

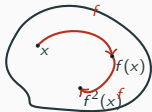
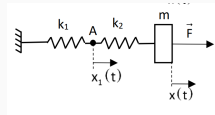
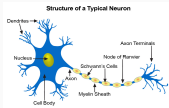
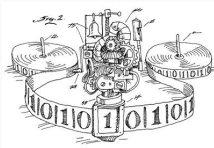
Complexity over the reals: characterising polynomial classes

Manon Blanc

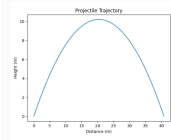
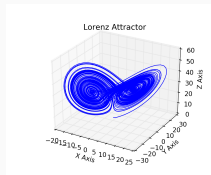
IT-UNIVERSITETET I KØBENHAVN

Introduction

Introduction: what are we studying?

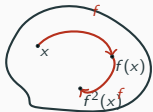
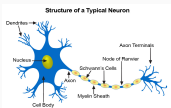
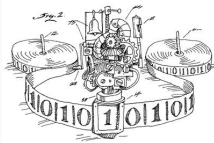


```
int x = 1;  
int y = 1;  
int sum = x + y;  
cout << sum;
```



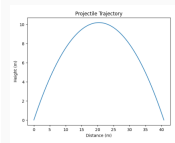
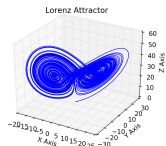
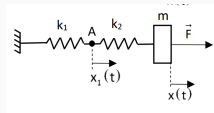
Introduction: what are we studying?

Discrete-time



```
int x = 1;  
int y = 1;  
int sum = x + y;  
cout << sum;
```

Continuous-time



Introduction

- Continuous-Time Systems: can be described by a (continuous) ODE

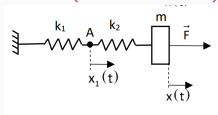
$$\frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x})$$

- Discrete-Time Systems: can be described by a discrete ODE

$$\frac{\delta \mathbf{f}(n, \mathbf{y})}{\delta n} = \mathbf{f}(n+1, \mathbf{y}) - \mathbf{f}(n, \mathbf{y}) = \mathbf{u}(\mathbf{f}(n, \mathbf{y}), n, \mathbf{y})$$

Introduction

- Continuous-Time Systems: can be described by a (continuous) ODE



$$\frac{\partial^2 x}{\partial^2 t}(t) + \frac{k}{m}x(t) = 0$$

- Discrete-Time Systems: can be described by a discrete ODE

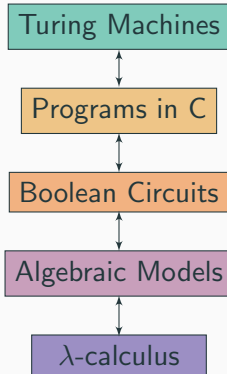
$$f : n \rightarrow 2^n$$

$$\frac{\delta f(n, \mathbf{y})}{\delta n} = f^2(n) - f(n)$$

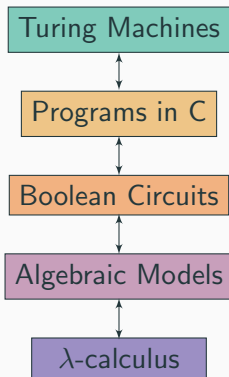
Motivations

- **Verification**: can a system reach an unsafe state from an initial state (reachability)
- **Models of computation**: what is the computational power of a given model
- **Ressources**: how to measure time, memory (space) used by a computation

Computational Power: model equivalence



Computational Power: model equivalence

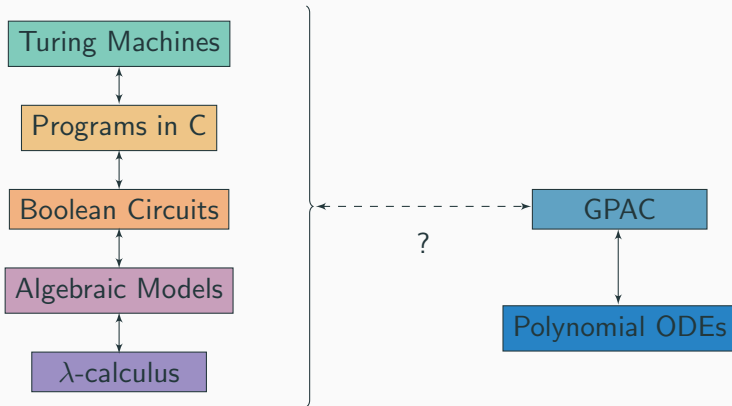


What about continuous-time models?

The General Purpose Analog Computer (GPAC)

- Introduced by Shannon (1941);
- Mathematical model of the *Differential Analyzer*;
- Model based on circuits, with several interconnected units doing basic operations
- Corresponds to polynomial ODEs: $y' = p(y)$

Computability: GPAC



Why algebraic characterisation?

$$[f_1, f_2, \dots, f_k; o_1, o_2, \dots, o_n]$$

- Allows a more denotational point of view on complexity classes

Why algebraic characterisation?

$$[f_1, f_2, \dots, f_k; o_1, o_2, \dots, o_n]$$

- Allows a more denotational point of view on complexity classes
- This is already done in computability:

$$[0, \mathbf{s}, \pi; \textit{composition}, \textit{minimisation}, \textit{primitive recursion}]$$

- Time?:
 - Time = Length
 - Continuous ODEs : Bournez, Graça, Pouly
 - Discrete ODEs : Bournez, Durand, Kontinen, Antonelli, ...

- Time?:
 - Time = Length
 - Continuous ODEs : Bournez, Graça, Pouly
 - Discrete ODEs : Bournez, Durand, Kontinen, Antonelli, ...

- Space?:
 - Effective Church thesis for probabilistic systems: Rojas
 - A characterisation of PSPACE: Gozzi, Graça

→ **Our approach: Space = Precision**

- Mathematical framework: **Computable Analysis**
- Complexity framework: **Algebraic Models**

The beginning of algebraic characterisations: PTIME

“Length ODE”?

Definition (Length ODE)

A function \mathbf{f} is “length-ODE” definable (from u , g and h) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

“Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

θ solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$

Motivation behind Length-ODEs

Now “time-complexity” is measured by the **length** of the solution curve of the ODE.

Invariance by rescaling

Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

Unique solution : $f(x) = 2^{2^{\ell(x)}}$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have $f(x) = F(z)$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$

Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^{z+1}} = F(z)F(z)$
- Then, $F(z+1) = F(z) + (z+1 - z)(F(z)F(z) - F(z))$

Characterisation of PTIME (the one you are used to)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

composition, linear length ODE]

Theorem (Bournez & Durand, '19)

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

Some basic definitions of computable analysis

Definitions (from computable analysis)

- What is a **computable real number** x ?
→ we can compute any digit of x .

Definitions (from computable analysis)

- What is a **computable real number** x ?
→ we can compute any digit of x .

Example

e , π are computable. $\sum_{i \geq 1} 2^{-BB(i)}$, where BB is the Busy Beavers function is not.

Definitions (from computable analysis)

- What is a **computable real number** x ?
→ we can compute any digit of x .

Example

e, π are computable. $\sum_{i \geq 1} 2^{-BB(i)}$, where BB is the Busy Beavers function is not.

- What is a **computable function** f over the reals?
→ on an input $x \in \mathbb{R}$, we produce any digit of $f(x)$.

Definitions (from computable analysis)

- What is a **computable real number** x ?
→ we can compute any digit of x .

Example

e, π are computable. $\sum_{i \geq 1} 2^{-BB(i)}$, where BB is the Busy Beavers function is not.

- What is a **computable function** f over the reals?
→ on an input $x \in \mathbb{R}$, we produce any digit of $f(x)$.
- How do we define **complexity over the reals**?

PTIME with Discrete ODEs

First step: real sequences

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is an **effective limit** of $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation : \bar{f} computable in polynomial time $\Rightarrow f$ computable in polynomial time.

Algebraic characterisation of PTIME

We consider:

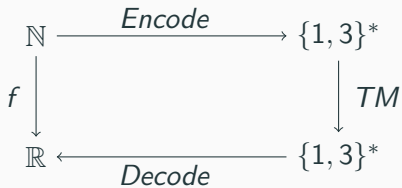
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MCU22)

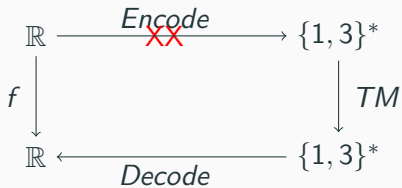
$$\text{FPTime} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

“Proof”



Second step: functions over the reals with discrete ODEs

What we cannot do



FPTIME for computable functions over the reals

We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MFCS23)

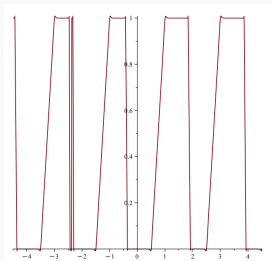
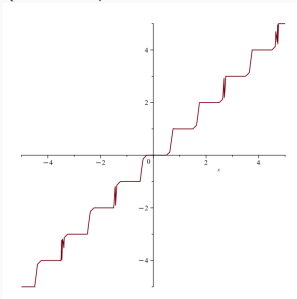
$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...):

Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...):



Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
- We *encode each step of the transition of the Turing machine* into the algebra, using the previous functions : Next;

Proof ideas (\supseteq)

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
- We *encode each step of the transition of the Turing machine* into the algebra, using the previous functions : Next;
- We obtain a *linear length ODE* encoding the execution of the Turing machine: $State(t + 1) = Next(State(t))$ so

$$\frac{\delta \overline{State}(t, y)}{\delta \ell(t)} = \overline{Next(State(t, y))}$$

PSPACE with Discrete ODEs

Definition (Robust linear ODE)

A bounded function \mathbf{f} is a robustly linear ODE definable from g and u (with \mathbf{u} essentially linear in $\mathbf{f}(x, \mathbf{y})$: $\mathbf{u} = \mathbf{A}(\dots)\mathbf{f}(x) + \mathbf{B}(\dots)$) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, robust linear ODE, ELim]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Functions over the reals with continuous ODEs

Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)

Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)
- Computable for \mathbf{u} **computable** and unique solution (Collins, Graça, Ruohonen)

Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)
- Computable for \mathbf{u} **computable** and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a **bounded domain** for \mathbf{u} computable in polynomial-time (Kawamura, Ko)

Robust Continuous ODEs (1/2)

A function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ is robustly ODE definable (from initial condition \mathbf{g} , and dynamic \mathbf{u}) if

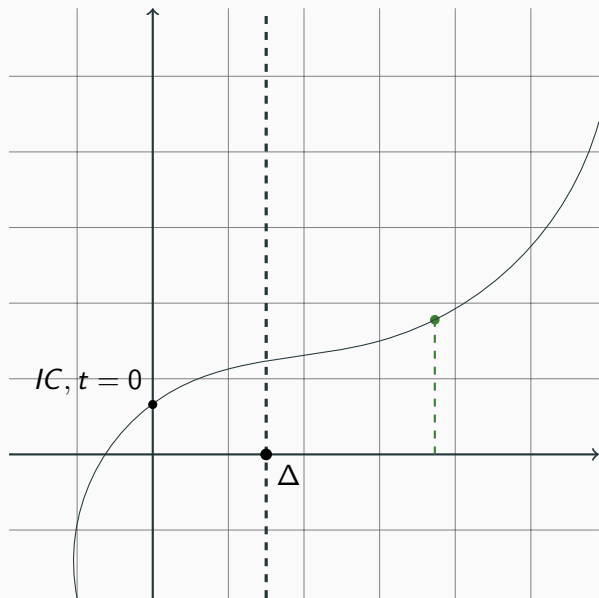
- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

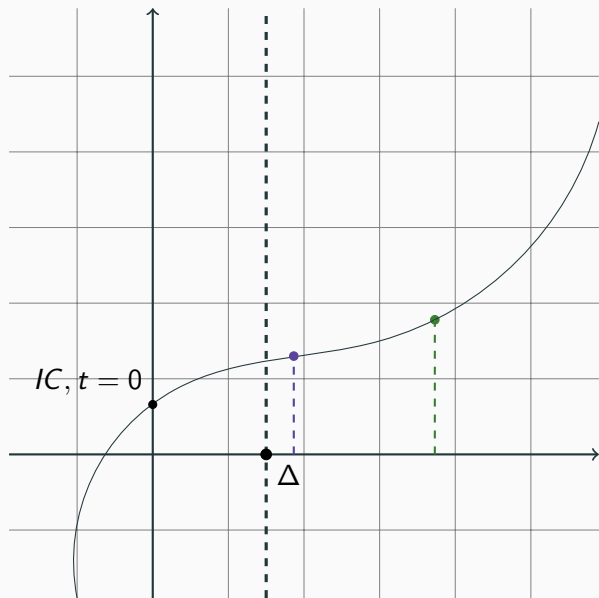
- And polynomially numerically stable (to be defined : next slides)

→ we allow computation by **dichotomy**.

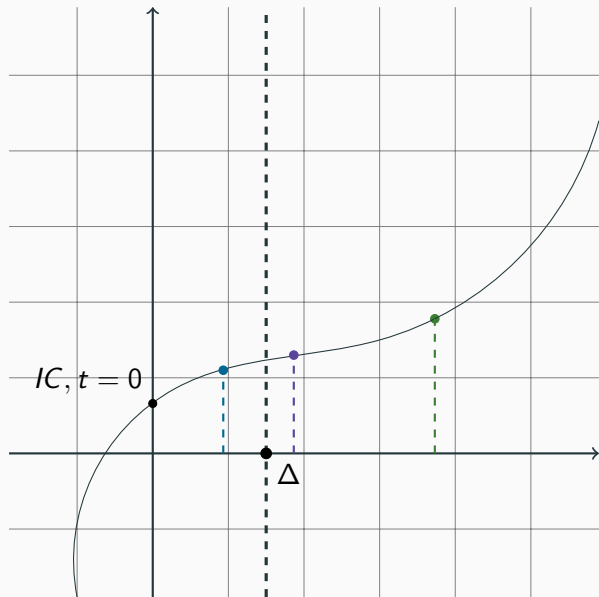
An unusual way of solving an ODE



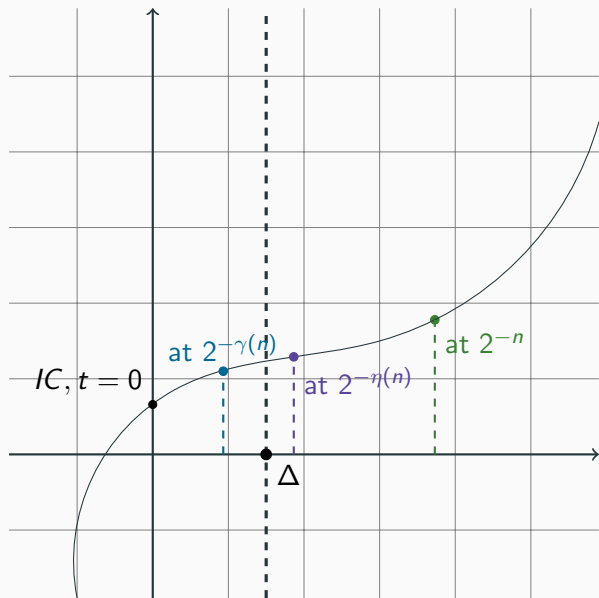
An unusual way of solving an ODE



An unusual way of solving an ODE



An unusual way of solving an ODE



Robust Continuous ODEs (2/2)

More “formally” :

- $\exists \Delta \in \mathbb{Q}_+^*$, such that the previous ODE is (polynomially spaced) **solvable** on $[0, \Delta]$.
- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ at 2^{-n} by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $2^{-\eta(n)}$.

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

composition, robust ODE, ELim]

Theorem (B., Bournez, ICALP24)

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Continuous \Rightarrow Discrete settings:**

From our definitions of robustness of ODEs

Proof ideas (\supseteq): going to continuous world

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete \Rightarrow Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

Proof ideas (\supseteq): going to continuous world

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete \Rightarrow Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

$$\frac{\delta f}{\delta t}(t) = u(y(t), t) \quad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0 \\ \frac{\partial z_2}{\partial t}(t) = c_2 (r(z_1(t)) - z_2(t))^3 \theta(-\cos 2\pi t) \\ \frac{\partial z_1}{\partial t}(t) = c_1 (\bar{u}(r(z_2(t))) - z_1(t))^3 \theta(\cos 2\pi t) \end{cases}$$

Conclusion

Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- The last algebra might not be minimal
- Further work :
 - talking about NP (non-determinism)?
 - having completeness results

Conclusion: what about now?

- My work currently: communication complexity with real numbers