

# Characterisations of polynomial-time and -space complexity classes over the reals: ODEs and Robustness

---

Manon Blanc

MC2 Seminar



université  
PARIS-SACLAY



# Introduction

---

I focus on complexity over the reals and notions of robustness.

- **Having algebraic characterisations of FPTIME and FPSPACE with ODEs;**

I focus on complexity over the reals and notions of robustness.

- Having algebraic characterisations of FPTIME and FPSPACE with ODEs;
- Having computability and complexity results for the reachability problem in dynamical systems;

I focus on complexity over the reals and notions of robustness.

- **Having algebraic characterisations of FPTIME and FPSPACE with ODEs;**
- **Having computability and complexity results for the reachability problem in dynamical systems;**
- Having computability and complexity results for tiling problems.

- We are interested in having *algebraic characterisations* of FPTIME and FPSPACE.

→ Why algebraic characterisations?

- We are interested in having *algebraic characterisations* of  $\text{FPTIME}$  and  $\text{FPSPACE}$ .

→ Why algebraic characterisations?

- We already do it for computability classes

- We are interested in having *algebraic characterisations* of FPTIME and FPSPACE.

→ Why algebraic characterisations?

- We already do it for computability classes
- Model : Framework of *computable analysis*



- Framework of **implicit complexity** (Bellantoni, Cook, Levant, Marion, de Naurois...)
- **Time = Length:**
  - Continuous ODEs : Bournez, Graça, Pouly (ICALP 2017)
  - Discrete ODEs : Bournez, Durand (MFCS 2019)

## Warm-up: algebraic characterisations of PTIME

---

## Discrete derivation

$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

→ It is a derivation with a step of one

# “Length ODE”?

## Definition (Length ODE)

A function  $\mathbf{f}$  is “length-ODE” definable (from  $u$ ,  $g$  and  $h$ ) if it is a solution of:

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

# “Length ODE”?

## Definition (Length ODE)

A function  $\mathbf{f}$  is “length-ODE” definable (from  $u$ ,  $g$  and  $h$ ) if it is a solution of:

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x+1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x+1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

## “Length ODE”?

$$\mathbf{f}(x+1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x+1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

- Derivation with respect to the length = change of variable.
- Variation when:

$$\ell(x+1) - \ell(x) \neq 0$$

- Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

# Motivation behind Length-ODEs

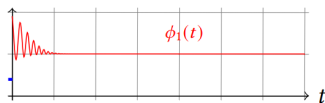
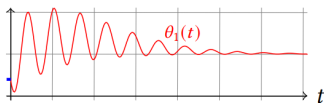


Fig. 5. A continuous system before and after an exponential speed-up.

$\theta$  solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$\phi$  solution of:

$$z = z'$$

$$y' = f(y)z$$

# Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$



# Motivation behind Length-ODEs

“Time-complexity” is measured by the **length** of the solution curve of the ODE:

$$length_I(\phi) = \int_I \|\phi'(t)\| dt$$

Invariance by rescaling

## Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

## Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

Unique solution :  $f(x) = 2^{2^{\ell(x)}}$

## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have  $f(x) = F(z)$

## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have  $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$



## Example of length-ODE

$$f(0) = 2; \quad \frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$
$$f(x+1) - f(x) = (\ell(x+1) - \ell(x))(f(x)f(x) - f(x))$$

$$z = \ell(x)$$

$$F(z) = 2^{2^z}$$

- We have  $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$
- Then,  $F(z+1) = F(z) + (z+1-z)(F(z)F(z) - F(z))$

$u$  is essentially linear in  $f$  iff:

$$u = A(\dots, \sigma(f)) \cdot f(x) + B(\dots, \sigma(f))$$

with  $A$  and  $B$  that may depend on a sigmoid over  $f$ .

# Characterisation of PTIME (the one you are used to)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

*composition, linear length ODE]*

**Theorem (Bournez & Durand, '19)**

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

## **Some basic definitions of computable analysis**

---

## Definitions (from computable analysis): intuition

- What is a **computable real number**  $x$ ?  
→ we can compute a representation of  $x$ .

## Definitions (from computable analysis): intuition

- What is a **computable real number**  $x$ ?  
→ we can compute a representation of  $x$ .

### Example

$e$ ,  $\pi$  are computable.  $\sum_{i \geq 1} 2^{-BB(i)}$ , where  $BB$  is the Busy Beavers function is not.

## Definitions (from computable analysis): intuition

- What is a **computable real number**  $x$ ?  
→ we can compute a representation of  $x$ .

### Example

$e, \pi$  are computable.  $\sum_{i \geq 1} 2^{-BB(i)}$ , where  $BB$  is the Busy Beavers function is not.

- What is a **computable function**  $f : \mathbb{R} \rightarrow \mathbb{R}$ ?  
→ on a representation of  $x \in \mathbb{R}$ , we produce a representation of  $f(x)$ .

## Definitions (from computable analysis): intuition

- What is a **computable real number**  $x$ ?  
→ we can compute a representation of  $x$ .

### Example

$e, \pi$  are computable.  $\sum_{i \geq 1} 2^{-BB(i)}$ , where  $BB$  is the Busy Beavers function is not.

- What is a **computable function**  $f : \mathbb{R} \rightarrow \mathbb{R}$ ?  
→ on a representation of  $x \in \mathbb{R}$ , we produce a representation of  $f(x)$ .
- How do we define **complexity over the reals**?



## Definitions (from computable analysis): formally

- What is a **computable real number**  $x$ ?

There exists a Cauchy sequence  $\phi : \mathbb{N} \rightarrow \mathbb{D}$  such that  
for all  $n \in \mathbb{N}$ ,  $|\phi(n) - x| \leq 2^{-n}$ .

## Definitions (from computable analysis): formally

- What is a **computable real number**  $x$ ?

There exists a Cauchy sequence  $\phi : \mathbb{N} \rightarrow \mathbb{D}$  such that for all  $n \in \mathbb{N}$ ,  $|\phi(n) - x| \leq 2^{-n}$ .

- What is a **computable function**  $f : \mathbb{R} \rightarrow \mathbb{R}$ ?

There exists an oracle Turing machine  $M$  such that, on a Cauchy sequence  $\phi$  converging to  $x$ ,  $M$  queries the oracle to have  $m$  such that  $|\phi(m) - x| \leq 2^{-m}$  and computes  $d \in \mathbb{D}$  such that  $|d - f(x)| \leq 2^{-n}$ .

# FPTIME over the reals with Discrete ODEs

---

## **First step: real sequences**

---

## Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$  is an **effective limit** of  $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$  if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation :  $\bar{f}$  computable in polynomial time  $\Rightarrow f$  computable in polynomial time.

# Algebraic characterisation of PTIME for sequences

We consider:

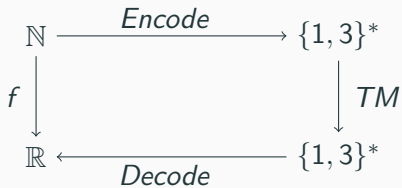
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

*composition, linear length ODE, ELim]*

**Theorem (B., Bournez, MCU22 Best Student Paper Award)**

$$\text{PTIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

# “Proof”

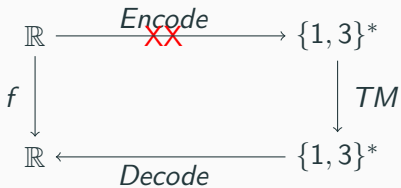


**Second step: functions over the  
reals with discrete ODEs**

---



## What we cannot do



# FPTIME for computable functions over the reals

We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, linear length ODE, ELim]*

**Theorem (B., Bournez, MFCS23 Best Paper Award)**

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

## Proof ideas ( $\supseteq$ )

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);

## Proof ideas ( $\supseteq$ )

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
- We *encode each step of the transition of the Turing machine* into the algebra, using the previous functions : Next;

## Proof ideas ( $\supseteq$ )

- We give *continuous approximations* of some basic functions (integer/fractional part, modulo 2, ...);
- We *encode each step of the transition of the Turing machine* into the algebra, using the previous functions :  $\text{Next}$ ;
- We obtain a *linear length ODE* encoding the execution of the Turing machine:  $\text{State}(t+1) = \text{Next}(\text{State}(t))$  so

$$\frac{\delta \overline{\text{State}}(2^{t+1}, y)}{\delta \ell} = \overline{\text{Next}}(\overline{\text{State}}(2^t, y))$$

.

## FPSPACE with Discrete ODEs

---

## Definition (Robust linear ODE)

A bounded function  $\mathbf{f}$  is a robustly linear ODE definable from  $g$  and  $u$  (with  $\mathbf{u}$  *essentially linear* in  $\mathbf{f}(x, \mathbf{y})$ ) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) **numerically stable**.

# FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, robust linear ODE, ELim]*

**Theorem (B., Bournez, MFCS23 Best Paper Award)**

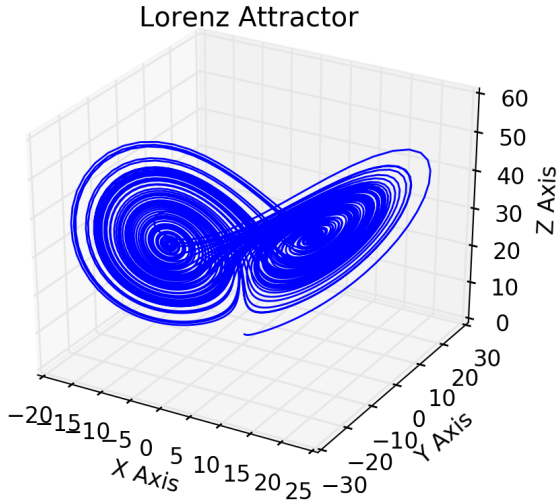
$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$



# Functions over the reals with continuous ODEs

---

# Why it is not easy to solve ODEs



## Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for  $\mathbf{u}$  computable (Pour-El, Richards)

## Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for  $\mathbf{u}$  computable (Pour-El, Richards)
- Computable for  $\mathbf{u}$  **computable** and unique solution (Collins, Graça, Ruohonen)

## Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for **u** computable (Pour-El, Richards)
- Computable for **u** **computable** and unique solution (Collins, Graça, Ruohonen)
- FPSPACE-completeness on **a bounded domain** for **u** computable in polynomial-time (Kawamura, Ko)

# Robust Continuous ODEs

## Definition

$\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$  is robustly ODE definable (from initial condition  $\mathbf{g}$ , and dynamic  $\mathbf{u}$ ) if

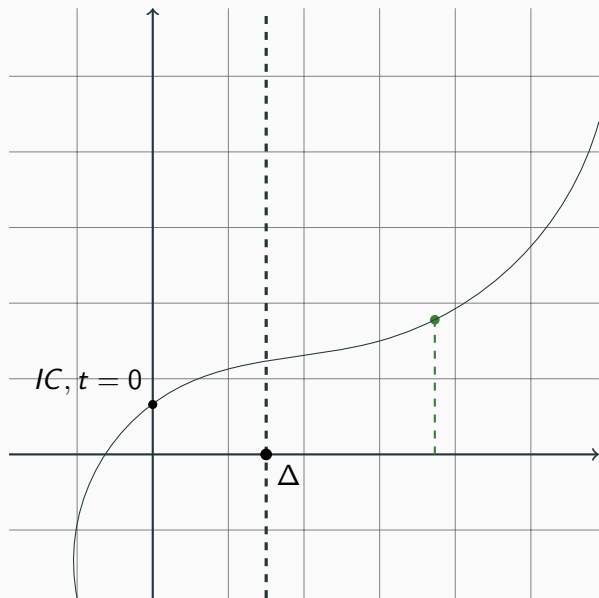
- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

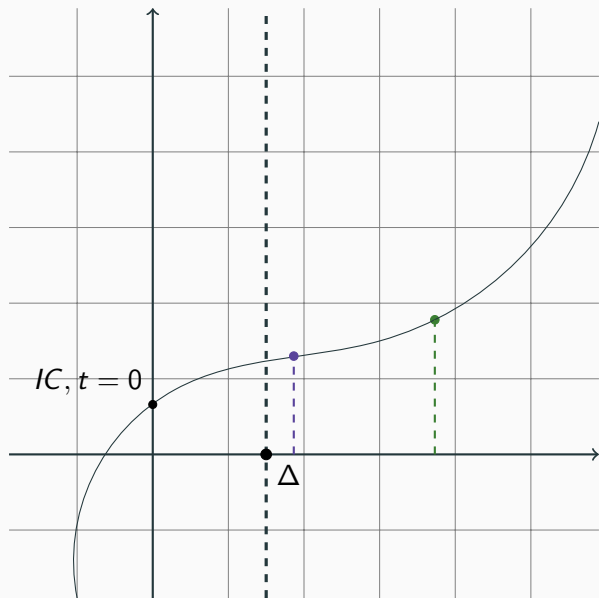
- And polynomially numerically stable (to be defined : next slides)

→ this allows computation by **dichotomy**.

## An unusual way of solving an ODE

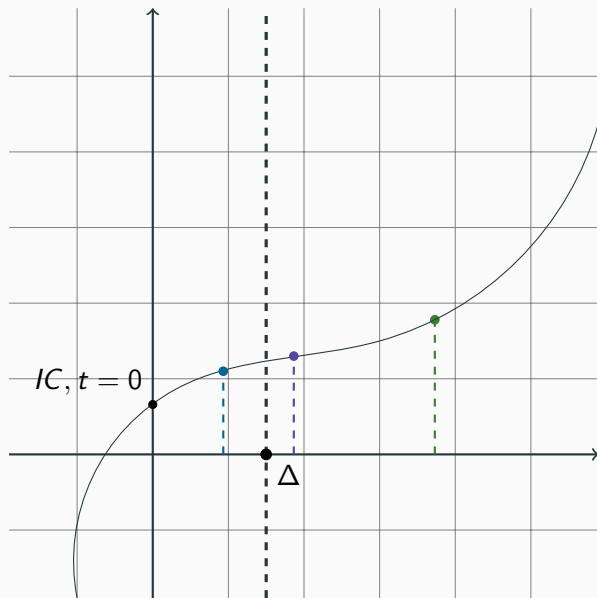


## An unusual way of solving an ODE

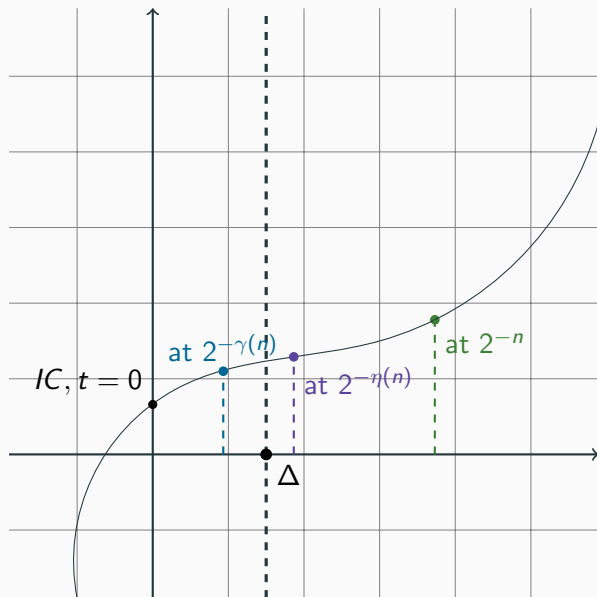




## An unusual way of solving an ODE



## An unusual way of solving an ODE



## An unusual way of solving an ODE

- Computation at  $t = x$  at precision  $2^{-n}$ , with initial condition  $t = 0$
- Computation at  $t = t_0 + \frac{x}{2}$  at precision  $2^{-\eta(n)}$ , with initial condition  $t_0 = 0$  and  $t_0 = \frac{x}{2}$
- Computation at  $t = t_0 + \frac{x}{4}$  at precision  $2^{-\gamma(n)}$ , with initial condition  $t_0 = 0$  and  $t_0 = \frac{x}{4}$  and  $t_0 = \frac{x}{2}$  and  $t_0 = \frac{3x}{4}$

## Robust Continuous ODEs (2/2)

More “formally” :

- $\exists \Delta \in \mathbb{Q}_+^*$ , such that the previous ODE is (polynomially spaced) **solvable** on  $[0, \Delta]$ .
- For  $t \geq \Delta$ , we can compute  $\mathbf{f}(t, \mathbf{x})$  at  $2^{-n}$  by computing some approximation  $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$  of  $\mathbf{f}(t/2, \mathbf{x})$  at precision  $2^{-\eta(n)}$ .

# Characterisation of PSPACE with continuous ODEs

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

*composition, robust continuous ODE, ELim]*

**Theorem (B., Bournez, ICALP24)**

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete  $\Rightarrow$  Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

$$\overline{\text{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Discrete  $\Rightarrow$  Continuous settings:**

Adaptation of the a trick due to Branicky ('95)

- **Continuous  $\Rightarrow$  Discrete settings:**

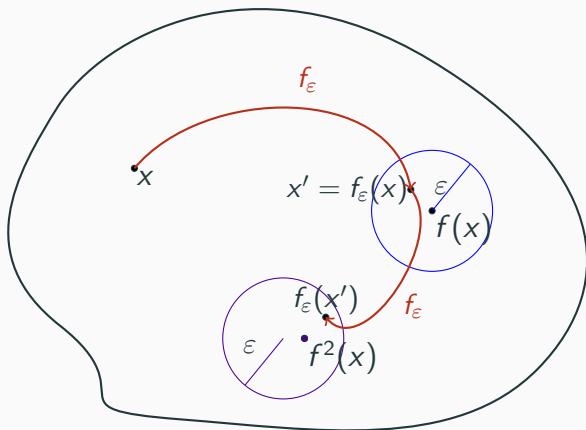
From our definitions of robustness of ODEs



## **Further work: PSPACE-completeness of Reach**

---

# PSPACE on robust dynamical systems



- Reachability in dynamical systems is *undecidable*

- Reachability in dynamical systems is *undecidable*
- Say  $R^{\mathcal{P}}$  is robust when  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$  :  $R^{\mathcal{P}}$  robust  $\Rightarrow R^{\mathcal{P}}$  **computable**.
- Reachability in **robust** dynamical systems is *decidable* (Asarin & Bouajjani)

## Theorem (B., Bournez, CSL24)

Take a locally *Lipschitz* system, with  $f : X \rightarrow X$  *polynomial-time* computable, with  $X$  a *closed rational box*. Then, for  $p : \mathbb{N} \rightarrow \mathbb{N}$  a *polynomial*,  $R_p^{\mathcal{P}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \text{PSPACE}$ .

→ works also for dynamical systems over the reals

## Conclusion

---

# Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous* ODEs.
- Time = Length and Space = Precision;

# Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- Time = Length and Space = Precision;
- Further work :
  - talking about NP (non-determinism)?
  - talking about probabilistic classes
  - talking about distributions