

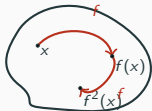
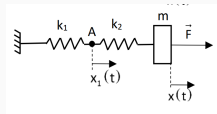
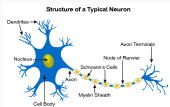
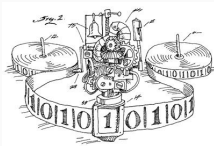
Computing with real numbers: characterising complexity classes

Manon Blanc

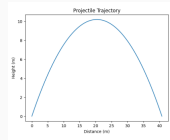
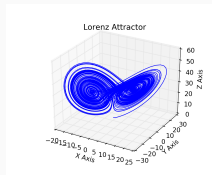
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Introduction

Introduction: what are we studying?

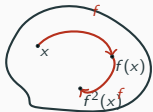
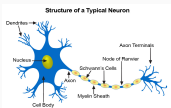
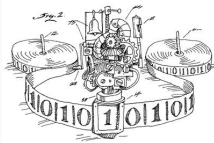


```
int x = 1;
int y = 1;
int sum = x + y;
cout << sum;
```



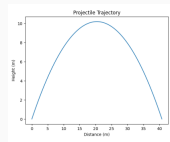
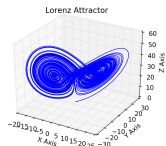
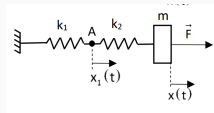
Introduction: what are we studying?

Discrete-time



```
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cout << sum;
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Continuous-time



Introduction

- Continuous-Time Systems: can be described by a (continuous) ODE

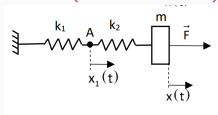
$$\frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x})$$

- Discrete-Time Systems: can be described by a discrete ODE

$$\frac{\delta \mathbf{f}(n, \mathbf{y})}{\delta n} = \mathbf{f}(n + 1, \mathbf{y}) - \mathbf{f}(n, \mathbf{y}) = \mathbf{u}(\mathbf{f}(n, \mathbf{y}), n, \mathbf{y})$$

Introduction

- Continuous-Time Systems: can be described by a (continuous) ODE



$$\frac{\partial^2 x}{\partial t^2}(t) + \frac{k}{m}x(t) = 0$$

- Discrete-Time Systems: can be described by a discrete ODE

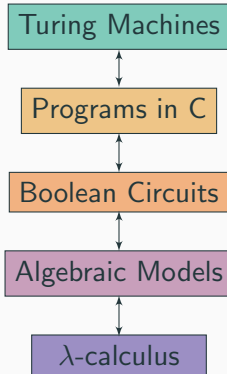
$$f : n \rightarrow 2^n$$

$$\frac{\delta f(n, \mathbf{y})}{\delta n} = f^2(n) - f(n)$$

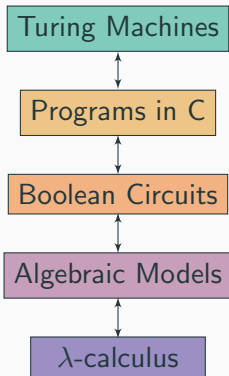
Motivations

- **Verification**: can a system reach an unsafe state from an initial state (reachability)
- **Models of computation**: what is the computational power of a given model
- **Ressources**: how to measure time, memory (space) used by a computation

Computational Power: model equivalence



Computational Power: model equivalence

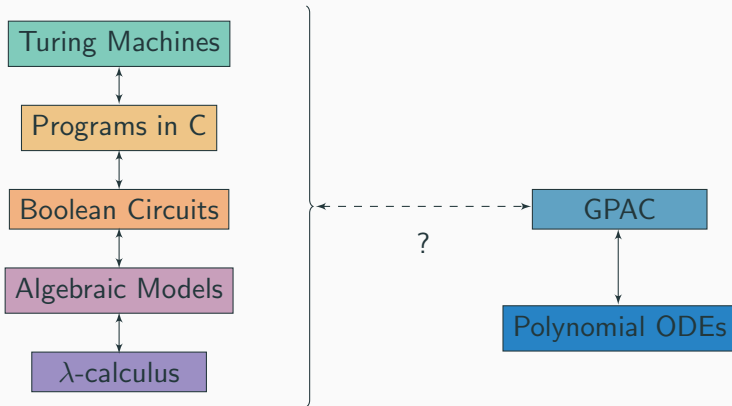


What about continuous-time models?

The General Purpose Analog Computer (GPAC)

- Introduced by Shannon (1941);
- Mathematical model of the *Differential Analyzer*;
- Model based on circuits, with several interconnected units doing basic operations
- Corresponds to polynomial ODEs: $y' = p(y)$

Computability: GPAC



Why algebraic characterisation?

$$[f_1, f_2, \dots, f_k; o_1, o_2, \dots, o_n]$$

- Allows a more denotational point of view on complexity classes

Why algebraic characterisation?

$$[f_1, f_2, \dots, f_k; o_1, o_2, \dots, o_n]$$

- Allows a more denotational point of view on complexity classes
- This is already done in computability:

$$[0, \mathbf{s}, \pi; \textit{composition}, \textit{minimisation}, \textit{primitive recursion}]$$

- Time?:
 - Time = Length
 - Continuous ODEs : Bournez, Graça, Pouly
 - Discrete ODEs : Bournez, Durand, Kontinen, Antonelli, ...

- Time?:
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- Space?:
 - Effective Church thesis for probabilistic systems: Rojas
 - A characterisation of PSPACE: Gozzi, Graça

→ **Our approach: Space = Precision**

- Mathematical framework: **Computable Analysis**
- Complexity framework: **Algebraic Models**

The beginning of algebraic characterisations: PTIME

“Length ODE”?

Definition (Length ODE)

A function \mathbf{f} is “length-ODE” definable (from u , g and h) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

“Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

θ solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$

Motivation behind Length-ODEs

Now “time-complexity” is measured by the **length** of the solution curve of the ODE.

Invariance by rescaling

Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

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Unique solution : $f(x) = 2^{2^{\ell(x)}}$

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- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$
- Then, $F(z+1) = F(z) + (z+1 - z)(F(z)F(z) - F(z))$

Characterisation of PTIME (the one you are used to)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

composition, linear length ODE]

Theorem (Bournez & Durand, '19)

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

Some basic definitions of computable analysis

Definitions (from computable analysis)

- What is a **computable real number** x ?
→ we can compute any digit of x .

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- What is a **computable function** f over the reals?
→ on an input $x \in \mathbb{R}$, we produce any digit of $f(x)$.
- How do we define **complexity over the reals**?

PTIME with Discrete ODEs

First step: real sequences

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is an **effective limit** of $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation : \bar{f} computable in polynomial time $\Rightarrow f$ computable in polynomial time.

Algebraic characterisation of PTIME

We consider:

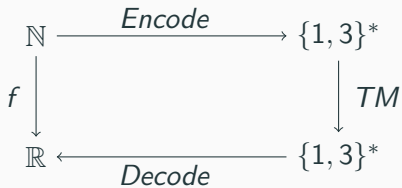
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MCU22)

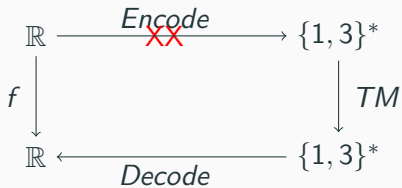
$$\text{FPTime} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

“Proof”



Second step: functions over the reals with discrete ODEs

What we cannot do



FPTIME for computable functions over the reals

We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MFCS23)

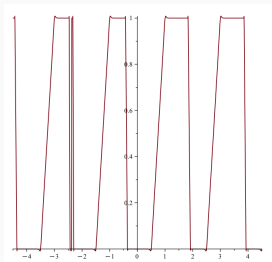
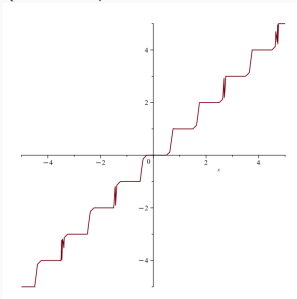
$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...):

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- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
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Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
- We *encode each step of the transition of the Turing machine* into the algebra, using the previous functions : Next;
- We obtain a *linear length ODE* encoding the execution of the Turing machine: $State(t + 1) = Next(State(t))$ so

$$\frac{\delta \overline{State}(t, y)}{\delta \ell(t)} = \overline{Next}(\overline{State}(t, y))$$

PSPACE with Discrete ODEs

Definition (Robust linear ODE)

A bounded function \mathbf{f} is a robustly linear ODE definable from g and u (with \mathbf{u} essentially linear in $\mathbf{f}(x, \mathbf{y})$: $u = A(\dots)\mathbf{f}(x) + B(\dots)$) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, robust linear ODE, ELim]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Functions over the reals with continuous ODEs

Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)

Why it is not easy to solve ODEs

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- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)
- Computable for \mathbf{u} **computable** and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a **bounded domain** for \mathbf{u} computable in polynomial-time (Kawamura, Ko)

Robust Continuous ODEs (1/2)

A function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ is robustly ODE definable (from initial condition \mathbf{g} , and dynamic \mathbf{u}) if

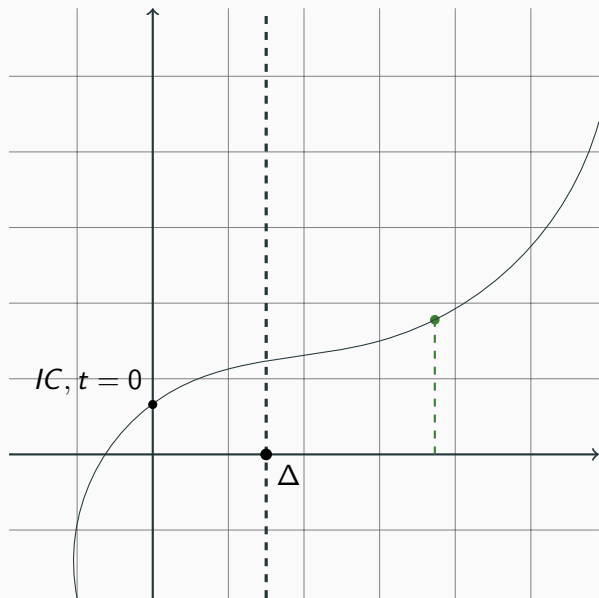
- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

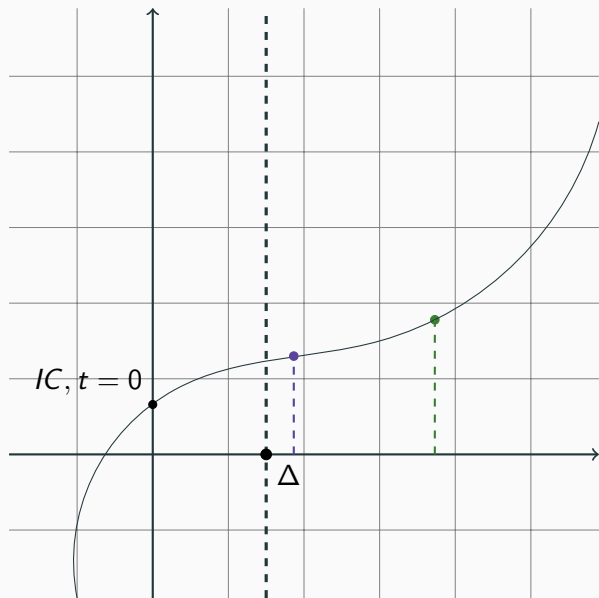
- And polynomially numerically stable (to be defined : next slides)

→ we allow computation by **dichotomy**.

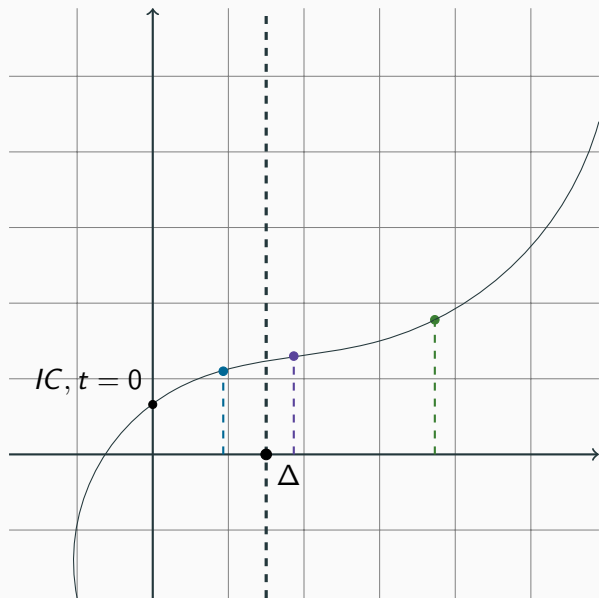
An unusual way of solving an ODE



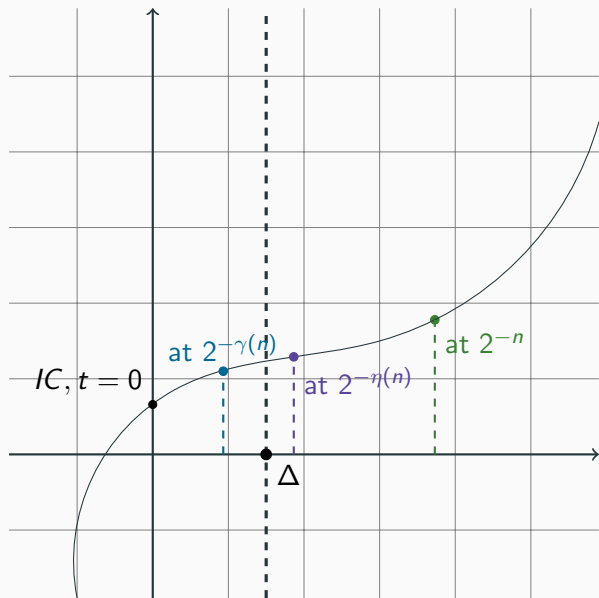
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Robust Continuous ODEs (2/2)

More “formally” :

- $\exists \Delta \in \mathbb{Q}_+^*$, such that the previous ODE is (polynomially spaced) **solvable** on $[0, \Delta]$.
- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ at 2^{-n} by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $2^{-\eta(n)}$.

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

composition, robust ODE, ELim]

Theorem (B., Bournez, ICALP24)

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Continuous \Rightarrow Discrete settings:**

From our definitions of robustness of ODEs

Proof ideas (\supseteq): going to continuous world

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete \Rightarrow Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

Proof ideas (\supseteq): going to continuous world

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete \Rightarrow Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

$$\frac{\delta f}{\delta t}(t) = u(y(t), t) \quad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0 \\ \frac{\partial z_2}{\partial t}(t) = c_2 (r(z_1(t)) - z_2(t))^3 \theta(-\cos 2\pi t) \\ \frac{\partial z_1}{\partial t}(t) = c_1 (\bar{u}(r(z_2(t))) - z_1(t))^3 \theta(\cos 2\pi t) \end{cases}$$

Conclusion

Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- The last algebra might not be minimal
- Further work :
 - talking about NP (non-determinism)?
 - having completeness results

Conclusion: what about now?

- My work currently: communication complexity with real numbers