Computability and complexity over the reals

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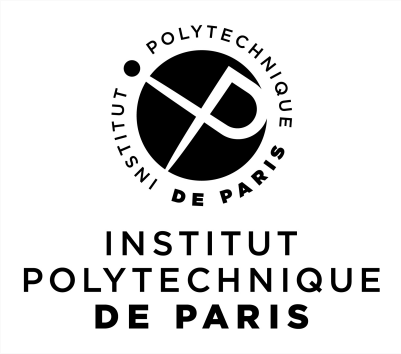

Introduction

We want to algebraically characterise PTIME and PSPACE...

Working on real numbers

Example: The set of computable functions

- constant function 0,
- the projection functions π_i^p,
- the function successor **s**,
- and that is closed under composition, minimisation and primitive recursion.

$$[O, \pi_i^p, \mathbf{s}; \textit{composition}, \textit{primitive recursion}, \textit{minimisation}]$$

Computability and complexity over the reals : how?

Computable real numbers

- What is a computable real number?

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- What is a computable function over the reals?
- How do we define complexity over the reals?

The begining of algebraic characterisation: a class you know well

Discrete derivation

$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

“Length ODE”?

Definition (Length ODE)

A function $\mathbf{f}$ is “length-ODE” definable (from u, g and h) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \textit{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \qquad (1)$$

Formal synonym for right-hand side of (1):
$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$

“Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x+1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

Motivation behind Length-ODEs

Fig. 5. A continuous system before and after an exponential speed-up.

θ solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \to \mathbb{R}$$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

Motivation behind Length-ODEs

Fig. 5. A continuous system before and after an exponential speed-up.

Re-scaling: $\phi_1(t) = \theta(e^t)$

Motivation behind Length-ODEs

Now "time-complexity" is measured by the length of the solution curve of the ODE.

Invariance by rescaling

Example of length-ODE

$$f(0) = 2$$
$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

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Unique solution : $f(x) = 2^{2^{\ell(x)}}$

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- $F(z+1) = 2^{2^z + 2^z} = F(z)F(z)$

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- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^z + 2^z} = F(z)F(z)$
- Then, $F(z+1) = F(z) + (z+1-z)(F(z)F(z) - F(z))$

Characterisation of PTIME (the one you are used to)

We consider

$$\mathbb{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \mathit{cond}(x);$$

$$\mathit{composition}, \mathit{linear\ length\ ODE}]$$

Theorem (Bournez & Durand, '19)

$$\mathbb{LDL} \cap \mathbb{N}^{\mathbb{N}} = \mathsf{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

First step: real sequences

Effective Limit

Definition
$f : \mathbb{R} \to \mathbb{R}$ is an effective limit of $\bar{f} : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ if

$$\left| f(x) - \bar{f}(x, 2^n) \right| \leq 2^{-n}$$

Key observation : $\bar{f}$ computable in polynomial time $\Rightarrow$ f computable in polynomial time.

Algebraic characterisation of PTIME

We consider:

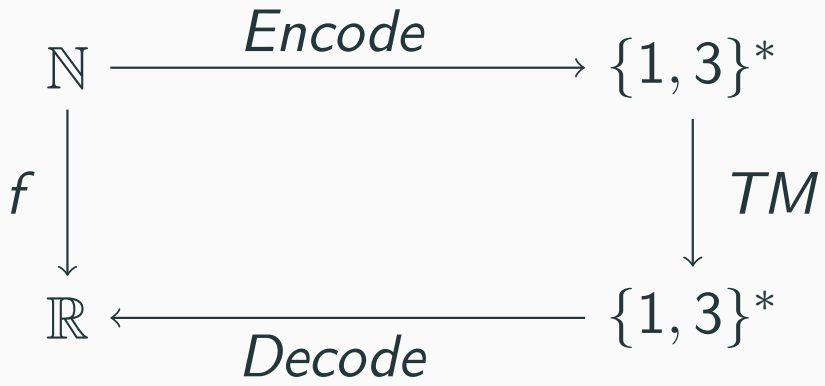
$$\overline{\mathbb{LDL}^\bullet} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\mathrm{cond}}(x), \frac{x}{2};$$

$$\textit{composition}, \textit{linear length ODE}, \textit{effective limit}]$$

Theorem (B., Bournez, MCU22)

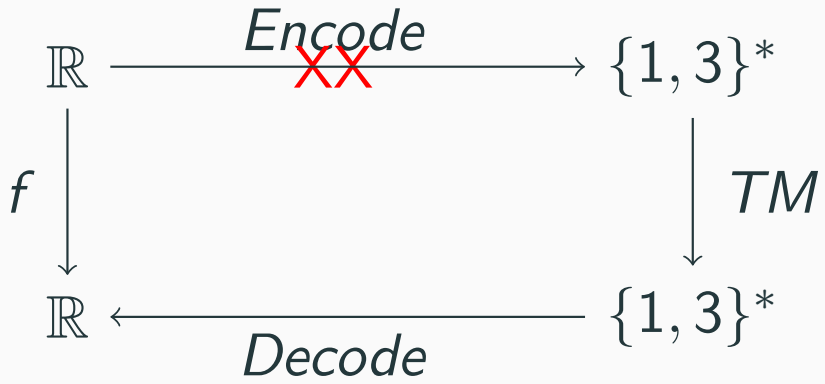
$$\mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\mathbb{LDL}^\bullet} \cap \mathbb{R}^{\mathbb{N}}$$

“Proof”



Second step: functions over the reals with discrete ODEs

What we cannot do



FPTIME for computable functions over the reals

We consider:

$$\overline{\mathbb{LDL}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \mathsf{tanh}, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{linear length ODE}, \textit{effective limit}]$$

Theorem (B., Bournez, MFCS23)

$$\overline{\mathbb{LDL}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Toward FPSPACE: Robust linear ODE

Definition (Robust linear ODE)

A bounded function $\mathbf{f}$ is a robustly linear ODE definable (with $\mathbf{u}$ *essentially linear* in $\mathbf{f}(x, \mathbf{y})$, $\mathbf{g}$ and $\mathbf{h}$) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \textit{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

FPSPACE for computable functions over the reals

We consider:

$$\overline{\mathbb{RLD}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \mathsf{tanh}, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{robust linear ODE}, \textit{effective limit}]$$

Theorem (B., Bournez, MFCS23)

$$\overline{\mathbb{RLD}^\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Third step: Functions over the reals with continuous ODEs

Robust Continuous ODEs

A function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is robustly ODE definable (from initial condition $\mathbf{g}$, and dynamic $\mathbf{u}$) if

- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), \mathbf{h}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

Robust Continuous ODEs

- $\exists \Delta \in \mathbb{Q}_+^*$, and some polynomial p such that the previous ODE is (polynomially) numerically stable on $[0, \Delta]$.

- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $\epsilon(n)$.

Characterisation of PSPACE with continuous ODEs

We consider :

$$\overline{\mathbb{RCD}} = [0, 1, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{robust ODE}, \textit{ELim}]$$

Theorem (B., Bournez, submitted)

$$\overline{\mathbb{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \mathsf{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Conclusion

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- We now have "natural" algebraic characterisations for functions over the reals computable in FPTIME and FPSPACE.

- Further works:

    - Cover other complexity classes
    - See what we can say for more general functions