T; L^q(\mathbb{R}^N))$ for all $1 \leq q \leq \infty$. Then, passing weakly to the limit in (99) and using the uniqueness results of section II, we deduce that ψ_ε converges weakly in $L^\infty(0, T; L^q(\mathbb{R}^N))$ ($1 < q \leq \infty$) to the solution $\psi \in C([0, T]; L^q(\mathbb{R}^N)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^N))$ ($\forall 1 \leq q < \infty$)

$$-\frac{\partial \psi}{\partial t} - \operatorname{div}(b\psi) = \chi \quad \text{in } (0, T) \times \mathbb{R}^N, \quad \psi|_{t=T} = 0 \quad \text{on } \mathbb{R}^N . \tag{101}$$

To prove strong convergence, we first observe that the same proof as the one used to show (98) yields

$$-\frac{\partial}{\partial t} \int_{\mathbb{R}^N} |\psi_\varepsilon|^q dx - \int_{\mathbb{R}^N} (q-1) \operatorname{div} b |\psi_\varepsilon|^q dx \leq 0 , \tag{102}$$

while we already know (see section II) that ψ satisfies

$$-\frac{\partial}{\partial t} \int_{\mathbb{R}^N} |\psi|^q dx - \int_{\mathbb{R}^N} (q-1) \operatorname{div} b |\psi|^q dx = 0 . \tag{103}$$

This, exactly as we did in the proof of the stability result Theorem II.4, implies that ψ_ε converges to ψ in $C([0, T]; L^q(\mathbb{R}^N))$ for all $1 \leq q < \infty$.

Now, if u_ε converges weakly in $L^\infty(0, T; L^p(\mathbb{R}^N))$ to some u , we may pass to the limit in (100) and we deduce

$$\int_0^T \int_{\mathbb{R}^N} u \psi dt dx - \int_0^T \int_{\mathbb{R}^N} u^0 \psi(0) dt dx = \int_0^T \int_{\mathbb{R}^N} u \chi dt dx . \tag{104}$$

And by the results of section II.5, we know that u is the unique renormalized solution of (11) with $c = 0$, for the initial condition u^0 .

The proof of the strong convergence follows then from (98) and the fact that u satisfies

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} (\operatorname{div} b) |u|^p dx = 0 ,$$

by the same arguments as those used in the proof of Theorem II.4. △

IV.4 Remarks

In this section, we just want to indicate some variants or extensions of the results presented above.

First of all, we begin with transport equations or ODE's in a bounded smooth domain Ω of \mathbb{R}^N ; then, let $b \in L^1(0, T; W^{1,1}(\Omega))$ be such that

$$b \cdot n = 0 \quad \text{on } \partial\Omega, \tag{105}$$

where n denotes the unit outward normal—recall that b has a trace on $\partial\Omega$ in $L^1(0, T; L^1(\partial\Omega))$.

All the results presented in the sections above may be adapted for the study of

$$\frac{\partial u}{\partial t} - b \cdot \nabla u = 0 \quad \text{in } (0, T) \times \Omega, \quad u|_{t=0} = u^0 \quad \text{on } \Omega \tag{106}$$

and

$$\dot{X} = b(t, X) \quad \text{for } t \geq s, \quad X(s, x) = x \in \bar{\Omega}. \tag{107}$$

Then, (105) is the condition which prevents the necessity of using boundary conditions for u and which makes X stay in $\bar{\Omega}$ for all t .

We may thus prove the existence, uniqueness and stability of distributional and renormalized solutions of (106); and the existence, uniqueness and stability of solutions X of (107) in $C([0, T] \times [0, T]; L^1(\Omega))$ which leave $\bar{\Omega}$ and the restriction of Lebesgue measure to $\bar{\Omega}$ invariant (up to an exponential factor if $\text{div } b \neq 0$). Let us only mention that the proofs still rely on the regularizing result Theorem II.1 which now provides a local regularization. Boundary effects are then taken care of by the following observation based upon (105) and the regularity of b :

$$\int_0^T \int_{\Gamma_\varepsilon} |b \cdot n| \, dS \, dt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0_+, \tag{108}$$

where $\Gamma_\varepsilon = \{y \in \Omega / \text{dist}(y, \partial\Omega) = \varepsilon\}$.

Let us also briefly mention that existence, uniqueness and stability results can also be obtained (in fact, in a simpler way due to the entropy formulations) for scalar conservation laws like

$$\frac{\partial u}{\partial t} + \text{div} \{b(x)f(u)\} = 0 \quad \text{in } (0, T) \times \mathbb{R}^N$$

where $b(t, x) \in L^1(0, T; W_{\text{loc}}^{1,1}(\mathbb{R}^N))$ and one deals with entropy solutions “à la Kruzkov” . . .

Our next remark concerns possible localizations of all the global results we presented: since b was not required to be bounded (or in $L^1(0, T; L^\infty)$), the speed of propagation was not finite and we were obliged to study global situations. However, if we assume that $b \in L^1(0, T; L^\infty)$ (for instance)—instead of (**)—then it is possible to localize all our results and then our uniqueness results for ODE's can be localized: in this way, one obtains almost a pathwise uniqueness in the sense that one still cannot prove by our methods uniqueness of a given trajectory starting from a given point but one may prove uniqueness by “flattening” a bit this trajectory in an arbitrary neighborhood of the initial point. In a very vague sense, trajectories for which trajectories exist for close enough initial points—one could call stable trajectories such trajectories—with a local invariance of the Lebesgue measure are unique.

Finally, let us mention some remarks concerning ODE's that we formulate to simplify in the context of Hamiltonian systems:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p) , \tag{109}$$

for some Hamiltonian H on $\mathbb{R}^N \times \mathbb{R}^N$.

First of all, since H is constant on trajectories, one sees that formally if $u(x, p, t)$ solves

$$\frac{\partial u}{\partial t} - \frac{\partial H}{\partial p}(x, p) \cdot \frac{\partial u}{\partial x} + \frac{\partial H}{\partial x}(x, p) \cdot \frac{\partial u}{\partial p} = 0 \tag{110}$$

then

$$\frac{\partial}{\partial t} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x, p, t) H(x, p) dx dp = 0 . \tag{111}$$

This formal estimate may then be used (and justified) in order to weaken the assumptions made on the behavior of $b = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x} \right)$ at infinity. Next, if H has some singularities on a “small set”, one may use our results in the following way: assume that $\{(x, p) \in \mathbb{R}^N \times \mathbb{R}^N / |H(x, p)| < R\}$ is open for R large enough and that $H \in W^{2,1}(Q_R)$ where Q_R denotes the above set. Then, one can apply the results mentioned above with $\Omega = Q_R$ since (105) clearly holds. By letting R go to ∞ , this allows to “solve” (109) for almost all initial conditions. But we will not pursue in that direction here.

References

1. Beck, A.: Uniqueness of flow solutions of differential equations. In: Recent Advances in Topological Dynamics, (Lect. Notes Math. 318). Berlin, Heidelberg, New York: Springer 1973
2. DiPerna, R.J., Lions, P.L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. Ann. Math. (to appear); see also C.R. Acad. Sci. Paris **306**, 343–346, (1988) and In Séminaire EDP, Ecole Polytechnique, 1987–88, Palaiseau
3. DiPerna, R.J., Lions, P.L.: On Fokker-Planck-Boltzmann equations. Commun. Math. Phys. (1989)
4. DiPerna, R.J., Lions, P.L.: Global weak solutions of Vlasov- Maxwell systems. Commun. Pure Appl. Math. (to appear)
5. DiPerna, R.J., Lions, P.L.: In preparation, see also C.R. Acad. Sci. Paris **307**, 655–658 (1988)
6. DiPerna, R.J., Lions, P.L.: In preparation, see also in Séminaire EDP, Ecole Polytechnique, 1988–89, Palaiseau
7. DiPerna, R.J., Lions, P.L.: In preparation, see also in Séminaire EDP, Ecole Polytechnique, 1988–89, Palaiseau