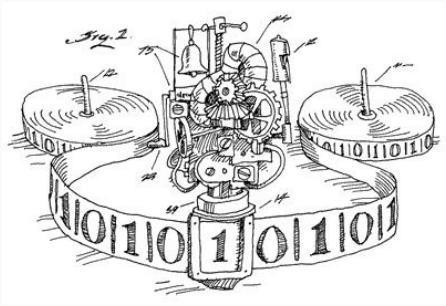
Discrete-Time and Continuous-Time Systems over the Reals: Relating Complexity with Robustness, Length and Precision

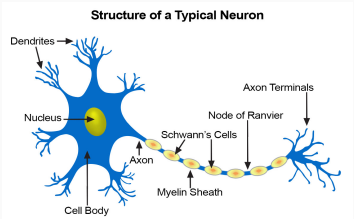
PhD Defence **Manon Blanc**

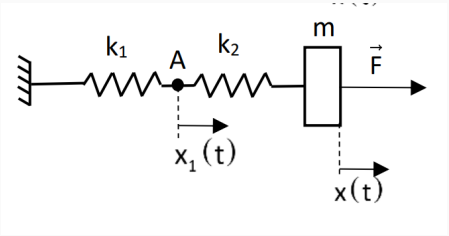
Jury: **Enrico Formenti**, **Juha Kontinen**, **Elvira Mayordomo**, **Cristóbal Rojas**, **Pierre Valarcher**

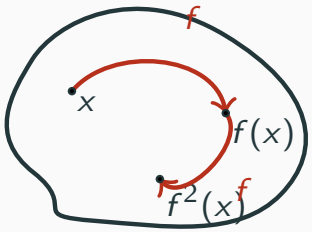
Introduction

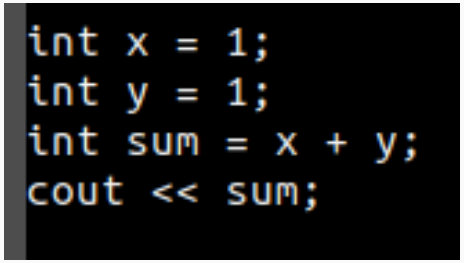
Introduction: what are we studying?

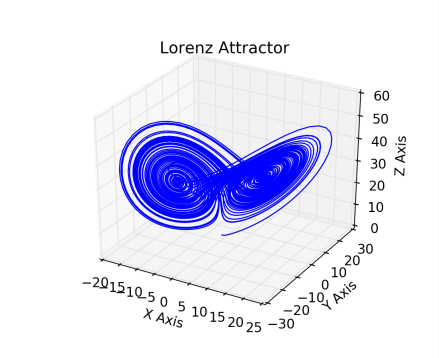


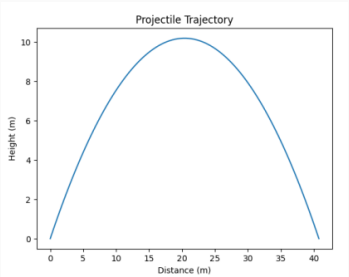












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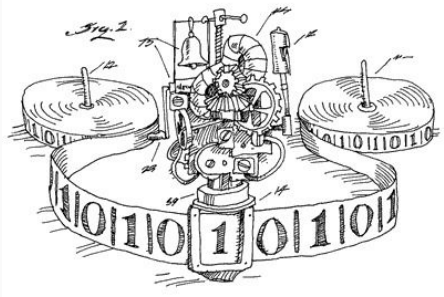
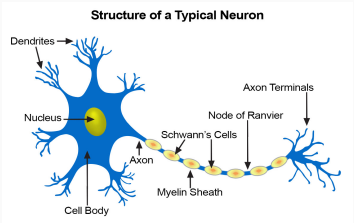
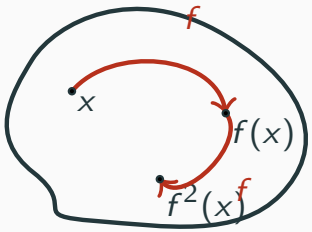
Discrete-time

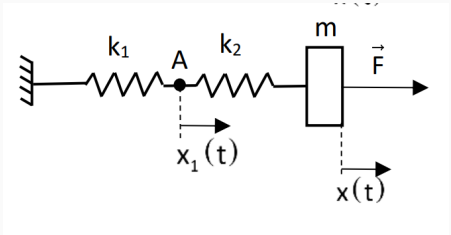


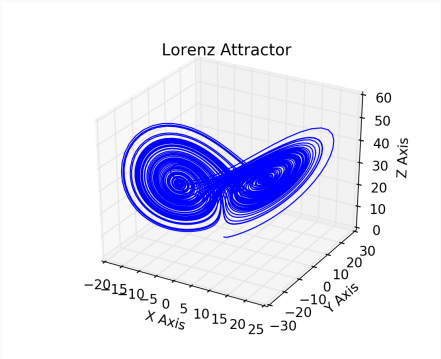


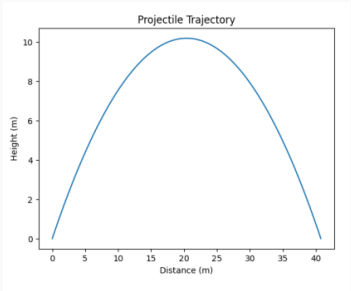



```
int x = 1;
int y = 1;
int sum = x + y;
cout << sum;
```

Continuous-time







Introduction

- Continuous-Time Systems: can be described by a (continuous) ODE

$$\frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x})$$

- Discrete-Time Systems: can be described by a discrete ODE

$$\frac{\delta \mathbf{f}(n, \mathbf{y})}{\delta n} = \mathbf{f}(n+1, \mathbf{y}) - \mathbf{f}(n, \mathbf{y}) = \mathbf{u}(\mathbf{f}(n, \mathbf{y}), n, \mathbf{y})$$

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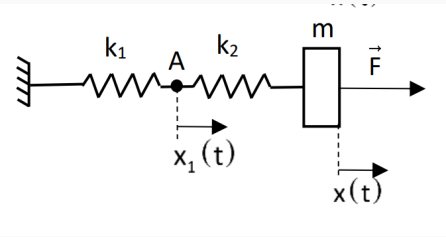
- Continuous-Time Systems: can be described by a (continuous) ODE



$$\frac{\partial^2 x}{\partial^2 t}(t) + \frac{k}{m}x(t) = 0$$

- Discrete-Time Systems: can be described by a discrete ODE

$$f : n \to 2^n$$

$$\frac{\delta f(n, \mathbf{y})}{\delta n} = f^2(n) - f(n)$$

Content

- Characterisations of polynomial-time and polynomial-space;
- Complexity of *robust* dynamical systems;
- Robustness of tilings.

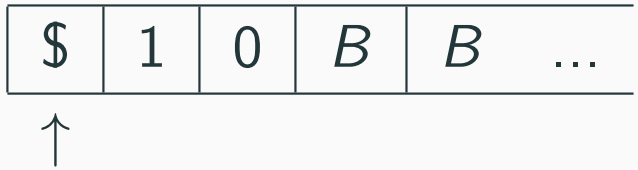
Motivations

Motivations

- Verification: can a system reach an unsafe state from an initial state (reachability)
- Models of computation: what is the computational power of a given model
- Ressources: how to measure time, memory (space) used by a computation

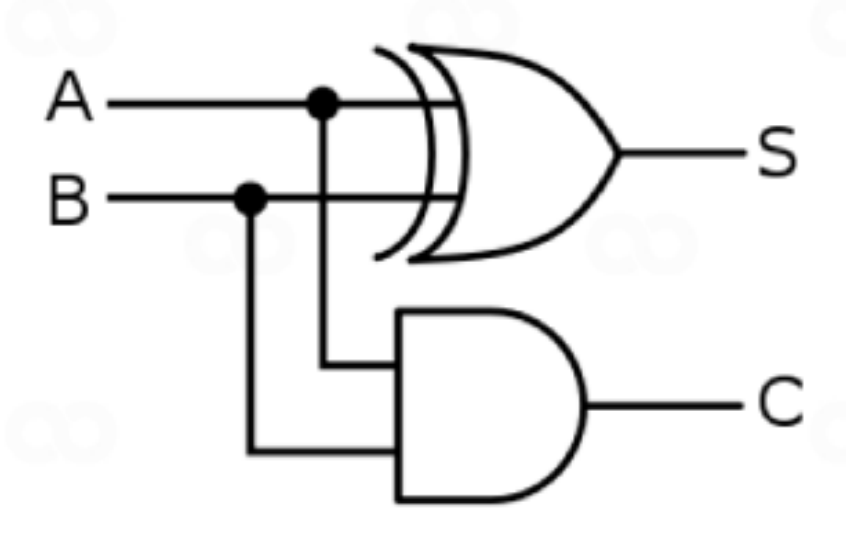
Discrete models of computation: Turing machines

The model:



Physical World	Mathematical Model
Computer	Turing machines, boolean circuits...

Example: the addition



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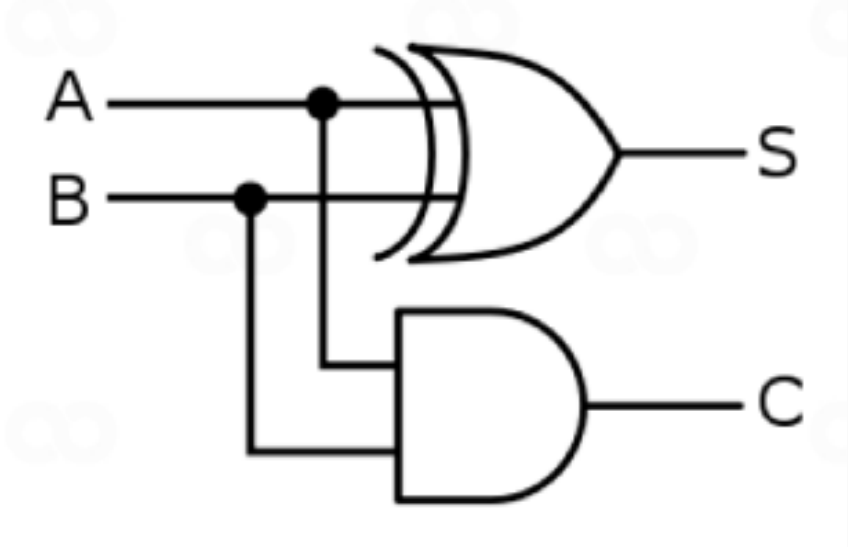
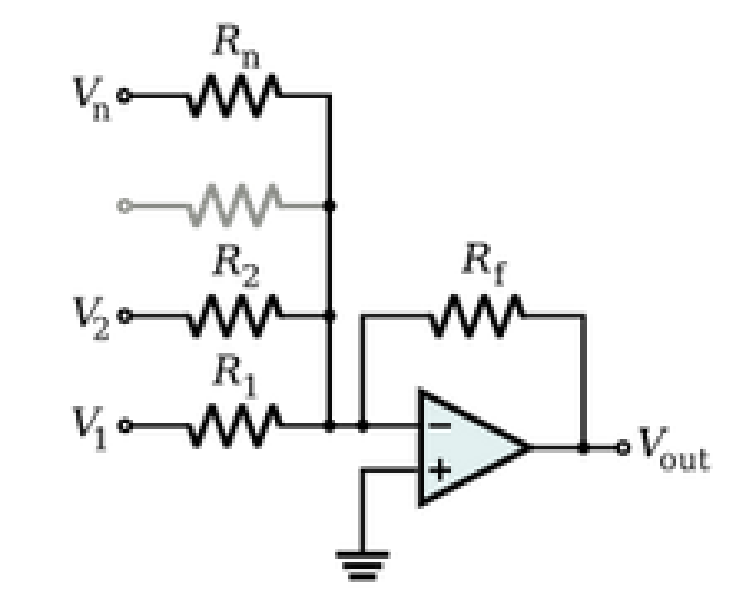
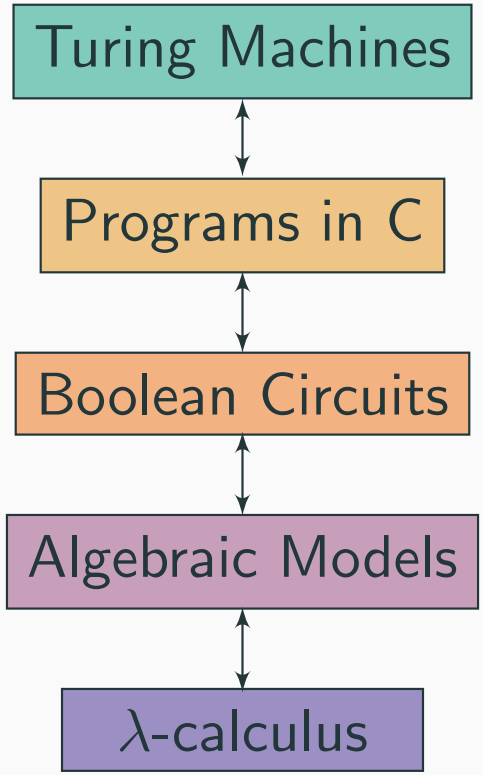




Computationnal Power: model equivalence



Why algebraic characterisation?

$$[f_1, f_2, \cdots, f_k; o_1, o_2, \cdots, o_n]$$

- Allows a more denotational point of view on complexity classes

Why algebraic characterisation?

$$[f_1, f_2, \cdots, f_k; o_1, o_2, \cdots, o_n]$$

- Allows a more denotational point of view on complexity classes
- This is already done in computability:

$$[0, \mathbf{s}, \pi; \textit{composition}, \textit{minimisation}, \textit{primitive recursion}]$$

On our example

Effective Church-Turing thesis: Every effectively and *reasonable* models of computation yields to the same complexity classes

$\rightarrow$ Notions of time and space do not depend on the model, up to some polynomial

$\rightarrow$ Verification is hard

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Effective Church-Turing thesis: Every effectively and *reasonable* models of computation yields to the same complexity classes

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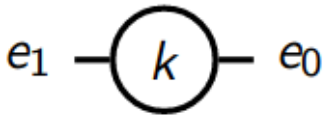
What about continuous-time models?

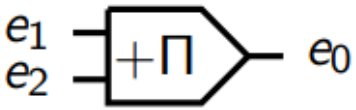
The General Purpose Analog Computer (GPAC)

- Introduced by Shannon (1941);
- Mathematical model of the *Differential Analizer*;
- Model based on circuits, with several interconnected units doing basic operations
- Corresponds to polynomial ODEs: $y' = p(y)$

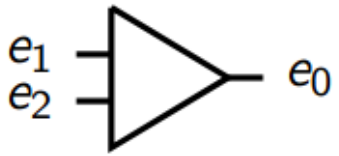
GPAC



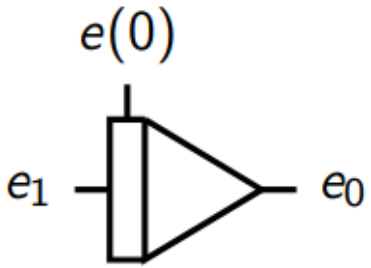
constant: $e_0 = ke_1$



product: $e_0 = e_1 e_2$



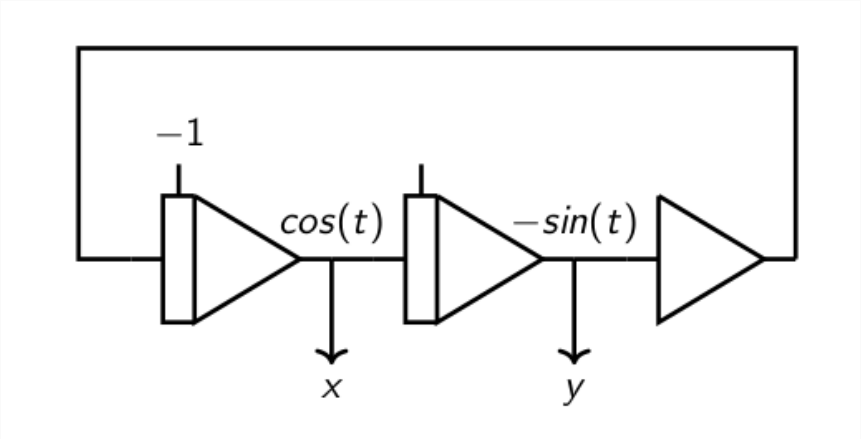
summer: $e_0 = -(e_1 + e_2)$



integrator:

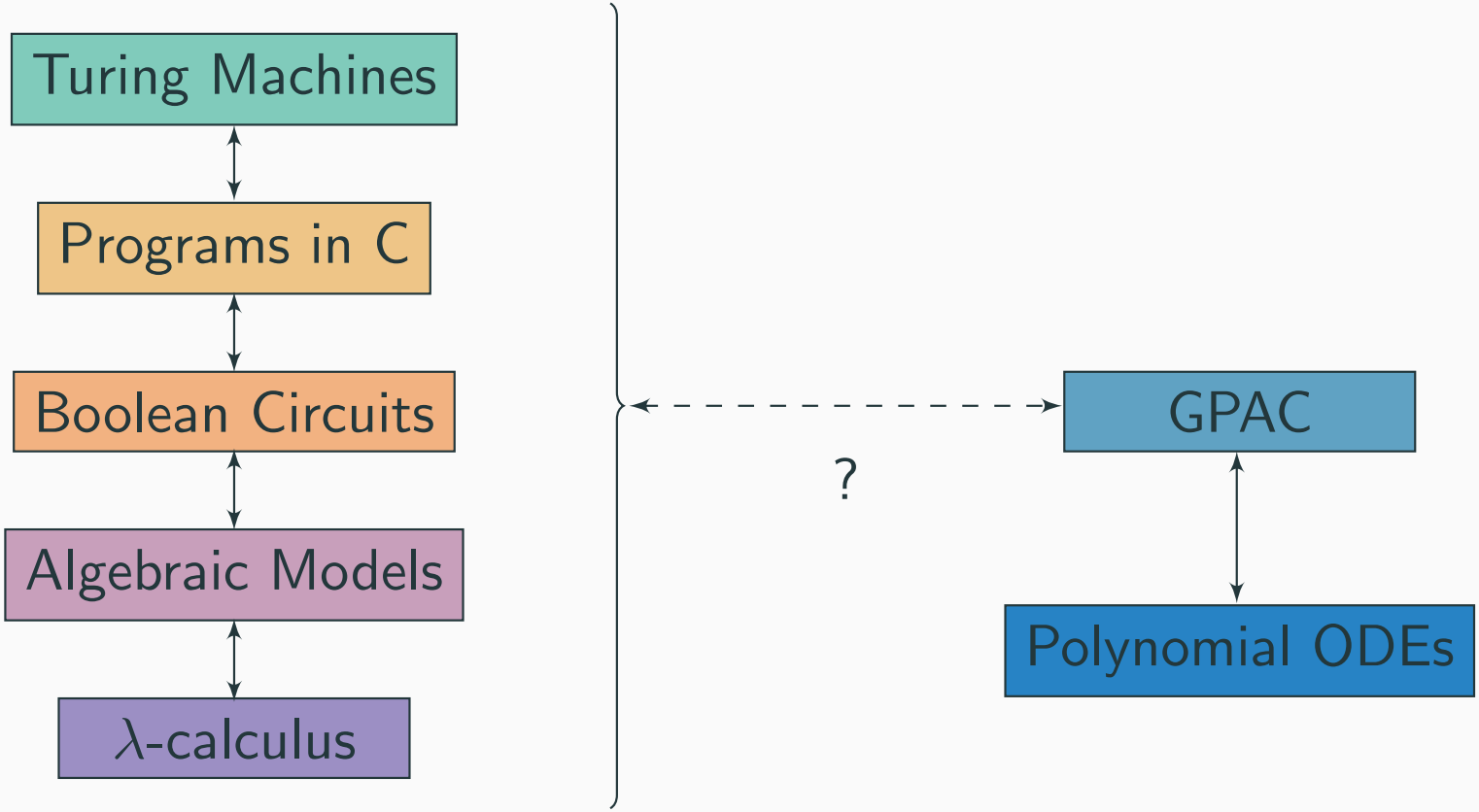
$$e_0 = -\int_0^t (e_1(u)du + e(0))$$

GPAC: how does it work?



$$\begin{cases} x'(t) &= y(t) \\ y'(t) &= -x(t) \\ x(0) &= -1 \\ y(0) &= 0 \end{cases} \Rightarrow \begin{cases} x(t) &= \cos(t) \\ y(t) &= -\sin(t) \end{cases}$$

Computability: GPAC



Is verification simpler for continuous-time systems?

- Difficulty: simulating Turing machines with continuous settings (e.g. Graça, Rojas) with
  - generalised shifts (Moore)
  - recurrent neural networks (Siegelmann, Sontag)
  - analytic functions (Graca, Buescu, Campagnolo)
  - hydrostatic equations in Beltrami fields (Cardona, Miranda, Peralta-Salas)
  - 2d continuous PAMs (Koiran, Cosnard, Garzon)
  - ...

Solving continuous ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the general case, for $\mathbf{u}$ computable (Pour-El, Richards)

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- Computable for $\mathbf{u}$ computable and unique solution (Collins, Graça, Ruohonen)
- FPSPACE-completeness on a bounded domain for $\mathbf{u}$ computable in polynomial-time (Kawamura, Ko)

Complexity

But now, how do we define the complexity?

- Measuring the complexity so it corresponds to the digital world

Complexity

- Time?:

  $\rightarrow$ Time = Length

  - Continuous ODEs : Bournez, Graça, Pouly
  - Discrete ODEs : Bournez, Durand, Kontinen, Antonelli, ...

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- Space?:

  - Effective Church thesis for probabilistic systems: Rojas
  - A characterisation of PSPACE: Gozzi, Graça

  $\rightarrow$ **Our approach: Space = Precision**

Our frameworks

- Mathematical framework: Computable Analysis
- Complexity framework: Algebraic Models

Definitions (from computable analysis): intuition

- What is a computable real number x?

  $\rightarrow$ we can compute a representation of x.

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Example

e, π are computable. $\sum_{i \geq 1} 2^{-BB(i)}$, where BB is the Busy Beavers function is not.

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  $\rightarrow$ on a representation of $x \in \mathbb{R}$, we produce a representation of $f(x)$.
- How do we define complexity over the reals?

Algebraic Models: for complexity classes

- Characterisation of polynomial time with explicit bounds: e.g. Cobham

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Here: describing complexity without explicit bounds (e.g. Bellantoni, Cook, Levant, Marion, Jacobé de Naurois, ...):

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Here: describing complexity without explicit bounds (e.g. Bellantoni, Cook, Levant, Marion, Jacobé de Naurois, ...):

- With BSS model (Jacobé de Naurois, Marion, ...)
- Characterisation of FPTIME with linear length ODEs (Bournez, Durand, Kontinen, ...)

Time and Space as algebras

Algebraic characterisations: Chapters 3 and 5 in one slide

	With Discrete ODEs	With ODEs
FPTIME for $\mathbb{N}^{\mathbb{N}}$	(Bournez, Durand)	
FPTIME for $\mathbb{R}^{\mathbb{N}}$	$\overline{\mathbb{LDL}^{\bullet}}$ (thm. 3.1.6)	
FPTIME for $\mathbb{R}^{\mathbb{R}}$	$\overline{\mathbb{LDL}^{\circ}}$ (thm. 3.2.3)	(Bournez, Graça, Pouly)
FPSPACE for $\mathbb{R}^{\mathbb{R}}$	$\overline{\mathbb{RLD}^{\circ}}$ (thm. 5.1.6)	$\overline{\mathbb{RCD}}$ (thm. 5.2.2)

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- $\overline{\mathbb{LDL}^{\bullet}}$, $\overline{\mathbb{LDL}^{\circ}}$ based on Linear Length ODEs;
- $\overline{\mathbb{RLD}^{\circ}}$ based on Robust Linear Discrete ODEs;
- $\overline{\mathbb{RCD}}$ based on Robust Continuous ODEs.

Robust Continuous ODEs

Definition (5.2.3)
$\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is robustly ODE definable (from initial condition $\mathbf{g}$, and dynamic $\mathbf{u}$) if

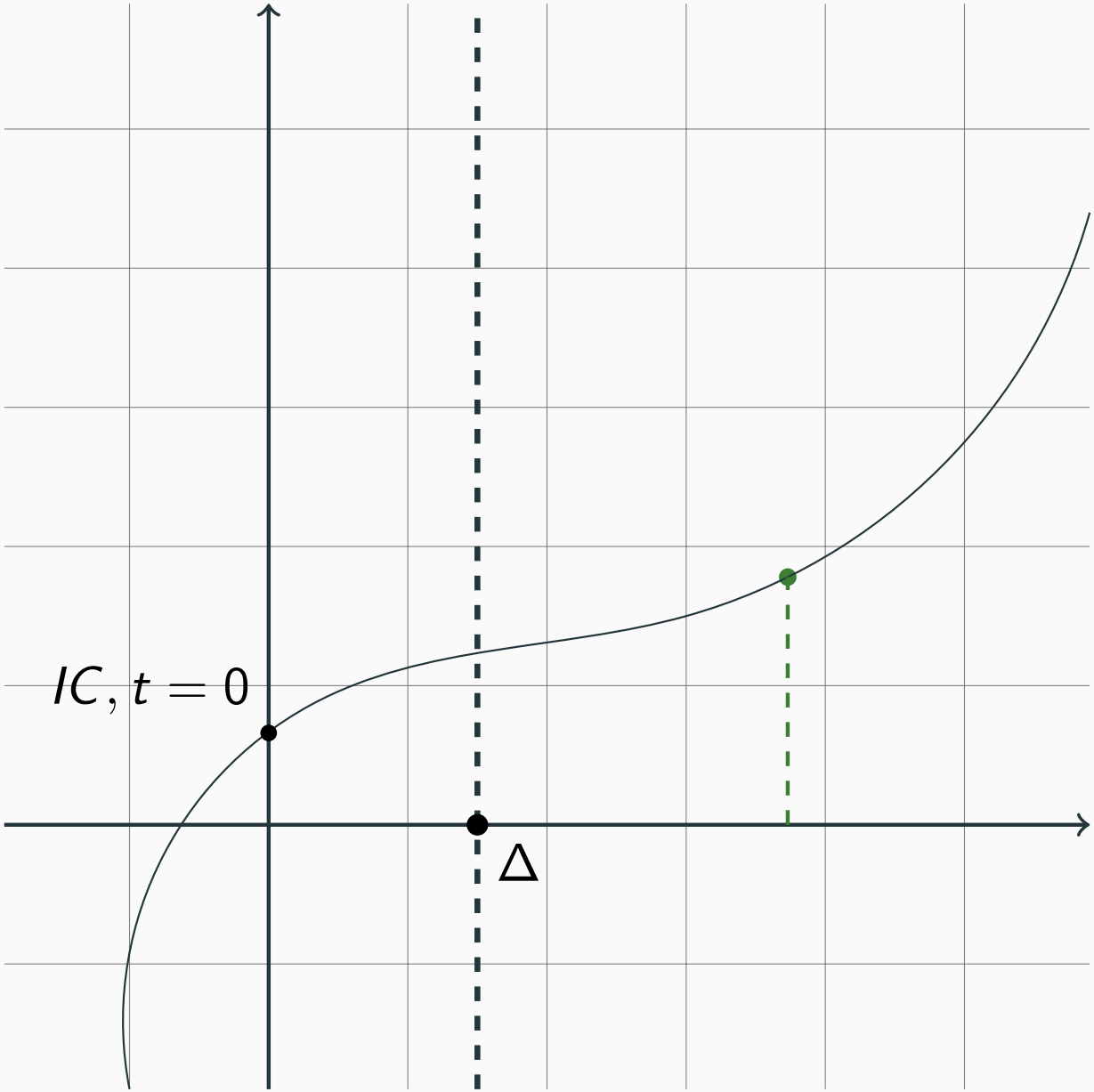
- it is the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x});$$

- And polynomially numerically stable (to be defined : next slides)

$\to$ this allows computation by dichotomy.

An unusual way of solving an ODE



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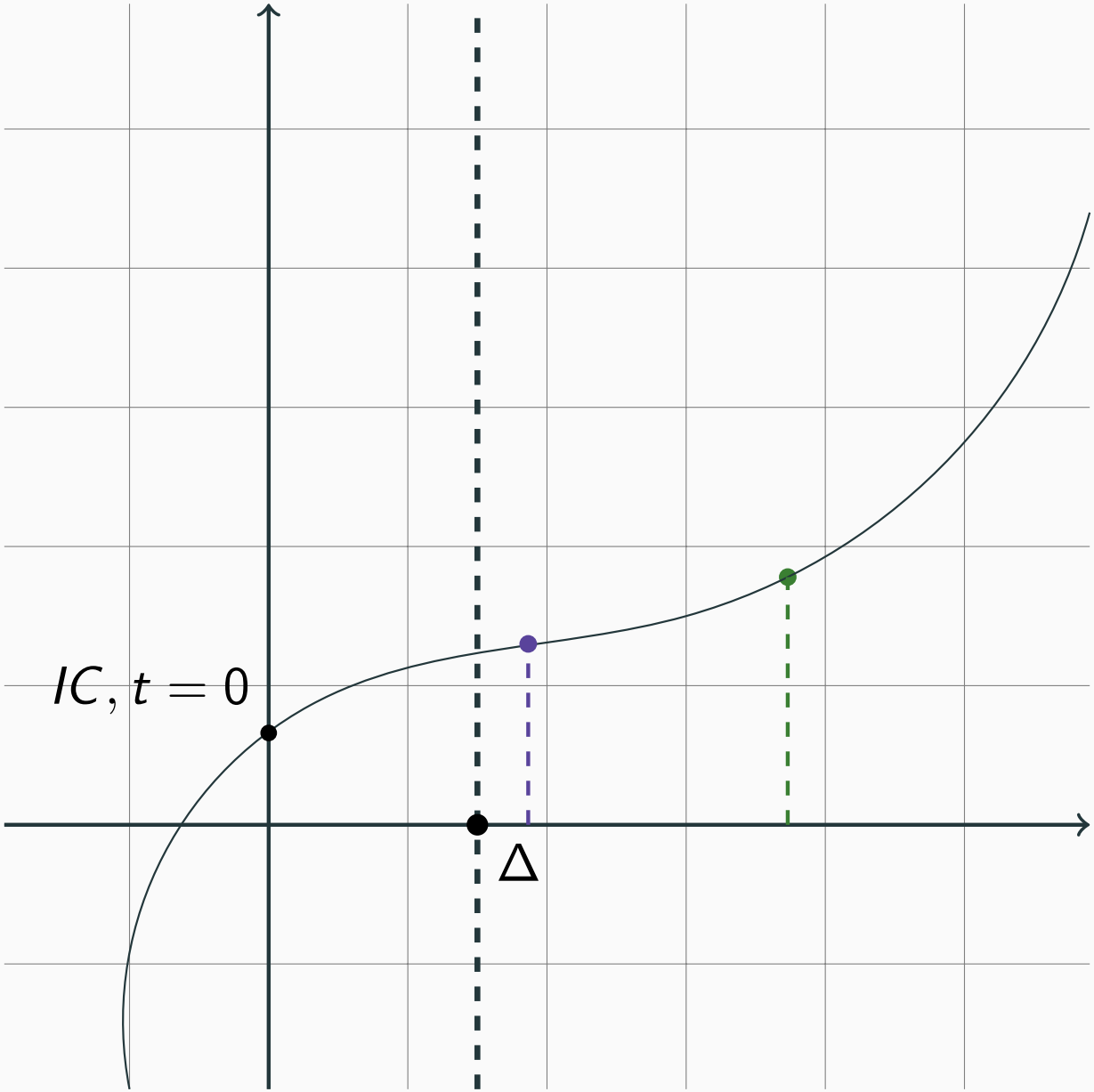

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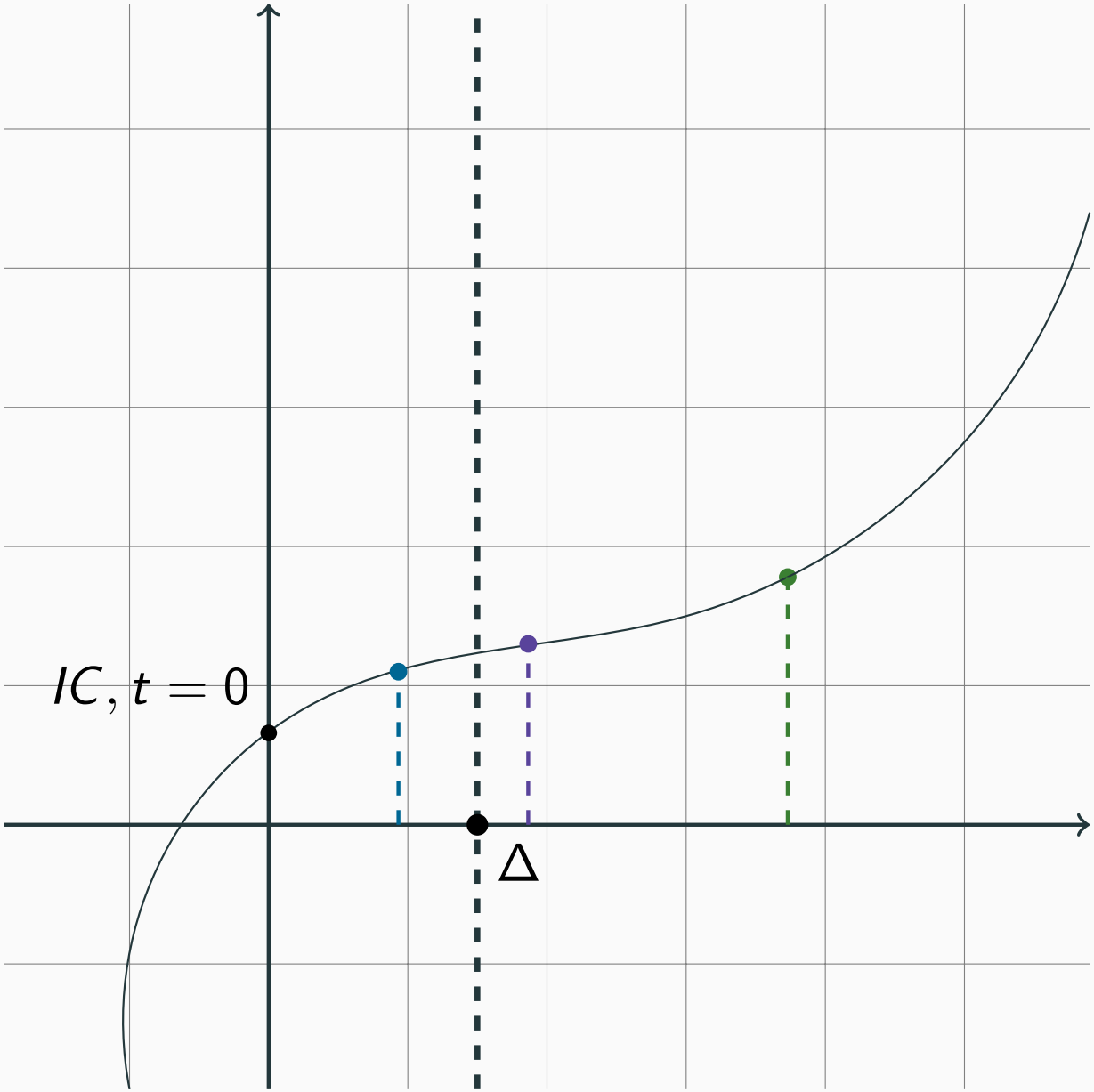

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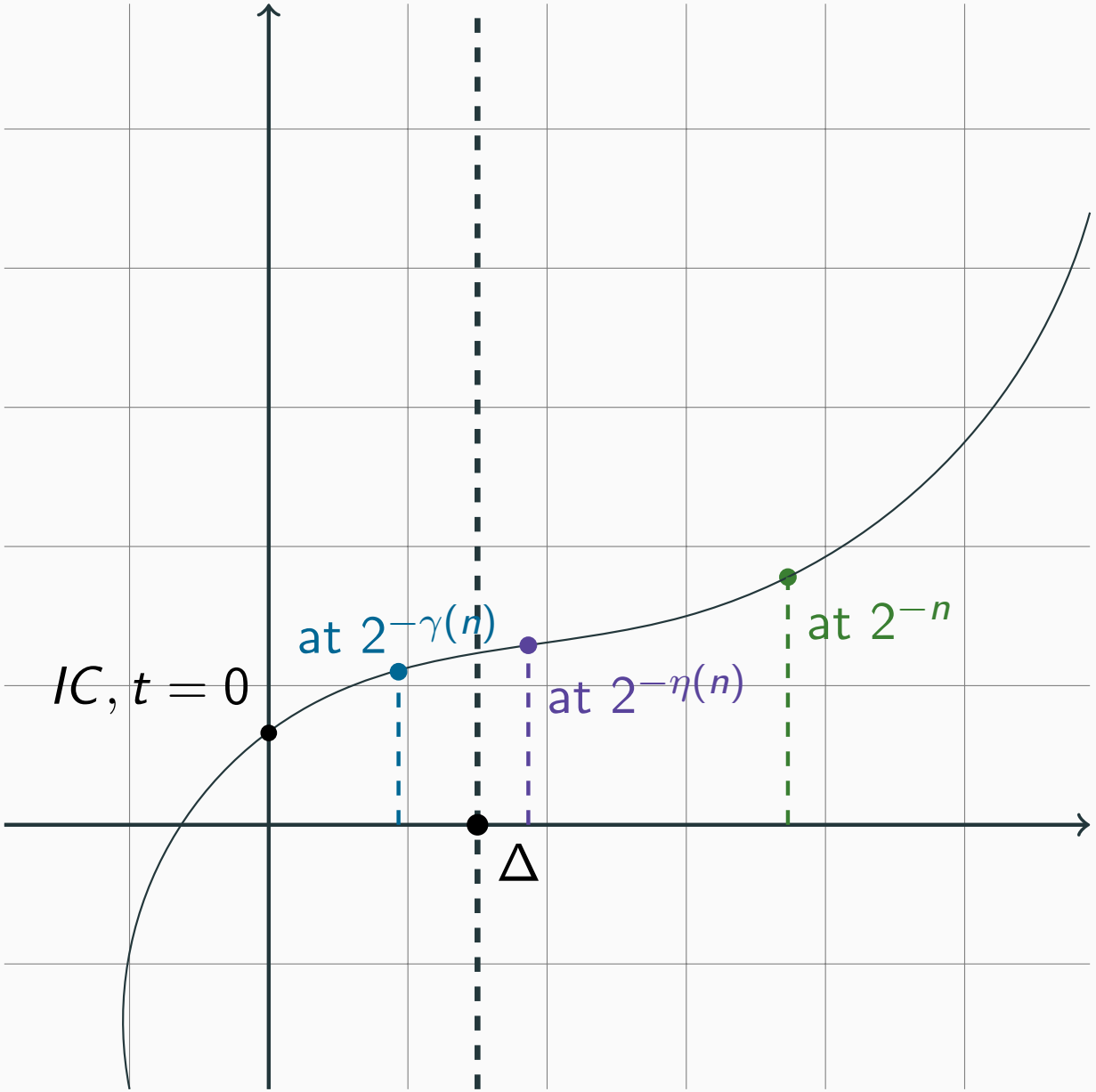

Robust Continuous ODEs (2/2)

More formally:

- $\exists \Delta \in \mathbb{Q}_+^*$, such that the previous ODE is (polynomially space) solvable on $[0, \Delta]$.

- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ at 2^{-n} by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $2^{-\eta(n)}$.

Characterisation of PSPACE with continuous ODEs

We consider :

$$\overline{\mathbb{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{robust continuous ODE}, \textit{ELim}]$$

Theorem (5.2.2)

$$\overline{\mathbb{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Proof ideas ($\subseteq$)

$$\overline{\mathbb{RLD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \overline{\mathbb{RCD}} \cap \mathbb{R}^{\mathbb{R}}$$

- **Continuous $\Rightarrow$ Discrete settings:**
  From our definitions of robustness of ODEs

Proof ideas ($\subseteq$)

All the functions and operators in $\overline{\mathbb{RCD}}$ are computable in polynomial-space.

Proof ideas ($\supseteq$): with discrete ODEs

- We give *continuous* approximations of some basic functions (integer/fractional part, modulo 2, ...):

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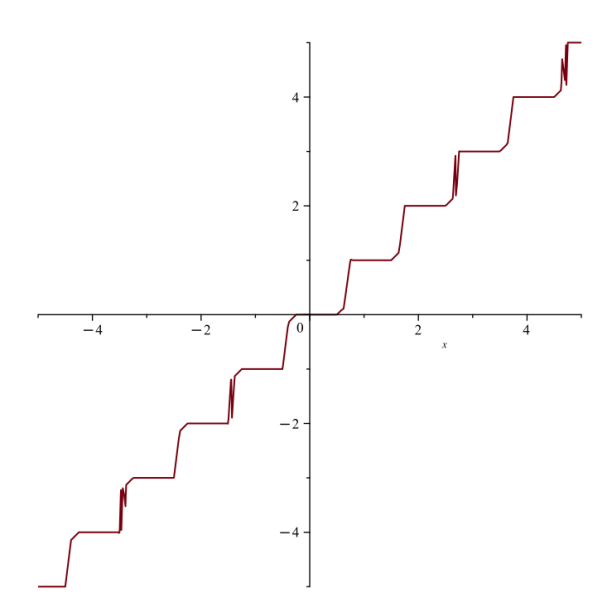

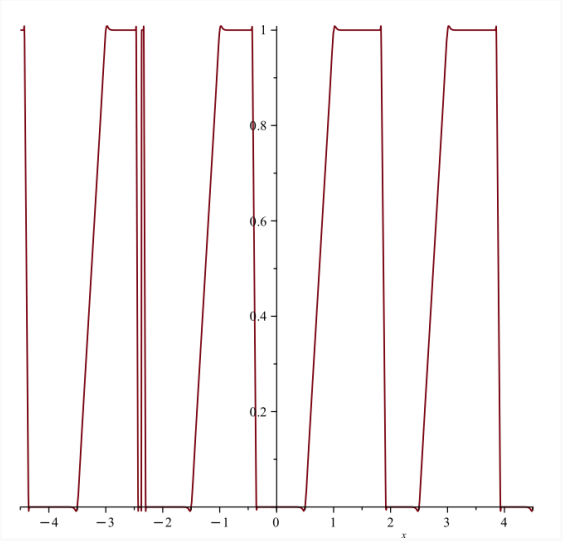

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Proof ideas ($\supseteq$): with discrete ODEs

- We give *continuous* approximations of some basic functions (integer/fractional part, modulo 2, ...);
- We encode each step of the transition of the Turing machine into the algebra, using the previous functions : Next;
- We obtain a *robust linear discrete ODE* encoding the execution of the Turing machine:
  $State(t+1) = Next(State(t))$ so

$$\frac{\delta \overline{State}(t,y)}{\delta t} = \overline{Next}(\overline{State}(t,y))$$

.

Proof ideas ($\supseteq$): going to continuous world

$$\overline{\mathbb{R}\mathrm{LD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \overline{\mathbb{R}\mathrm{CD}} \cap \mathbb{R}^{\mathbb{R}}$$

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  Adaptation of the a trick due to Branicky ('95)

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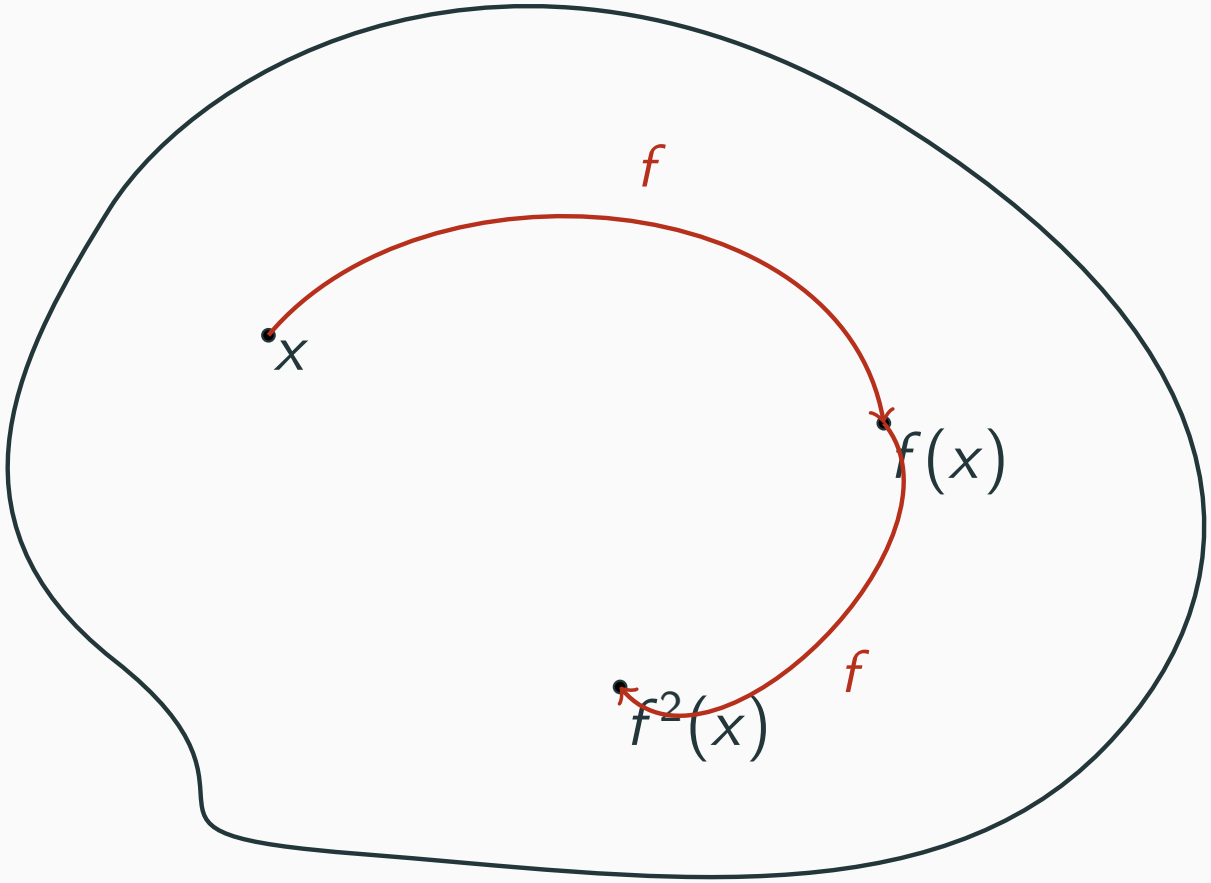
- **Discrete $\Rightarrow$ Continuous settings:**
  Adaptation of the a trick due to Branicky ('95)

$$\frac{\delta f}{\delta t}(t) = u(y(t), t) \qquad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0 \\ \begin{cases} \frac{\partial z_2}{\partial t}(t) = c_2 \left(r(z_1(t)) - z_2(t)\right)^3 \theta\left(-\cos 2\pi t\right) \\ \frac{\partial z_1}{\partial t}(t) = c_1 \left(\bar{u}\left(r(z_2(t))\right) - z_1(t)\right)^3 \theta\left(\cos 2\pi t\right) \end{cases} \end{cases}$$

Robustness in Dynamical Systems

Dynamical Systems



Computability Properties

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- It becomes decidable for a sub-class of systems: Robust Dynamical Systems.

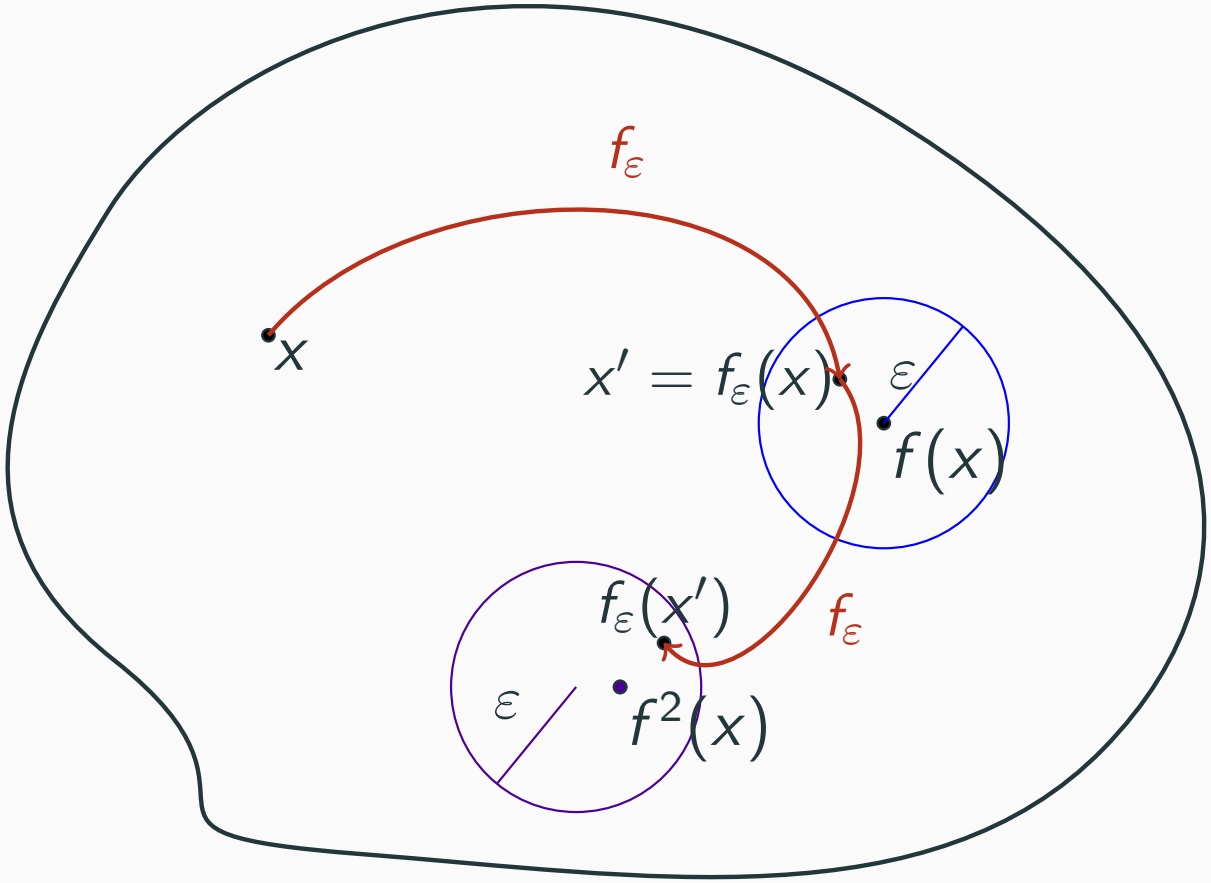
Computability

- Informal conjecture: undecidability in verification does not happen for robust systems.

- Undecidabilty is related to non-robustness of the systems (Asarin, Bouajjani)

  $\rightarrow$ with proper notion of robusteness: no sensitivity to an arbitrarly small perturbation.

Robustness in Dynamical Systems



Complexity for Reach

- go from computability to complexity,
- quantifying the robustness to characterise PSPACE,
- having a notion of robustness to characterise PTIME.

Robustness

$$\text{Consider } R_\omega^{\mathcal{H}} = \bigcap_{\epsilon>0} R_\epsilon^{\mathcal{H}}(x, y)$$

- Say $R^{\mathcal{H}}$ is robust when $R_\omega^{\mathcal{H}} = R^{\mathcal{H}}$:

$$R^{\mathcal{H}} \text{ robust} \Rightarrow R^{\mathcal{H}} \text{ computable}.$$

- Reachability in robust dynamical systems is *decidable* (Asarin & Bouajjani)

PSPACE **on robust dynamical systems**

Theorem (4.2.33)

Take a locally Lipschitz system, with $f : X \to X$ *polynomial-time computable, with* X *a closed rational box. Then, for* $p : \mathbb{N} \to \mathbb{N}$ *a polynomial,* $R_p^{\mathcal{H}} \subseteq \mathbb{Q}^d \times \mathbb{Q}^d \times \mathbb{N} \in$ PSPACE.

$\to$ works also for dynamical systems over the reals

Example
Simulation of Turing machines with continuous PAMs.

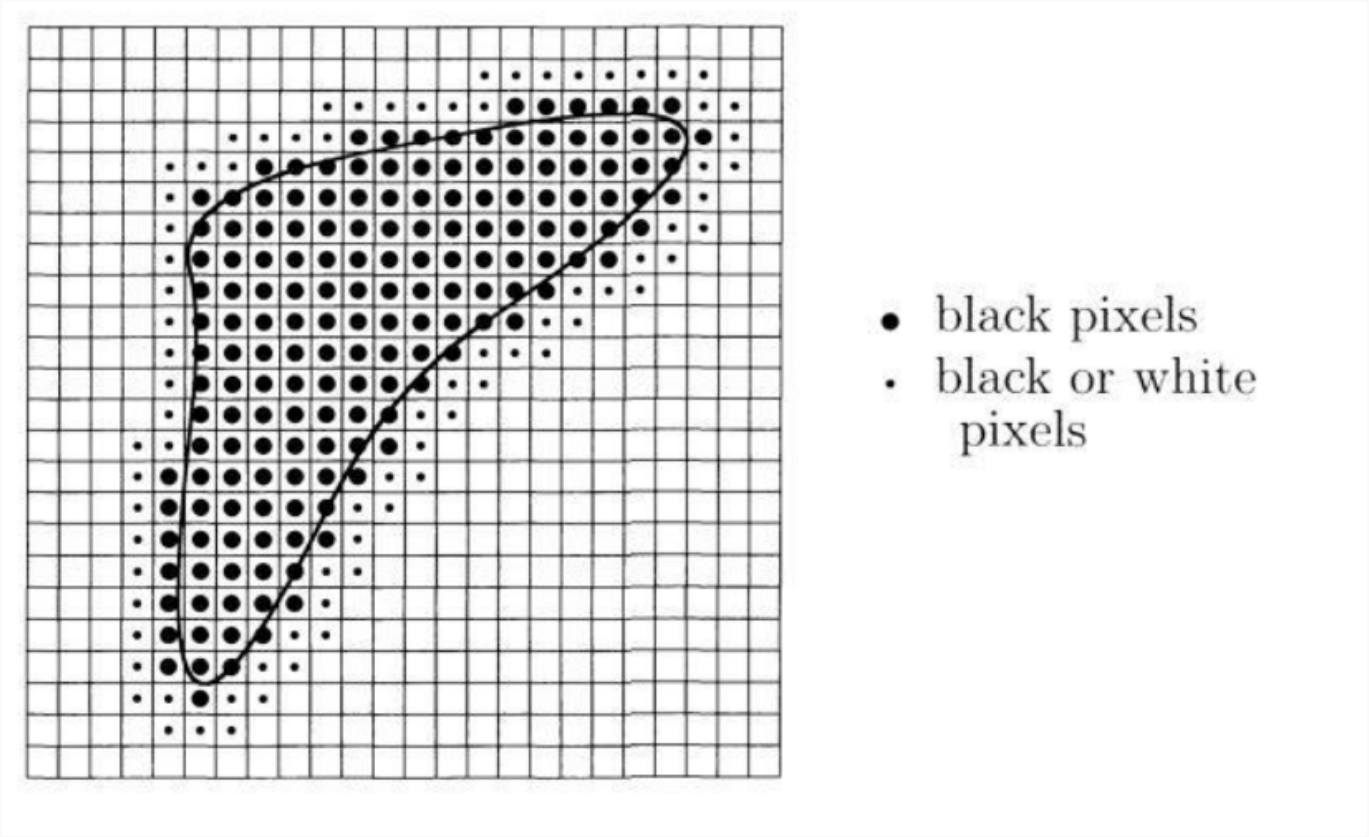
And With Reachability Properties

Theorem (4.3.2)

Consider a dynamical system, with $\mathbf{f}$ *locally Lipschitz, computable, whose domain is a computable compact, then, for all* $\mathbf{x}$, $cls(R_\omega^{\mathcal{H}}(\mathbf{x})) \subseteq \mathbb{R}^d$ *is a co-computably enumerable closed subset.*

Plot twist

From CA, Computable $\Leftrightarrow$ can be plotted.

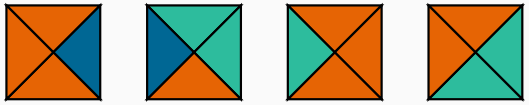


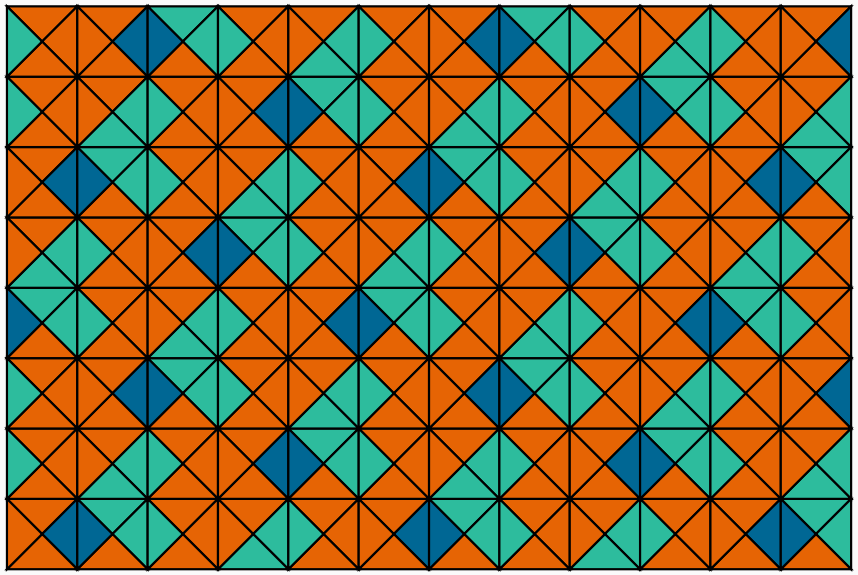
Theorem (4.3.5)

$R^{\mathcal{H}}$ closed and can be plotted in a name of $\mathbf{f}$ $\Leftrightarrow$ *the system is robust, i.e.* $R^{\mathcal{H}}_{\omega} = R^{\mathcal{H}}$.

Toward Complexity for tilings

Wang tilesets





Robinson

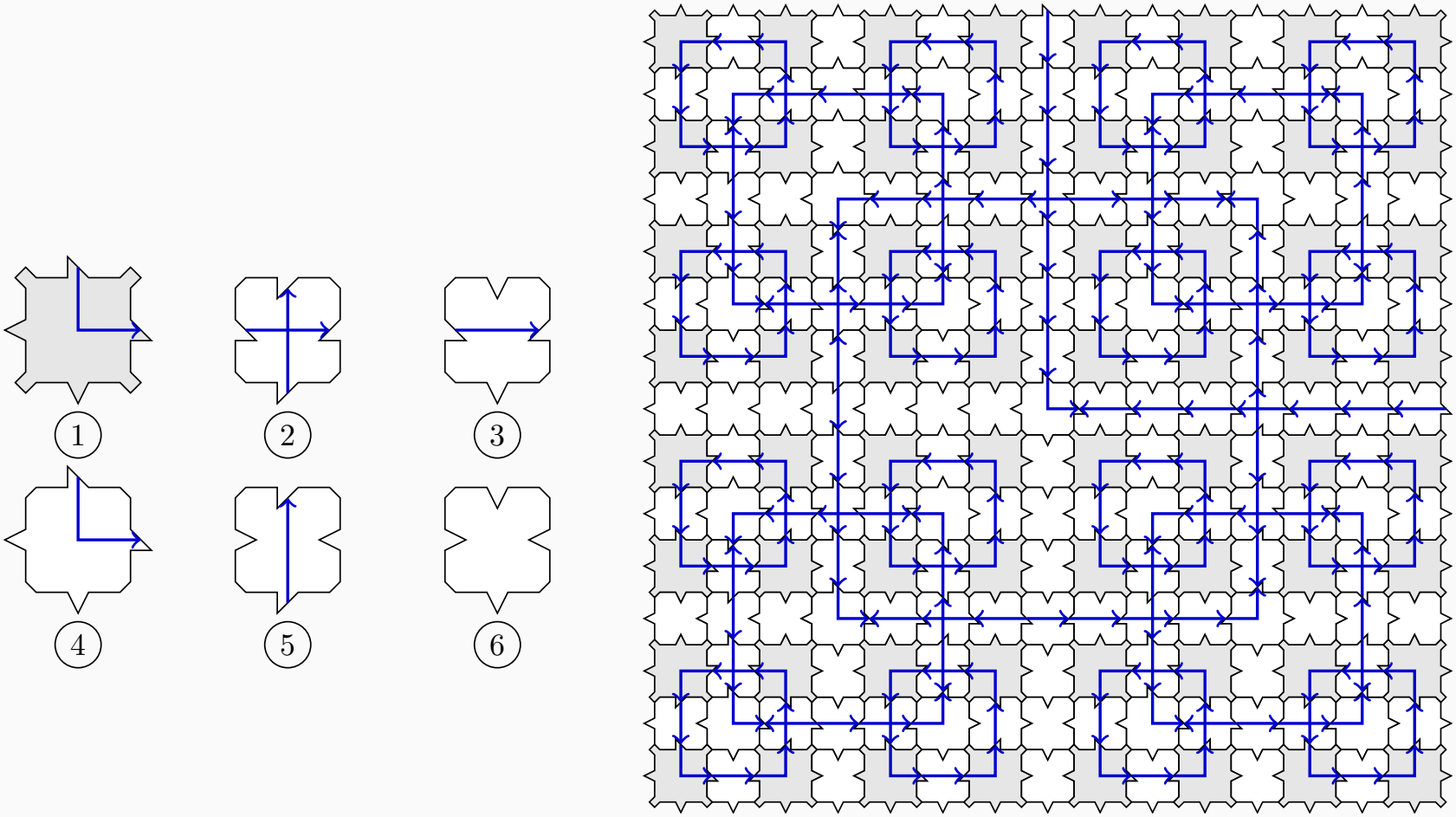


Figure 1: To the left is Robinson's tileset, where tiles can be rotated and reflected. To the right a pattern that appears in every tiling by Robinson's tileset.

Computability Properties

- The domino problem is undecidable...
- ... more precisely, it is not computably enumerable;
- It becomes decidable for a sub-class of tilesets: robust tilesets.

$\rightarrow$ Chapter 5

Conclusion

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Conclusion

- We extended the motto **time = length** to discrete ODEs (chapter 3);
- We proved that **space = precision** (chapter 5);
  $\rightarrow$ proper notions of costs for analogue models of computation;
- We studied what makes reachability decidable for robust dynamical systems and prove that space complexity is linked to the precision (chapter 4);
- We apply this principle to tilings (chapter 6).

Conclusion: Perspectives

- Minimising the algebras?
- Would it work for GPAC and polynomial ODEs?
- Can we relate this approch to second-order complexity?
- Can we relate more thinly our notions of robustness?

Conclusion: Perspective

- Studying non-determinism, and polynomial hierarchy;
- Algebraically characterising probabilistic classes;
- Studying the complexity for more general differential equations, e.g. distributions.

Appendix: Algebraic characterisation of PTIME for sequences

We consider:

$$\overline{\mathbb{LDL}^\bullet} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\mathrm{cond}}(x), \frac{x}{2};$$

composition, *linear length ODE*, *ELim*]

Theorem (B., Bournez, MCU22 Best Student Paper Award)

$$\mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\mathbb{LDL}^\bullet} \cap \mathbb{R}^{\mathbb{N}}$$

Appendix: FPTIME for computable functions over the reals

We consider:

$$\overline{\mathbb{LDL}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{linear length ODE}, \textit{ELim}]$$

Theorem (B., Bournez, MFCS23 Best Paper Award)

$$\overline{\mathbb{LDL}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Appendix: FPSPACE for computable functions over the reals

We consider:

$$\overline{\mathbb{RLD}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{robust linear ODE}, \textit{ELim}]$$

Theorem (B., Bournez, MFCS23 Best Paper Award)

$$\overline{\mathbb{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Appendix: An unusual way of solving an ODE

- Computation at $t = x$ at precision 2^{-n}, with initial condition $t = 0$
- Computation at $t = t_0 + \frac{x}{2}$ at precision $2^{-\eta(n)}$, with initial condition $t_0 = 0$ and $t_0 = \frac{x}{2}$
- Computation at $t = t_0 + \frac{x}{4}$ at precision $2^{-\gamma(n)}$, with initial condition $t_0 = 0$ and $t_0 = \frac{x}{4}$ and $t_0 = \frac{x}{2}$ and $t_0 = \frac{3x}{4}$