

The Domino Problem is Decidable for Robust Tilesets

Nathalie Aubrun, **Manon Blanc**, Olivier Bournez



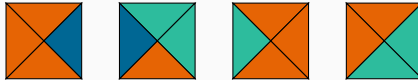
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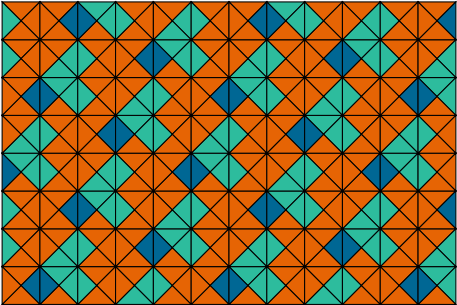
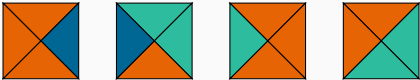
Wang Tileset

- A set of tiles with compatibility rules



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Theorem (Easy implication, compacity argument)

*Given a set of tiles, the problem of knowing whether it tiles the plan is **co-computably enumerable**.*

Conjecture (Wang '61) : all the tiling of the plan are **periodic**.

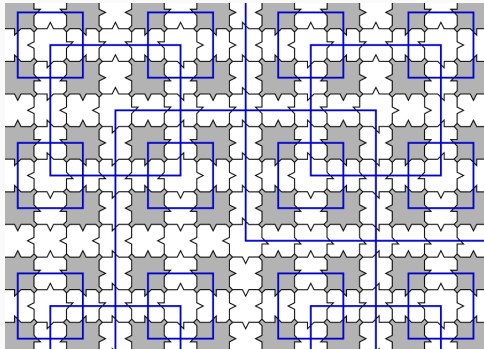
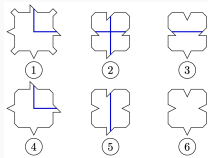
Theorem (Easy implication, compacity argument)

*Given a set of tiles, the problem of knowing whether it tiles the plan is *co-computably enumerable*.*

Conjecture (Wang '61) : all the tiling of the plan are *periodic*.

FALSE!!

Counter-example : Robinson tileset '78



Berger Theorem

Theorem (Berger '64)

*Given a set of tiles, the problem of knowing whether it tiles the plane is **undecidable**.*

→ The proof uses an aperiodic tiling

Motivation

- Applying a notion of robustness to tilings :
what makes the problem of tiling *undecidable*?
- What can make the problem *decidable for some* tileset?

The tilesets we studied

- Jeandel-Rao:



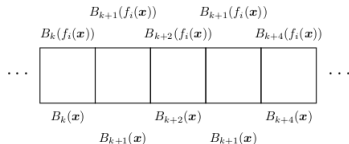
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- Jeandel-Rao:



- Kari-Čulík:

$$\begin{array}{ccc}
 & B_k(f_i(\mathbf{x})) & \\
 C_{f_i, k-1}(\mathbf{x}) & \square & C_{f_i, k}(\mathbf{x}) \\
 & B_k(\mathbf{x}) &
 \end{array}$$

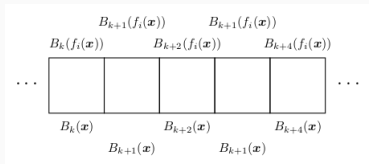
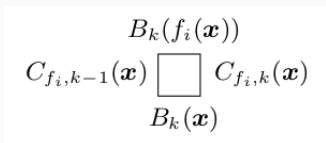


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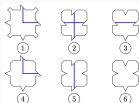
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- Kari-Čulík:



- Robinson:



Tell me why?

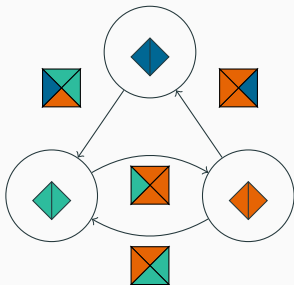
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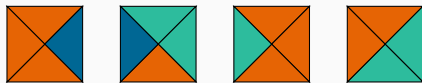
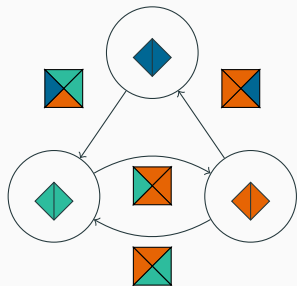
The proofs are similar... related to the existence of an invariant

The formalism of transducers

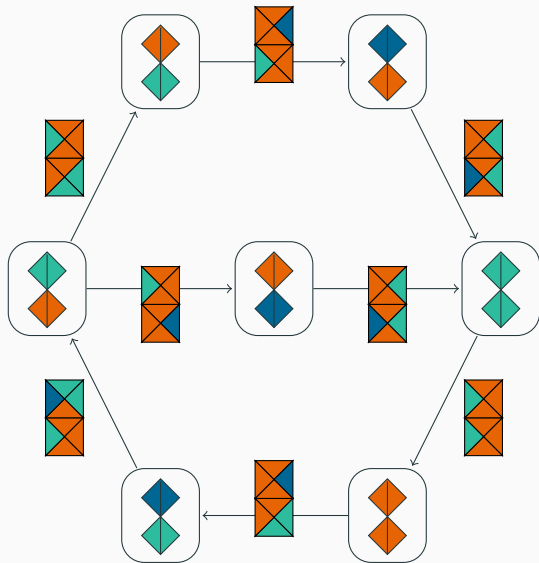
Transducers



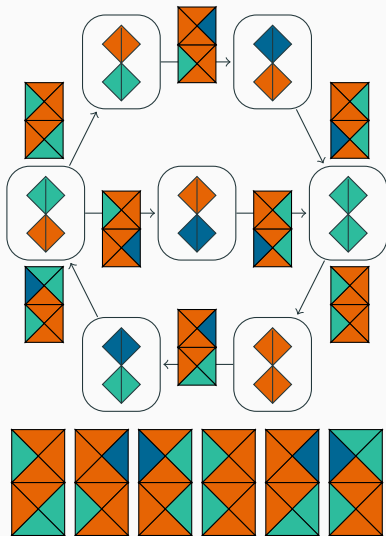
Example (Wang)



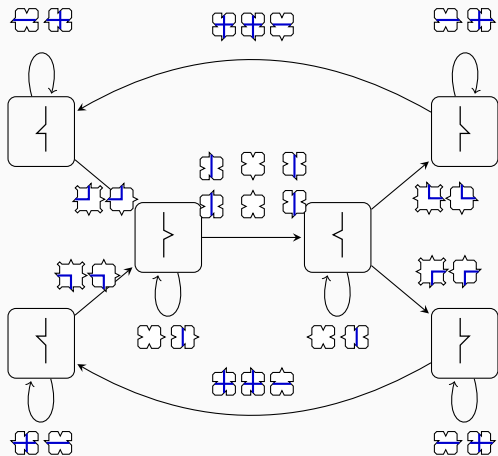
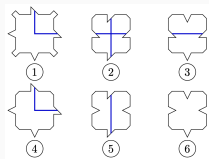
Meta Transducer



Example (Wang)



Example (Robinson)



The notions of robustness

Definition (Notation $\mathcal{T} \succ_{\varepsilon} \mathcal{T}'$)

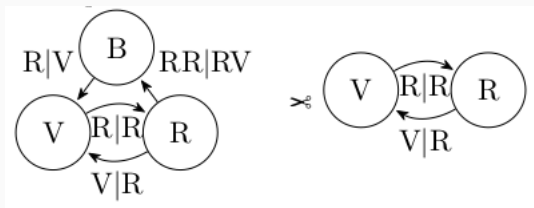
Given two transducers \mathcal{T} and \mathcal{T}' , we write $\mathcal{T} \succ_{\varepsilon} \mathcal{T}'$ iff every edge of the transducer \mathcal{T}' is an edge of \mathcal{T} .

Some notations

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Example



Definition (Composition pattern)

Consider a finite set S of transducers. Consider the signature made of the symbols of S (with arity 0) and the symbol \circ (of arity 2). A *composition pattern* F over S is a term over this signature.

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Example

$H \circ \tau \circ H$ is a composition pattern over $\{H, \tau\}$. Such a composition pattern is interpreted as expected: \circ corresponds to composition.

Definition

A tileset τ is **semantically robust** (or *inductive*) if there exists a family of transducers $(\mathcal{T}_n)_n$, of respective heights $(g(n))_n$, with $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ increasing and injective with $g(1) = 1$ and some (fixed) composition pattern $F = F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n)$ over $\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n$ for some integer k , such that:

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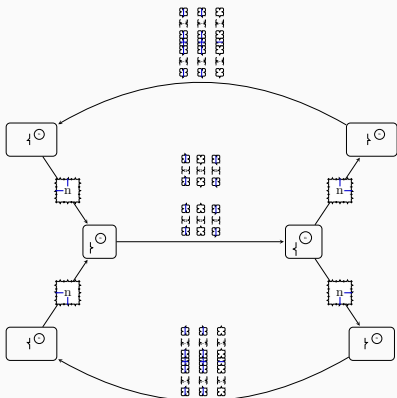
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3. For all n , \mathcal{T}_n contains a loop.

τ_n of the Robinson



Theorem (Aubrun, B., Bournez)

Let τ a tileset. τ is *semantically robust* $\Leftrightarrow \tau$ admits a tiling.

\rightarrow the domino problem for semantically robust tileset is closely related to the existence of an invariant.

Definition

A tileset τ is *provably robust* if there exists a family of transducers $(\mathcal{T}_n)_n$, of heights $(g(n))_n$, $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ injective, $g(1) = 1$ and some (fixed) composition pattern $F = F(\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n)$ over $\mathcal{T}_1, \mathcal{T}_{n-k}, \dots, \mathcal{T}_n$ for some integer k , such that:

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→ the domino problem for provably robust tileset is decidable.

Conclusion

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- We have criteria to prove if a tileset can tile a plan.
- We applied it to some well-known tilesets, including Jeandel-Rao 2015 tileset and Robinson 1978 tileset.
- Extension : a criterion to prove if a tileset can *aperiodically* tiles.