

The Domino Problem is Decidable for Robust Tilesets

Nathalie Aubrun, **Manon Blanc**, Olivier Bournez



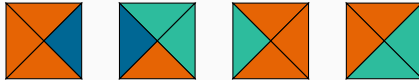
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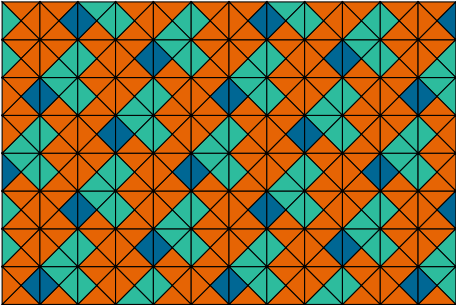
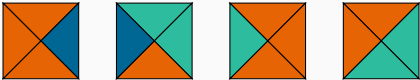
Wang Tileset

- A set of tiles with compatibility rules



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Domoni Problem: Computability result

Theorem (Easy implication, compacity argument)

*Given a set of tiles, the problem of knowing whether it tiles the plan is *co-computably enumerable*.*

Conjecture (Wang '61) : all the tiling of the plan are *periodic*.

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Conjecture (Wang '61) : all the tiling of the plan are *periodic*.

FALSE!!

Berger Theorem

Theorem (Berger '64)

*Given a set of tiles, the problem of knowing whether it tiles the plane is **undecidable**.*

→ The proof uses an aperiodic tiling

Motivation

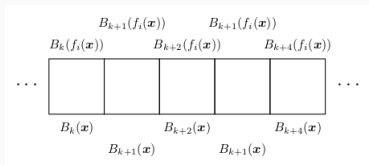
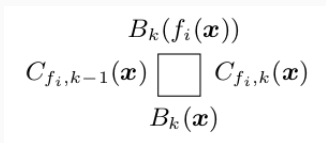
- Applying a notion of robustness to tilings :
what makes the problem of tiling *undecidable*?
- What can make the problem *decidable for some* tileset?

The tilesets we studied

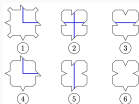
- Jeandel-Rao:



- Kari-Čulík:



- Robinson:



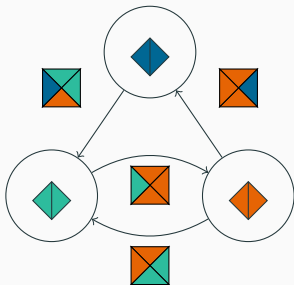
Tell me why!

The proofs are similar... related to the existence of an invariant

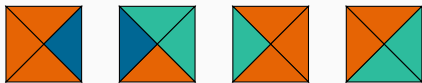
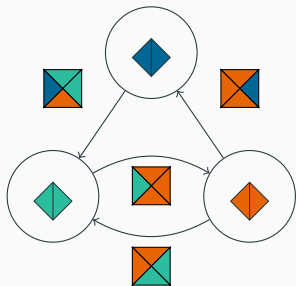
→ We give here 2 criteria

The formalism of transducers

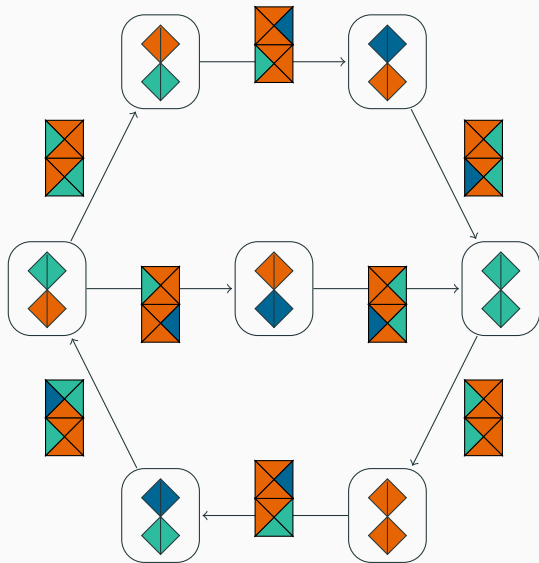
Transducers



Example (Wang)



Meta Transducer



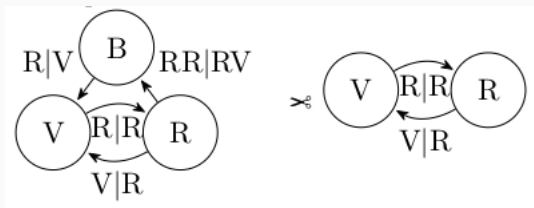
The notions of robustness

Some notations: “inclusion”

Definition (Notation $\mathcal{T} \succcurlyeq \mathcal{T}'$)

Given two transducers \mathcal{T} and \mathcal{T}' , we write $\mathcal{T} \succcurlyeq \mathcal{T}'$ iff every edge of the transducer \mathcal{T}' is an edge of \mathcal{T} .

Example



Some notations: “composition”

Definition (Composition pattern)

Consider a finite set S of transducers. Consider the signature made of the symbols of S (with arity 0) and the symbol \circ (of arity 2). A *composition pattern* F over S is a term over this signature.

Example

$H \circ \tau \circ H$ is a composition pattern over $\{H, \tau\}$. Such a composition pattern is interpreted as expected: \circ corresponds to composition.

Definition

A tileset τ is **semantically robust** (or *inductive*) if there exists a family of transducers $(\mathcal{T}_n)_n$, of respective heights $(g(n))_n$, with $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ increasing and injective with $g(1) = 1$ and some (fixed) composition pattern $F = F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n)$ over $\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n$ for some integer k , such that:

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1. Initialisation: $\tau \succ_{\circlearrowleft} \mathcal{T}_1, \dots, \tau^{g(k+1)} \succ_{\circlearrowleft} \mathcal{T}_{k+1}$

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1. Initialisation: $\tau \succ_{\circlearrowleft} \mathcal{T}_1, \dots, \tau^{g(k+1)} \succ_{\circlearrowleft} \mathcal{T}_{k+1}$
2. F provides an invariant: $\forall n, \quad F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n) \succ_{\circlearrowleft} \mathcal{T}_{n+1}$

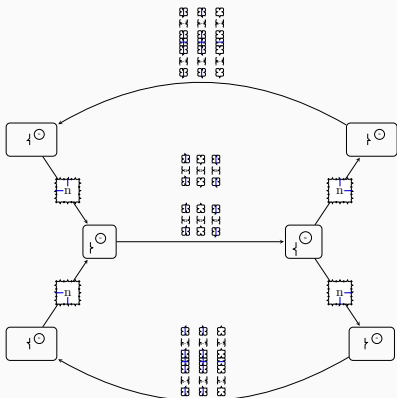
First criterion: Semantical Robustness

Definition

A tiling set τ is **semantically robust** (or *inductive*) if there exists a family of transducers $(\mathcal{T}_n)_n$, of respective heights $(g(n))_n$, with $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ increasing and injective with $g(1) = 1$ and some (fixed) composition pattern $F = F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n)$ over $\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n$ for some integer k , such that:

1. Initialisation: $\tau \succ \mathcal{T}_1, \dots, \tau^{g(k+1)} \succ \mathcal{T}_{k+1}$
2. F provides an invariant: $\forall n, F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n) \succ \mathcal{T}_{n+1}$
3. For all n , \mathcal{T}_n contains a loop.

τ_n of the Robinson



Theorem (Aubrun, B., Bournez)

Let τ a tileset. τ is *semantically robust* $\Leftrightarrow \tau$ admits a tiling.

\rightarrow the domino problem for semantically robust tileset is closely related to the existence of an invariant.

Second criterion: Provable Robustness

Definition

A tileset τ is *provably robust* if there exists a family of transducers $(\mathcal{T}_n)_n$, of heights $(g(n))_n$, $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ injective, $g(1) = 1$ and some (fixed) composition pattern $F = F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n)$ over $\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n$ for some integer k , such that:

1. Initialisation: $\tau \succ \mathcal{T}_1, \dots, \tau^{g(k+1)} \succ \mathcal{T}_{k+1}$
2. F provides an invariant holding provably:
$$\forall n, \quad F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n) \succ \mathcal{T}_{n+1}$$
3. For all n , \mathcal{T}_n contains a loop.

Theorem (Aubrun, B., Bournez)

Let τ a tileset. τ is *provably robust* $\Leftrightarrow \tau$ admits a tiling.

→ the underlying logic is Hoare logic;

→ the domino problem for provably robust tileset is decidable.

Conclusion

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- We have 2 criteria to prove if a tileset can tile a plan.
- We applied it to some well-known tilesets, including Jeandel-Rao 2015 tileset and Robinson 1978 tileset.
- Extension : a criterion to prove if a tileset can *aperiodically* tiles.