

# Measuring the robustness of dynamical systems

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ÉCOLE  
POLYTECHNIQUE



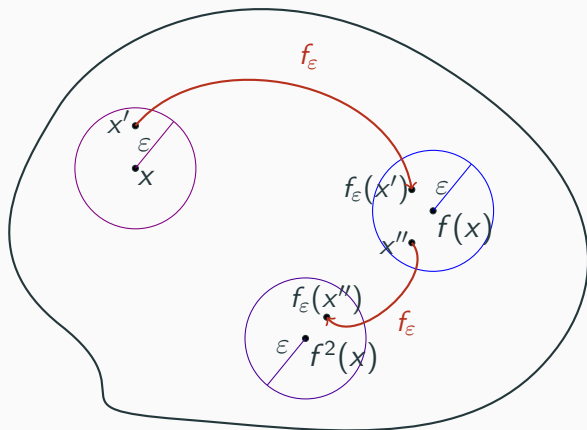
# Introduction

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- Motivated by the field of verification.
- Informal conjecture: undecidability in verification does not happen for robust systems.

- Approach: Asarin and Bouajjani (LICS '01).
- Our goal:
  - go from computability to complexity,
  - quantifying the robustness to characterise FPSPACE.

# Introduction



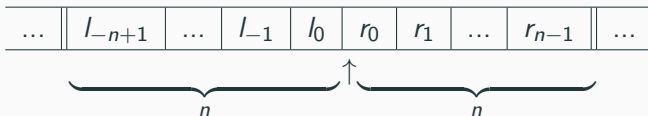
Frame : We want to talk about **discrete- and continuous-time dynamical systems**. We first use the frame of Turing machines (seen as a particular dynamical system).

# Space-perturbation of a Dynamical System

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## n-perturbation of a TM

Let  $\mathcal{M}_n$ , the *n-space-perturbed version* of TM  $\mathcal{M}$ : the idea is that the *n*-perturbed version of the machine  $\mathcal{M}$  is unable to remain correct at a distance more than *n* from the head of the machine.





## Some Properties (from Asarin and Bouajjani '01)

Let  $L_n(\mathcal{M})$  be the  $n$ -perturbed language of  $\mathcal{M}$ .

- a word accepted by  $\mathcal{M}$  is also recognised by all the  $\mathcal{M}_n$ 's.

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- $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \dots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$ .

**Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)**

$$L_\omega(\mathcal{M}) \in \Pi_1^0.$$

Say  $L$  is robust if  $L = L_\omega$ :

$L$  robust  $\Rightarrow L$  decidable.

- Definition of the notion of perturbation of a TM
- Some properties

# **A Characterisation of PSPACE**

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### Definition

For  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L_{\{f\}}(\mathcal{M})$  the set of words accepted by  $\mathcal{M}$  with a "space-perturbation"  $f$ :

$$L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}.$$



**Theorem (Polynomial precision robust  $\Leftrightarrow$  PSPACE)**

$L \in \text{PSPACE}$  iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  
 $L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M})$ .

- Characterisation of PSPACE with space-perturbated TMs

# Reachability Relations

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# Reachability Relation for Dynamical Systems

To each rational discrete time dynamical system  $\mathcal{P}$  is associated its reachability relation  $R^{\mathcal{P}}(\cdot, \cdot)$  on  $\mathbb{Q}^d \times \mathbb{Q}^d$ .

→ two rational points  $\mathbf{x}$  and  $\mathbf{y}$ ,  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  holds iff there exists a trajectory of  $\mathcal{P}$  from  $\mathbf{x}$  to  $\mathbf{y}$ .

## Reachability Relation With Small Perturbations

- Consider now a discrete-time dynamical system  $\mathcal{P}$  with function  $\mathbf{f}$  Lipschitz on a compact domain:  
For  $\varepsilon > 0$  we consider the  $\varepsilon$ -perturbed system  $\mathcal{P}_\varepsilon$ .
- Its trajectories are defined as sequences  $\mathbf{x}_t$  satisfying  $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$  for all  $t$ .

## Reachability Relation With Small Perturbations

- We denote reachability in the system  $\mathcal{P}_\epsilon$  by  $R_\epsilon^{\mathcal{P}}(\cdot, \cdot)$ .
- $R_\omega^{\mathcal{P}} = \bigcap_\epsilon R_\epsilon^{\mathcal{P}}(\cdot, \cdot)$  is co-c.e..
- Say  $R^{\mathcal{P}}$  is robust when  $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$ :  
 $R^{\mathcal{P}}$  robust  $\Rightarrow R^{\mathcal{P}}$  computable.

**Proposition** (Robust  $\Leftrightarrow$  reachability true or  $\epsilon$ -far from being true)

*We have  $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$  iff for all  $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$ , either  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  is true or  $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$  is false and there exists  $\epsilon > 0$  such that it is  $\epsilon$ -far from being true.*

$\Rightarrow$  relation with  $\delta$ -reachability (Gao, Kong, Chen, Clarke '06).

# Ball decision problem

## Input:

- A point  $x$
- A set  $B$
- A dynamic given by the function  $f$
- The promise that the dynamics starting from  $x$  never ends up on the border of  $B$ .



# Ball decision problem

## Input:

- A point  $x$
- A set  $B$
- A dynamic given by the function  $f$
- The promise that the dynamics starting from  $x$  never ends up on the border of  $B$ .

Output: Does the dynamics starting from  $x$  reach  $B$  ?

### Theorem (Polynomially robust to precision $\Rightarrow$ PSPACE)

*Take a Lipschitz system on a compact, with  $\mathbf{f}$  poly. time computable, whose domain  $X$  is a closed rational box, and that for all rational  $\mathbf{x}$ ,  $R^{\mathcal{P}}(\mathbf{x})$  is closed and  $R^{\mathcal{P}}(\mathbf{x}) = R^{\mathcal{P}}_{\{p\}}(\mathbf{x})$  for a polynomial  $p$ . Then the ball decision problem is in PSPACE.*

**Theorem (Polynomially robust to precision  $\Leftarrow$  PSPACE)**

*Any PSPACE language is reducible to PAM's reachability relation:*

$$R^{\mathcal{P}} = R^{\mathcal{P}}_{\{p\}}, \text{ for some polynomial } p.$$

- Perturbated reachability relation for rational dynamical systems
- Characterisation of PSPACE in that framework (with additionnal properties)

## **Alternative view: Drawability**

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But we could also see it as a relation over the reals and use the framework of CA, regarding subsets of  $\mathbb{R}^d \times \mathbb{R}^d$ .

### Definition

A closed subset of  $\mathbb{R}^d$  is said *c.e. closed* if we can effectively enumerate the rational open balls intersecting it.

From the statements of CA, the following holds:

### Theorem

*Consider a computable discrete time system  $\mathcal{P}$  whose domain is a computable compact. For all  $\mathbf{x}$ ,  $\text{cls}(R^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a c.e. closed subset.*

## And With Reachability Properties

### Theorem (Perturbed reachability is co-c.e.)

*Consider a dynamical system, with  $\mathbf{f}$  locally Lipschitz, computable, whose domain is a computable compact, then, for all  $\mathbf{x}$ ,*

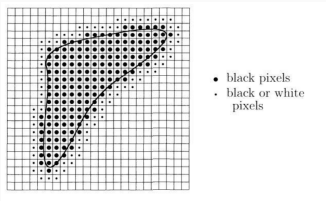
*$\text{cls}(R_{\omega}^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$  is a co-c.e. closed subset.*

Say  $R^{\mathcal{P}}$  is robust when  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$ :  $R^{\mathcal{P}}$  robust  $\Rightarrow R^{\mathcal{P}}$  computable.



# Plot twist

From CA, Computable  $\Leftrightarrow$  can be plotted.



## Theorem

Assume  $R^{\mathcal{P}}$  is closed and can be plotted effectively in a name of  $\mathbf{f}$ .

Then the system is robust, i.e.  $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$ .

- We have a nice extension to geometric properties

# Time-perturbation

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## Another type of perturbation

### Definition

Given  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $L^{\{f\}}(\mathcal{M})$  for the set of words accepted by  $\mathcal{M}$  with time perturbation  $f$ :

$$L^{\{f\}}(\mathcal{M}) = \{w \mid w \in L^{f(\ell(w))}(\mathcal{M})\}.$$

## Another Characterisation of PTIME with length

### Theorem (Polynomially robust to time $\Leftrightarrow$ PTIME)

- A language  $L \in \text{PTIME}$  iff for some  $\mathcal{M}$  and some polynomial  $p$ ,  $L = L(\mathcal{M}) = L^{\{p\}}(\mathcal{M})$ .
- Any PTIME language is reducible to PAM's reachability:  $R^{\mathcal{P}} = R^{\mathcal{P},(p)}$  for some polynomial  $p$ .

## Theorem (Polynomially length robust $\Rightarrow$ PTIME)

*Assume distance  $d$  is time metric and  $R^{\mathcal{P}} = R^{\mathcal{P},(p)}$  for some polynomial  $p$ . Then  $R^{\mathcal{P}} \in \text{PTIME}$ .*

- Definition of another type of perturbation
- Characterisation of PTIME

# Conclusion

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- Characterisation of FPSPACE with the suitable notion of perturbation.
- Characterisation of PTIME with the suitable notion of perturbation.
- Extension to drawability.