

# Robust Dynamical Systems: the complexity of the reachability relation

---

Manon Blanc

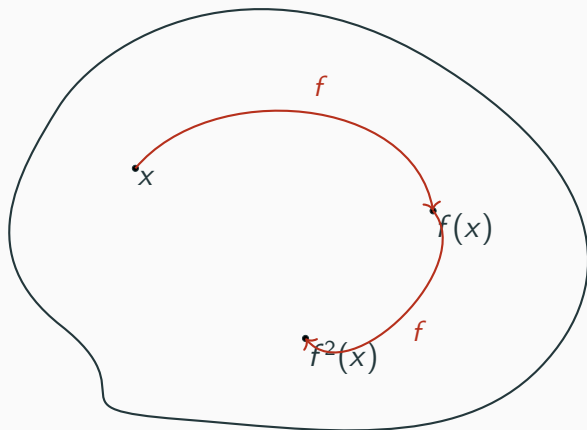
IT-UNIVERSITETET I KØBENHAVN

# Introduction

---

# Introduction: robustness in Dynamical Systems

- Motivated by the field of [verification](#).
- Informal conjecture: undecidability in verification does not happen for *robust systems*.



# Computability Properties

- Reachability is undecidable...

# Computability Properties

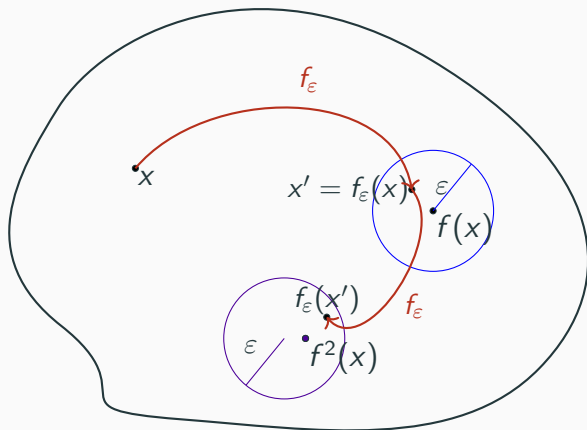
- Reachability is undecidable...
- ... more precisely, it is not co-computably enumerable;

# Computability Properties

- Reachability is undecidable...
- ... more precisely, it is not co-computably enumerable;
- It becomes decidable for a sub-class of systems: **Robust Dynamical Systems**.

- Undecidability is related to **non-robustness** of the systems (Asarin, Bouajjani)  
  
→ with proper notion of robustness: **no sensitivity to an arbitrarily small perturbation.**

# Robustness in Dynamical Systems



## Complexity for Reach

- go from computability to complexity,
- quantifying the robustness to characterise PSPACE,
- having a notion of robustness to characterise PTIME.

# Complexity for Reach

- go from computability to complexity,
- quantifying the robustness to characterise PSPACE,
- having a notion of robustness to characterise PTIME.
- We want to talk about **discrete- and continuous-time dynamical systems**.
  - We first use the framework of Turing machines (seen as a particular dynamical system).

# Robustness in Dynamical Systems

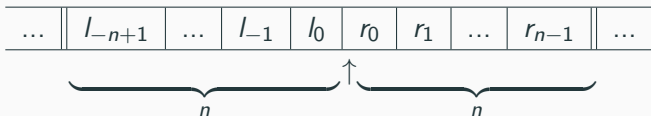
---

# Space-perturbation for Turing Machines

---

## n-perturbation of a TM

Let  $\mathcal{M}_n$ , the  $n$ -space-perturbed version of TM  $\mathcal{M}$ : the idea is that the  $n$ -perturbed version of the machine  $\mathcal{M}$  is unable to remain correct at a distance more than  $n$  from the head of the machine.



## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- By definition:
  - a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.

## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- By definition:
  - a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.
  - $L_{n+1}(\mathcal{M}) \subseteq L_n(\mathcal{M})$ .

## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- By definition:
  - a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.
  - $L_{n+1}(\mathcal{M}) \subseteq L_n(\mathcal{M})$ .
  - $L_n(\mathcal{M})$  is co-c.e.

## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- By definition:
  - a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.
  - $L_{n+1}(\mathcal{M}) \subseteq L_n(\mathcal{M})$ .
  - $L_n(\mathcal{M})$  is co-c.e.
- So we have
  - $L_\omega(\mathcal{M}) = \bigcap_n L_n(\mathcal{M})$ : this is the set of words accepted by  $\mathcal{M}$  when subject to arbitrarily “small” perturbations.

## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- By definition:
  - a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.
  - $L_{n+1}(\mathcal{M}) \subseteq L_n(\mathcal{M})$ .
  - $L_n(\mathcal{M})$  is co-c.e.
- So we have
  - $L_\omega(\mathcal{M}) = \bigcap_n L_n(\mathcal{M})$ : this is the set of words accepted by  $\mathcal{M}$  when subject to arbitrarily “small” perturbations.
  - $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \dots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$ .

**Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)**

$$L_\omega(\mathcal{M}) \in \Pi_1^0.$$

Say  $L$  is robust if  $L = L_\omega$ :

$L$  robust  $\Rightarrow L$  decidable.

- Definition of the notion of perturbation of a TM
- Some properties

# **A Characterisation of PSPACE for Turing machines**

---

### Definition

For  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L_{\{f\}}(\mathcal{M})$  the set of words accepted by  $\mathcal{M}$  with a "space-perturbation"  $f$ :

$$L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}.$$

**Theorem ( CSL24, B., Bournez)**

$L \in \text{PSPACE}$  iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  
 $L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M})$ .

- Characterisation of PSPACE with space-perturbated TMs

# **Dynamical Systems and Reachability Relations**

---

# Reachability Relation for Dynamical Systems

With each rational discrete-time dynamical system  $\mathcal{H}$  is associated its **reachability relation**  $R^{\mathcal{H}}(\cdot, \cdot)$  on  $\mathbb{Q}^d \times \mathbb{Q}^d$ .

→ two rational points  $\mathbf{x}$  and  $\mathbf{y}$ ,  $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$  holds iff there exists a trajectory of  $\mathcal{H}$  from  $\mathbf{x}$  to  $\mathbf{y}$ .

→ Associated problem:  $\mathfrak{R}^{\mathcal{H}}$

## Reachability Relation With Small Perturbations

- Consider now a discrete-time dynamical system  $\mathcal{H}$  with function  $\mathbf{f}$  Lipschitz on a compact domain:  
For  $\varepsilon > 0$  we consider the  $\varepsilon$ -perturbed system  $\mathcal{H}_\varepsilon$ .
- Its trajectories are defined as sequences  $\mathbf{x}_t$  satisfying  $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$  for all  $t$ .

# Reachability Relation With Small Perturbations

- Reachability relation in  $\mathcal{H}_\epsilon$ :  $R_\epsilon^{\mathcal{H}}(\cdot, \cdot)$ .  
Associated problem:  $\mathfrak{R}_\epsilon^{\mathcal{H}}$
- $R_\omega^{\mathcal{H}} = \bigcap_\epsilon R_\epsilon^{\mathcal{H}}(\cdot, \cdot)$  is co-c.e..
- Say  $\mathcal{H}$  is robust when  $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$ :  
 $\mathcal{H}$  robust  $\Rightarrow \mathfrak{R}^{\mathcal{H}}$  decidable
- Reachability in robust dynamical systems is *decidable* (Asarin & Bouajjani)

## Theorem

Take a locally *Lipschitz* system, with  $f : X \rightarrow X$  *polynomial-time* computable, with  $X$  a *closed rational box*. Then, for  $p : \mathbb{N} \rightarrow \mathbb{N}$  a *polynomial*,  $\mathfrak{R}_p^{\mathcal{H}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in \text{PSPACE}$ .

→ works also for dynamical systems over the reals

## Example

Simulation of Turing machines with continuous PAMs.

**Proposition** (Robust  $\Leftrightarrow$  reach true or  $\epsilon$ -far from being true)

We have  $R_{\omega}^{\mathcal{H}} = R^{\mathcal{H}}$  iff for all  $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ , either  $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$  is true or  $R^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$  is false and there exists  $\epsilon > 0$  such that it is  $\epsilon$ -far from being true.

$\Rightarrow$  relation with  $\delta$ -reachability (Gao, Kong, Chen, Clarke '06).

# Complexity of reachability relations

Polynomially robust to precision  $\Leftarrow$  PSPACE-completeness

**Theorem (CSL24, B., Bournez)**

*Any PSPACE language is reducible to PAM's reachability relation:*

$R^{\mathcal{H}} = R^{\mathcal{P}}_{\{p\}}$ , for some polynomial  $p$ .

- Perturbated reachability relation for rational dynamical systems
- Characterisation of PSPACE in that framework (with additionnal properties)

## **Alternative view: Drawability and extension to the reals**

---

⇒ we can also see it as a relation over the reals and use the framework of **computable analysis**

### **Definition**

A closed subset of  $\mathbb{R}^d$  is said *c.e. closed* if we can effectively enumerate the rational open balls intersecting it.

From the statements of CA, the following holds:

### Theorem

*Consider a computable discrete time system  $\mathcal{H}$  whose domain is a computable compact. For all  $\mathbf{x}$ ,  $\text{cls}(R^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a c.e. closed subset.*

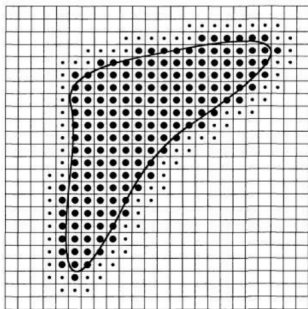
### Theorem (Perturbed reachability is co-c.e.)

Consider a dynamical system, with  $\mathbf{f}$  locally Lipschitz, computable, whose domain is a computable compact, then, for all  $\mathbf{x}$ ,  $\text{cls}(R_\omega^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a co-c.e. closed subset.

Say  $\mathcal{H}$  is robust when  $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$ :  $\mathcal{H}$  robust  $\Rightarrow \mathfrak{R}^{\mathcal{H}}$  decidable

# Plot twist

From CA, Computable  $\Leftrightarrow$  can be plotted.



- black pixels
- black or white pixels

## Theorem

$R^{\mathcal{H}}$  closed and can be plotted in a name of  $\mathbf{f} \Leftrightarrow$  the system is robust, i.e.  $R_{\omega}^{\mathcal{H}} = R^{\mathcal{H}}$ .

- We have a nice extension to geometric properties : what can be drawn is robust

# Time-perturbation

---

## Another type of perturbation

### Definition

Given  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L^{\{f\}}(\mathcal{M})$  for the set of words accepted by  $\mathcal{M}$  with time perturbation  $f$ :

$$L^{\{f\}}(\mathcal{M}) = \{w \mid w \in L^{f(\ell(w))}(\mathcal{M})\}.$$

### Theorem (Polynomially robust to time $\Leftrightarrow$ PTIME)

- A language  $L \in \text{PTIME}$  iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  $L = L(\mathcal{M}) = L^{\{p\}}(\mathcal{M})$ .
- Any PTIME language is reducible to PAM's reachability:  
 $R^{\mathcal{H}} = R^{\mathcal{H},(p)}$  for some polynomial  $p$ .

## A Characterisation of PTIME

**Theorem (Polynomially length robust  $\Rightarrow$  PTIME)**

*Assume distance  $d$  is **time-metric** and  $R^{\mathcal{H}} = R^{\mathcal{H},(p)}$  for some polynomial  $p$ . Then  $\mathfrak{R}^{\mathcal{H}} \in$  PTIME.*

- Definition of another type of perturbation
- Characterisation of PTIME

## Partial Conclusion

---

## Partial Conclusion

- Characterisation of **FPSPACE** with the suitable notion of perturbation.
- Extension to **drawability**.
- Characterisation of **PTIME** with the suitable notion of perturbation.

# **Toward PSPACE-Completeness: ongoing work**

---

## Definitions (from computable analysis): intuition

- What is a **computable real number**  $x$ ?  
→ we can compute a representation of  $x$ .

### Example

$e, \pi$  are computable.  $\sum_{i \geq 1} 2^{-BB(i)}$ , where  $BB$  is the Busy Beavers function is not.

- What is a **computable function**  $f : \mathbb{R} \rightarrow \mathbb{R}$ ?  
→ on a representation of  $x \in \mathbb{R}$ , we produce a representation of  $f(x)$ .

- $f : \Sigma^* \rightarrow \Sigma^*$  **regular**  $\Leftrightarrow$  for all  $u, v \in \text{Dom}(f)$  such that  $|u| \leq |v|$ , then  $|f(u)| \leq |f(v)|$ .
- **Reg** : set of regular functions
- **Pred** : set of  $\{0, 1\}$ -valued regular functions

- $f : \Sigma^* \rightarrow \Sigma^*$  **regular**  $\Leftrightarrow$  for all  $u, v \in \text{Dom}(f)$  such that  $|u| \leq |v|$ , then  $|f(u)| \leq |f(v)|$ .
- **Reg** : set of regular functions
- **Pred** : set of  $\{0, 1\}$ -valued regular functions

$\rightarrow$  regular functions give a **proper measure of length**, especially for oracles

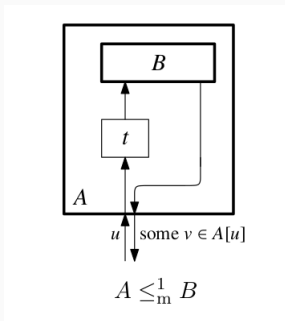
# Type-2 Reduction for Decision Problems

## Definition (Kawamura-Cook)

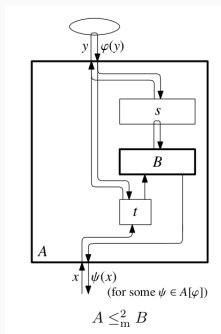
Let  $A$  and  $B$  be decision problems, or multi-functions from **Reg** to **Pred**. We write  $A \leq_m^2 B$  if there exists  $s, t \in \text{FPTIME}^2$  such that for any  $\phi \in \text{Dom}(A)$ , we have  $s(\phi) \in \text{Dom}(B)$  and for  $\theta = B(s(\phi))$  the function  $\theta \circ t(\phi) = A(\phi)$ .

$$\begin{array}{ccc} \phi & \xrightarrow{A} & A(\phi) \\ \downarrow t & & \parallel \\ t(\phi) & \xrightarrow{B \circ s \circ \phi} & \theta \circ t(\phi) \end{array} \quad \text{or} \quad \begin{array}{ccc} \phi & \xrightarrow{A} & A(\phi) \\ \downarrow t & \nearrow B \circ s \circ \phi & \\ t(\phi) & & \end{array}$$

# The original scheme (Kawamura-Cook)



**Figure 1:** Usual reduction



**Figure 2:** Type-2 reduction

## An example of complete problem

### **Basic PSPACE<sup>2</sup>:**

Input:  $\langle M, \bar{\mu}, \phi \rangle$ , with  $M$  a Type-2 TM,  $\bar{\mu} \in \mathbf{Reg}$  such that  $\bar{\mu}(u) = 0^{\mu(|u|)}$  where  $\mu : \mathbb{N} \rightarrow \mathbb{N}$  a non-decreasing polynomial bounding the space of  $M$ ,  $\phi \in \mathbf{Reg}$  and a string  $u$ .

Output: Does  $M$ , with an oracle  $\phi$ , accept  $u$ ?

## The case of Discrete-Time Dynamical Systems

## A previous result on FPTIME

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, linear length ODE, effective limit]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

## Discrete Reach Robust Lipschitz (DRRL):

Input: A discrete-time polynomially robust dynamical system

$\mathcal{H} = (X, g)$ ,  $X$  a closed rational box,  $(g : X \rightarrow X) \in \mathbb{L}\mathbb{D}\mathbb{L}^\circ + \Phi$

Lipschitz,  $p \in \mathbb{N}$ ,  $\mathbf{x}, \mathbf{y} \in X$

Output:  $R_{2^{-p}}^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$

with  $\Phi : \gamma(w) \rightarrow \gamma(\phi(w))$ ,  $\gamma : \Sigma^* \rightarrow \mathbb{R}$ ,  $w \in \Sigma^*$ ,  $\phi \in \mathbf{Reg}$

## Discrete Reach Robust Lipschitz (DRRL):

Input: A discrete-time polynomially robust dynamical system

$\mathcal{H} = (X, g)$ ,  $X$  a closed rational box,  $(g : X \rightarrow X) \in \mathbb{L}\mathbb{D}\mathbb{L}^\circ + \Phi$

Lipschitz,  $p \in \mathbb{N}$ ,  $\mathbf{x}, \mathbf{y} \in X$

Output:  $R_{2-p}^{\mathcal{H}}(\mathbf{x}, \mathbf{y})$

with  $\Phi : \gamma(w) \rightarrow \gamma(\phi(w))$ ,  $\gamma : \Sigma^* \rightarrow \mathbb{R}$ ,  $w \in \Sigma^*$ ,  $\phi \in \mathbf{Reg}$

## Theorem

**DRRL** is PSPACE<sup>2</sup> –  $\leq_m^2$ -hard.

## Proof Ideas: Reduction

**Basic PSPACE<sup>2</sup>:**

Input:  $\langle M, \bar{\mu}, \phi \rangle$ , with  $M$  a Type-2 TM,  $\bar{\mu} \in \mathbf{Reg}$  such that  $\bar{\mu}(u) = 0^{\mu(|u|)}$  where  $\mu : \mathbb{N} \rightarrow \mathbb{N}$  a non-decreasing polynomial bounding the space of  $M$ ,  $\phi \in \mathbf{Reg}$  and a string  $u$ .

Output: Does  $M$ , with an oracle  $\phi$ , accept  $u$ ?

We prove that:  $\mathbf{Basic PSPACE}^2 \leq_m^2 \mathbf{DRRL}$

Transforming the execution of the TM into a dynamical system:

- **Domain** → encoding of the space of the possible configurations
- **Dynamic** → encoding of the execution of the TM into a discrete ODE

## Proof Ideas: Encoding

Transforming the execution of the TM into a dynamical system:

- **Domain** → encoding of the space of the possible configurations
- **Dynamic** → encoding of the execution of the TM into a discrete ODE
  - $t \equiv$  encoding of the execution;
  - $s \equiv$  maps the real encoding the configuration  $C_i$  to the real encoding  $C_{i+1}$  (related to the oracle);

## Proof Ideas: Encoding

Transforming the execution of the TM into a dynamical system:

- **Domain** → encoding of the space of the possible configurations
- **Dynamic** → encoding of the execution of the TM into a discrete ODE
  - $t \equiv$  encoding of the execution;
  - $s \equiv$  maps the real encoding the configuration  $C_i$  to the real encoding  $C_{i+1}$  (related to the oracle);
  - Prove it is computable in polynomial time
  - Prove the underlying function is Lipschitz

## The case of Continuous-Time Dynamical Systems

## Continuous Reach Robust Lipschitz (CRRL):

Input: A continuous-time polynomially robust dynamical system

$\mathcal{H}_C = (X_C, g_C)$ , with  $X_C \subseteq \mathbb{R}$  a closed rational box,  $g_C : X_C \rightarrow X_C \in \text{LDL}^\circ$  Lipschitz,  $p \in \mathbb{N}$ ,  $\mathbf{x}, \mathbf{y} \in X_C$

Output:  $\mathfrak{R}_{2^{-p}}^{\mathcal{H}_C}(\mathbf{x}, \mathbf{y})$

## Theorem

**CRRL is  $\text{PSPACE}^2_{\leq m}$ -hard.**

# Proof Ideas: Reduction

We prove that:  $\text{DRRL} \leq_m^2 \text{CRRL}$

# Proof Ideas: Reduction

We prove that:  $\text{DRRL} \leq_m^2 \text{CRRL}$

- We use the Branicky trick:
  - $t \equiv$  encoding of the Branicky trick in our context;
  - $s \equiv$  maps a real in the solution of the discrete ODE to the next point
  - Prove it is computable in polynomial time

## Partial Conclusion

---

## Partial Conclusion

- **DRRL** is  $\text{PSPACE}^{2-\leq_m^2}$ -complete.
- **CRRL** is  $\text{PSPACE}^{2-\leq_m^2}$ -complete.

## Conclusion

---

# Conclusion

- We studied what makes **reachability decidable for robust dynamical systems** and prove that space complexity is linked to the precision;

# Conclusion

- We studied what makes **reachability decidable for robust dynamical systems** and prove that space complexity is linked to the precision;
- Under additional hypotheses, we prove the **PSPACE-hardness of Reach**.

