Characterisations of polynomial-time and -space complexity classes over the reals: ODEs and Robustness

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Introduction

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• Having algebraic characterisations of FPTIME and FPSPACE with ODEs;

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- Having computability and complexity results for the reachability problem in dynamical systems;

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- Having algebraic characterisations of FPTIME and FPSPACE with ODEs;
- Having computability and complexity results for the reachability problem in dynamical systems;
- Having computability and complexity results for tiling problems.

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- We already do it for computability classes
- Model : Framework of computable analysis

Introduction



Introduction





Recurrent Neural Network

Feed-Forward Neural Network

- Framework of implicit complexity (Bellantoni, Cook, Levant, Marion, de Naurois...)
- Time = Length:
 - Continuous ODEs : Bournez, Graça, Pouly (ICALP 2017)
 - Discrete ODEs : Bournez, Durand (MFCS 2019)

Warm-up: algebraic characterisations of PTIME

$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

 \rightarrow It is a derivation with a step of one

Definition (Length ODE)

A function **f** is "length-ODE" definable (from u, g and h) if it is a solution of:

$$f(0,\mathbf{y}) = \mathbf{g}(\mathbf{y})$$
 and $\frac{\delta \mathbf{f}(x,\mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}).$ (1)

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Formal synonym for right-hand side of (1): $\mathbf{f}(x+1,\mathbf{y}) = \mathbf{f}(x,\mathbf{y}) + (\ell(x+1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y})$

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- Derivation with respect to the length = change of variable.
- Variation when:

$$\ell(x+1) - \ell(x) \neq 0$$

• Inspired by:

$$\frac{\delta f(x,\mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x,\mathbf{y})}{\delta \ell(x)}.$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

| heta solution of: | ϕ solution of: |
|------------------------------|---------------------|
| y' = f(y) | z = z' |
| $f:\mathbb{R} 	o \mathbb{R}$ | y' = f(y)z |

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

Re-scaling: $\phi_1(t) = \theta(e^t)$

"Time-complexity" is measured by the length of the solution curve of the ODE: f

length_I(
$$\phi$$
) = $\int_I \|\phi'(t)\| dt$

Invariance by rescaling

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

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Unique solution : $f(x) = 2^{2^{\ell(x)}}$

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• We have
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$$F(z+1) = 2^{2^z+2^z} = F(z)F(z)$$

• Then, F(z+1) = F(z) + (z+1-z)(F(z)F(z) - F(z))

u is essentially linear in *f* iff:

$$u = A(..., \sigma(f)) \cdot f(x) + B(..., \sigma(f)))$$

with A and B that may depends on a sigmoid over f.

We consider

$$\mathbb{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, cond(x);$$

composition, linear length ODE]

Theorem (Bournez & Durand, '19) $\mathbb{LDL}\cap\mathbb{N}^{\mathbb{N}}=\mathsf{PTIME}\cap\mathbb{N}^{\mathbb{N}}$

Some basic definitions of computable analysis

- What is a computable real number x?
 - \rightarrow we can compute a representation of *x*.

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Example

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 → on a representation of x ∈ ℝ, we produce a representation of f(x).

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- What is a computable function f : ℝ → ℝ?
 → on a representation of x ∈ ℝ, we produce a representation of f(x).
- How do we define complexity over the reals?

Definitions (from computable analysis): formally

• What is a computable real number x?

There exists a Cauchy sequence $\phi : \mathbb{N} \to \mathbb{D}$ such that for all $n \in \mathbb{N}$, $|\phi(n) - x| \leq 2^{-n}$. • What is a computable real number x?

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• What is a computable function $f : \mathbb{R} \to \mathbb{R}$?

There exists an oracle Turing machine M such that, on a Cauchy sequence ϕ converging to x, M queries the oracle to have m such that $|\phi(m) - x| \leq 2^{-m}$ and computes $d \in \mathbb{D}$ such that $|d - f(x)| \leq 2^{-n}$.
FPTIME over the reals with Discrete ODEs

First step: real sequences

Definition $f : \mathbb{R} \to \mathbb{R}$ is an effective limit of $\overline{f} : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ if $|f(x) - \overline{f}(x, 2^n)| \le 2^{-n}$

<u>Key observation</u> : \overline{f} computable in polynomial time $\Rightarrow f$ computable in polynomial time.

We consider:

$$\overline{\mathbb{LDL}^{\bullet}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\mathrm{cond}}(x), \frac{x}{2};$$

composition, linear length ODE, ELim]

Theorem (B., Bournez, MCU22 Best Student Paper Award)

 $\mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\mathbb{LDL}^{\bullet}} \cap \mathbb{R}^{\mathbb{N}}$



Second step: functions over the reals with discrete ODEs



We consider:

$$\overline{\mathbb{LDL}^{\circ}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, linear length ODE, ELim]

Theorem (B., Bournez, MFCS23 Best Paper Award) $\overline{\mathbb{LDL}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$ • We give *continuous* approximations of some basic functions (integer/fractional part, modulo 2, ...);

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- We encode each step of the transition of the Turing machine into the algebra, using the previous functions : Next;

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- We encode each step of the transition of the Turing machine into the algebra, using the previous functions : Next;
- We obtain a *linear length ODE* encoding the execution of the Turing machine: State(t + 1) = Next(State(t)) so

$$\frac{\delta \overline{State}(2^{t+1}, y)}{\delta \ell} = \overline{\textit{Next}}(\overline{State}(2^t, y))$$

FPSPACE with Discrete ODEs

Definition (Robust linear ODE)

A bounded function **f** is a robustly linear ODE definable from g and u (with **u** essentially linear in $\mathbf{f}(x, \mathbf{y})$) if:

1. it is a solution of

$$\mathbf{f}(0,\mathbf{y}) = \mathbf{g}(\mathbf{y})$$
 and $\frac{\delta \mathbf{f}(x,\mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x,\mathbf{y}),\mathbf{h}(x,\mathbf{y}),x,\mathbf{y}),$

2. the ODE is (polynomially) numerically stable.

We consider:

$$\overline{\mathbb{RLD}^{\circ}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, robust linear ODE, ELim]

Theorem (B., Bournez, MFCS23 Best Paper Award)

 $\overline{\mathbb{RLD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$

PSPACE-completeness of Reach in Dynamical Systems

Dynamical System



PSPACE on robust dynamical systems



• Reachability in dynamical systems is undecidable

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- Say $R^{\mathcal{P}}$ is robust when $R^{\mathcal{P}}_{\omega} = R^{\mathcal{P}} : R^{\mathcal{P}}$ robust $\Rightarrow R^{\mathcal{P}}$ computable.
- Reachability in robust dynamical systems is *decidable* (Asarin & Bouajjani)

Theorem (B., Bournez, CSL24)

Take a locally Lipschitz system, with $f : X \to X$ polynomial-time computable, with X a closed rational box. Then, for $p : \mathbb{N} \to \mathbb{N}$ a polynomial, $R_p^{\mathcal{P}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \mathsf{PSPACE}.$

 \rightarrow works also for dynamical systems over the reals

Functions over the reals with continuous ODEs



$$\mathbf{f}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x})$$
 and $\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t,\mathbf{x}),t,\mathbf{x}),$

• Non-computable in the general case, for **u** computable (Pour-El, Richards)

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- Non-computable in the general case, for **u** computable (Pour-El, Richards)
- Computable for **u** computable and unique solution (Collins, Graça, Ruohonen)
- FPSPACE-completeness on a bounded domain for **u** computable in polynomial-time (Kawamura, Ko)

Definition

 $f:\mathbb{R}\to\mathbb{R}$ is robustly ODE definable (from initial condition g, and dynamic u) if

• it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x})$$
 and $\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t,\mathbf{x}),t,\mathbf{x}),$ (2)

• And polynomially numerically stable (to be defined : next slides)

 \rightarrow this allows computation by dichotomy.









- Computation at t = x at precision 2^{-n} , with initial condition t = 0
- Computation at t = t₀ + ^x/₂ at precision 2^{-η(n)}, with initial condition t₀ = 0 and t₀ = ^x/₂
- Computation at $t = t_0 + \frac{x}{4}$ at precision $2^{-\gamma(n)}$, with initial condition $t_0 = 0$ and $t_0 = \frac{x}{4}$ and $t_0 = \frac{x}{2}$ and $t_0 = \frac{3x}{4}$

More "formally" :

- $\exists \Delta \in \mathbb{Q}_+^*$, such that the previous ODE is (polynomially spaced) solvable on $[0, \Delta]$.
- For t ≥ Δ, we can compute f(t, x) at 2⁻ⁿ by computing some approximation f(t/2, x) of f(t/2, x) at precision 2^{-η(n)}.

We consider :

$$\overline{\mathbb{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

composition, robust continuous ODE, ELim]

Theorem (B., Bournez, ICALP24)

 $\overline{\mathbb{RCD}}\cap\mathbb{R}^{\mathbb{R}}=\mathsf{FPSPACE}\cap\mathbb{R}^{\mathbb{R}}$

$\overline{\mathbb{RLD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \overline{\mathbb{RCD}} \cap \mathbb{R}^{\mathbb{R}}$

$\overline{\mathbb{RLD}^{\circ}}\cap\mathbb{R}^{\mathbb{R}}=\overline{\mathbb{RCD}}\cap\mathbb{R}^{\mathbb{R}}$

• Discrete \Rightarrow Continuous settings:

Adaptation of the a trick due to Branicky ('95)

$\overline{\mathbb{RLD}^{\circ}}\cap\mathbb{R}^{\mathbb{R}}=\overline{\mathbb{RCD}}\cap\mathbb{R}^{\mathbb{R}}$

- Discrete \Rightarrow Continuous settings: Adaptation of the a trick due to Branicky ('95)
- Continuous \Rightarrow Discrete settings:

From our definitions of robustness of ODEs
Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- Time = Length and Space = Precision;

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- Time = Length and Space = Precision;
- Further work :
 - talking about NP (non-determinism)?
 - talking about probabilistic classes
 - talking about distributions