

Robust Dynamical Systems: the complexity of the reachability relation

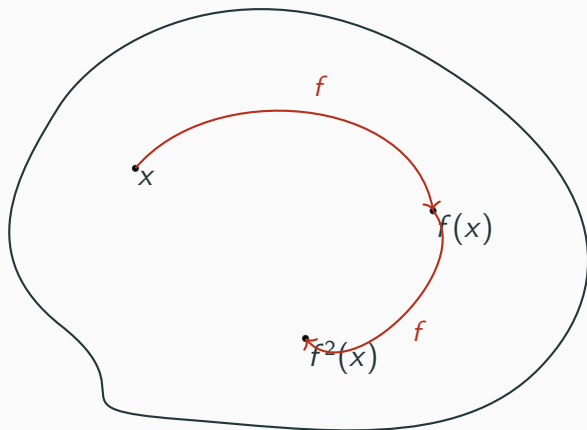
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Introduction

Introduction: robustness in Dynamical Systems

- Motivated by the field of [verification](#).
- Informal conjecture: undecidability in verification does not happen for *robust systems*.



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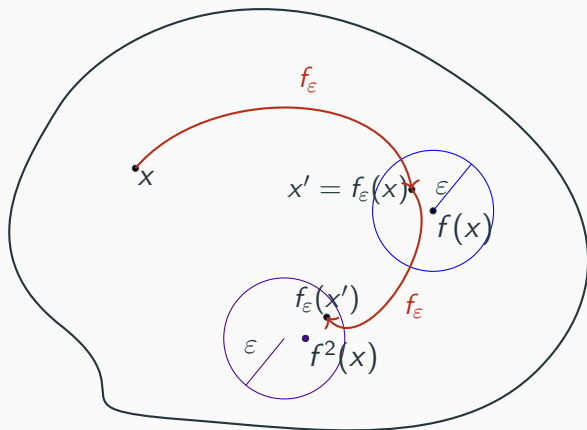
Computability Properties

- Reachability is undecidable...
- ... more precisely, it is not co-computably enumerable;
- It becomes decidable for a sub-class of systems: **Robust Dynamical Systems**.

- Undecidability is related to **non-robustness** of the systems (Asarin, Bouajjani)

→ with proper notion of robustness: **no sensitivity to an arbitrarily small perturbation.**

Robustness in Dynamical Systems



Complexity for Reach

- go from computability to complexity,
- quantifying the robustness to characterise PSPACE,
- having a notion of robustness to characterise PTIME.

Complexity for Reach

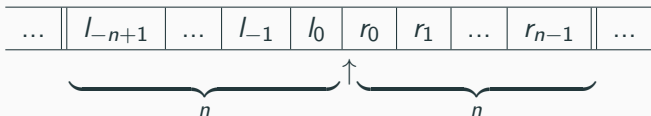
- go from computability to complexity,
- quantifying the robustness to characterise PSPACE,
- having a notion of robustness to characterise PTIME.
- We want to talk about **discrete- and continuous-time dynamical systems**.
 - We first use the framework of Turing machines (seen as a particular dynamical system).

Robustness in Dynamical Systems

Space-perturbation for Turing Machines

n-perturbation of a TM

Let \mathcal{M}_n , the *n-space-perturbed version* of TM \mathcal{M} : the idea is that the *n*-perturbed version of the machine \mathcal{M} is unable to remain correct at a distance more than *n* from the head of the machine.



Some Properties (from Asarin and Bouajjani '01)

Let $L_n(\mathcal{M})$ be the n -perturbed language of \mathcal{M} .

- By definition:
 - a word accepted by \mathcal{M} is also recognised by all the \mathcal{M}_n 's.

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- So we have
 - $L_\omega(\mathcal{M}) = \bigcap_n L_n(\mathcal{M})$: this is the set of words accepted by \mathcal{M} when subject to arbitrarily “small” perturbations.

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 - $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \dots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$.

Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)

$$L_\omega(\mathcal{M}) \in \Pi_1^0.$$

Say L is robust if $L = L_\omega$:

L robust $\Rightarrow L$ decidable.

- Definition of the notion of perturbation of a TM
- Some properties

A Characterisation of PSPACE for Turing machines

Definition

For $f : \mathbb{N} \rightarrow \mathbb{N}$, we write $L_{\{f\}}(\mathcal{M})$ the set of words accepted by \mathcal{M} with a "space-perturbation" f :

$$L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}.$$

Theorem (CSL24, B., Bournez)

$L \in \text{PSPACE}$ iff for some \mathcal{M} and some polynomial p ,
 $L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M})$.

- Characterisation of PSPACE with space-perturbed TMs

Dynamical Systems and Reachability Relations

Reachability Relation for Dynamical Systems

With each rational discrete-time dynamical system \mathcal{H} is associated its **reachability relation** $R^{\mathcal{H}}(\cdot, \cdot)$ on $\mathbb{Q}^d \times \mathbb{Q}^d$.

→ two rational points \mathbf{x} and \mathbf{y} , $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ holds iff there exists a trajectory of \mathcal{H} from \mathbf{x} to \mathbf{y} .

→ Associated problem: $\mathfrak{R}^{\mathcal{H}}$

Reachability Relation With Small Perturbations

- Consider now a discrete-time dynamical system \mathcal{H} with function \mathbf{f} Lipschitz on a compact domain:
For $\varepsilon > 0$ we consider the ε -perturbed system \mathcal{H}_ε .
- Its trajectories are defined as sequences \mathbf{x}_t satisfying $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$ for all t .

Reachability Relation With Small Perturbations

- Reachability relation in \mathcal{H}_ϵ : $R_\epsilon^{\mathcal{H}}(\cdot, \cdot)$.
Associated problem: $\mathfrak{R}_\epsilon^{\mathcal{H}}$
- $R_\omega^{\mathcal{H}} = \bigcap_\epsilon R_\epsilon^{\mathcal{H}}(\cdot, \cdot)$ is co-c.e..
- Say \mathcal{H} is robust when $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$:
 \mathcal{H} robust $\Rightarrow \mathfrak{R}^{\mathcal{H}}$ decidable
- Reachability in robust dynamical systems is *decidable* (Asarin & Bouajjani)

Theorem

Take a locally *Lipschitz* system, with $f : X \rightarrow X$ *polynomial-time* computable, with X a *closed rational box*. Then, for $p : \mathbb{N} \rightarrow \mathbb{N}$ a *polynomial*, $\mathfrak{R}_p^{\mathcal{H}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \text{PSPACE}$.

→ works also for dynamical systems over the reals

Example

Simulation of Turing machines with continuous PAMs.

Proposition (Robust \Leftrightarrow reach true or ϵ -far from being true)

We have $R_{\omega}^{\mathcal{H}} = R^{\mathcal{H}}$ iff for all $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$, either $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ is true or $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$ is false and there exists $\epsilon > 0$ such that it is ϵ -far from being true.

\Rightarrow relation with δ -reachability (Gao, Kong, Chen, Clarke '06).

Complexity of reachability relations

Polynomially robust to precision \Leftarrow PSPACE-completeness

Theorem (CSL24, B., Bournez)

Any PSPACE language is reducible to PAM's reachability relation:

$R^{\mathcal{H}} = R^{\mathcal{P}}_{\{p\}}$, for some polynomial p .

- Perturbed reachability relation for rational dynamical systems
- Characterisation of PSPACE in that framework (with additional properties)

Alternative view: Drawability and extension to the reals

⇒ we can also see it as a relation over the reals and use the framework of **computable analysis**

Definition

A closed subset of \mathbb{R}^d is said *c.e. closed* if we can effectively enumerate the rational open balls intersecting it.

From the statements of CA, the following holds:

Theorem

Consider a computable discrete time system \mathcal{H} whose domain is a computable compact. For all \mathbf{x} , $\text{cls}(R^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$ is a c.e. closed subset.

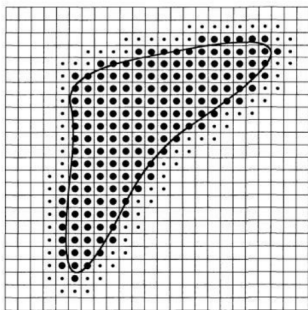
Theorem (Perturbed reachability is co-c.e.)

Consider a dynamical system, with \mathbf{f} locally Lipschitz, computable, whose domain is a computable compact, then, for all \mathbf{x} , $\text{cls}(R_\omega^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$ is a co-c.e. closed subset.

Say \mathcal{H} is robust when $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$: \mathcal{H} robust $\Rightarrow \mathfrak{R}^{\mathcal{H}}$ decidable

Plot twist

From CA, Computable \Leftrightarrow can be plotted.



- black pixels
- black or white pixels

Theorem

$R^{\mathcal{H}}$ closed and can be plotted in a name of $\mathbf{f} \Leftrightarrow$ the system is robust, i.e. $R_{\omega}^{\mathcal{H}} = R^{\mathcal{H}}$.

- We have a nice extension to geometric properties : what can be drawn is robust

Partial Conclusion

- Characterisation of **FPSPACE** with the suitable notion of perturbation.
- Extension to **drawability**.
- Characterisation of **PTIME** with the suitable notion of perturbation.

Toward PSPACE-Completeness: ongoing work

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→ we can compute a representation of x .

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e, π are computable. $\sum_{i \geq 1} 2^{-BB(i)}$, where BB is the Busy Beavers function is not.

- What is a **computable function** $f : \mathbb{R} \rightarrow \mathbb{R}$?
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- What is a **computable function** $f : \mathbb{R} \rightarrow \mathbb{R}$?
→ on a representation of $x \in \mathbb{R}$, we produce a representation of $f(x)$ → **uses oracle Turing machines**

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- $f : \Sigma^* \rightarrow \Sigma^*$ **regular** \Leftrightarrow for all $u, v \in \text{Dom}(f)$ such that $|u| \leq |v|$, then $|f(u)| \leq |f(v)|$.
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\rightarrow regular functions give a **proper measure of length**, especially for oracles

Type-2 Reduction for Decision Problems

Definition (Kawamura-Cook)

Let A and B be decision problems, or multi-functions from **Reg** to **Pred**. We write $A \leq_m^2 B$ if there exists $s, t \in \text{FPTIME}^2$ such that for any $\phi \in \text{Dom}(A)$, we have $s(\phi) \in \text{Dom}(B)$ and for $\theta = B(s(\phi))$ the function $\theta \circ t(\phi) = A(\phi)$.

$$\begin{array}{ccc} \phi & \xrightarrow{A} & A(\phi) \\ \downarrow t & & \parallel \\ t(\phi) & \xrightarrow{B \circ s \circ \phi} & \theta \circ t(\phi) \end{array} \quad \text{or} \quad \begin{array}{ccc} \phi & \xrightarrow{A} & A(\phi) \\ \downarrow t & \nearrow B \circ s \circ \phi & \\ t(\phi) & & \end{array}$$

The original scheme (Kawamura-Cook)

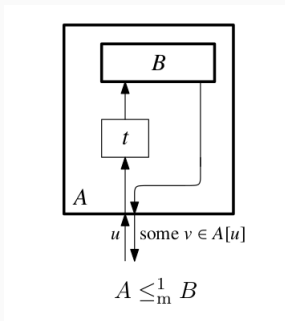


Figure 1: Usual reduction

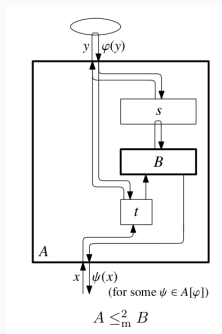


Figure 2: Type-2 reduction

An example of complete problem

Basic PSPACE²:

Input: $\langle M, \bar{\mu}, \phi \rangle$, with M a Type-2 TM, $\bar{\mu} \in \mathbf{Reg}$ such that $\bar{\mu}(u) = 0^{\mu(|u|)}$ where $\mu : \mathbb{N} \rightarrow \mathbb{N}$ a non-decreasing polynomial bounding the space of M , $\phi \in \mathbf{Reg}$ and a string u .

Output: Does M , with an oracle ϕ , accept u ?

The case of Discrete-Time Dynamical Systems

A previous result on FPTIME

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Some proof ideas

Discrete Reach Robust Lipschitz (DRRL):

Input: A discrete-time polynomially robust dynamical system

$\mathcal{H} = (X, g)$, X a closed rational box, $(g : X \rightarrow X) \in \mathbb{L}\mathbb{D}\mathbb{L}^\circ + \Phi$

Lipschitz, $p \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X$

Output: $R_{2^{-p}}^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$

with $\Phi : \gamma(w) \rightarrow \gamma(\phi(w))$, $\gamma : \Sigma^* \rightarrow \mathbb{R}$, $w \in \Sigma^*$, $\phi \in \mathbf{Reg}$

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Theorem

DRRL is $\text{PSPACE}^2 - \leq_m^2$ -hard.

Proof Ideas: Reduction

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We prove that: $\mathbf{Basic PSPACE}^2 \leq_m^2 \mathbf{DRRL}$

Transforming the execution of the TM into a dynamical system:

- **Domain** → encoding of the space of the possible configurations
- **Dynamic** → encoding of the execution of the TM into a discrete ODE

Proof Ideas: Encoding

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 - $t \equiv$ encoding of the execution;
 - $s \equiv$ maps the real encoding the configuration C_i to the real encoding C_{i+1} (related to the oracle);
 - Prove it is computable in polynomial time
 - Prove the underlying function is Lipschitz

The case of Continuous-Time Dynamical Systems

Continuous Reach Robust Lipschitz (CRRL):

Input: A continuous-time polynomially robust dynamical system

$\mathcal{H}_C = (X_C, g_C)$, with $X_C \subseteq \mathbb{R}$ a closed rational box, $g_C : X_C \rightarrow X_C \in \text{LDL}^\circ$ Lipschitz, $p \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in X_C$

Output: $R_{2^{-p}}^{\mathcal{H}_C}(\mathbf{x}, \mathbf{y})$

Theorem

CRRL is $\text{PSPACE}^2_{\leq m}$ -hard.

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- We use the Branicky trick:
 - $t \equiv$ encoding of the Branicky trick in our context;
 - $s \equiv$ maps a real in the solution of the discrete ODE to the next point
 - Prove it is computable in polynomial time

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- We studied what makes **reachability** decidable for robust **dynamical systems** and prove that space complexity is linked to the precision;
- Under additional hypotheses, we prove the **PSPACE-hardness of Reach**
 - **DRRL** is $\text{PSPACE}^2_{\leq m}$ -complete.
 - **CRRL** is $\text{PSPACE}^2_{\leq m}$ -complete.