Analogue models of computation: computing with real numbers

Manon BLANC

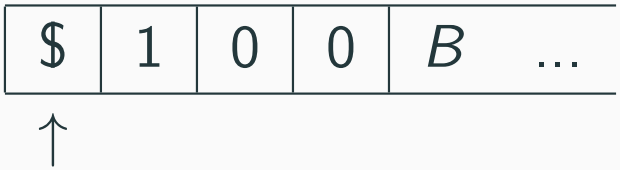
Practices around Computation

We need to talk about a discrete model

An example of non-analogue model

A mathematical representation: the Turing machine

The model:



A mathematical representation: the Turing machine

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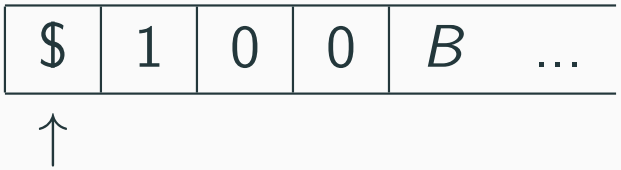

Physical World	**Mathematical Model**
Laptop	Turing machines

Example: the addition +1

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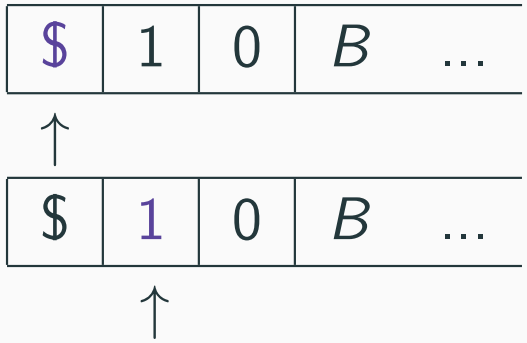

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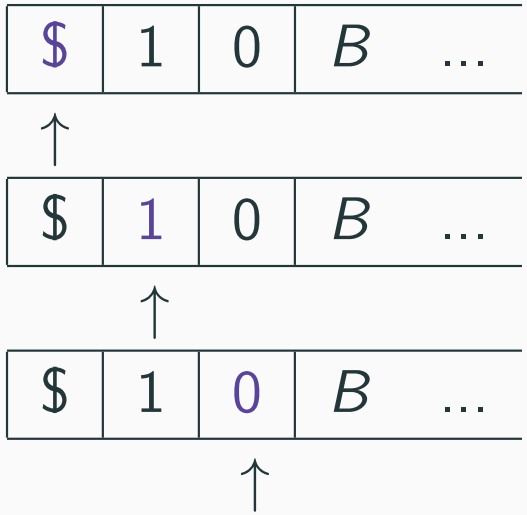

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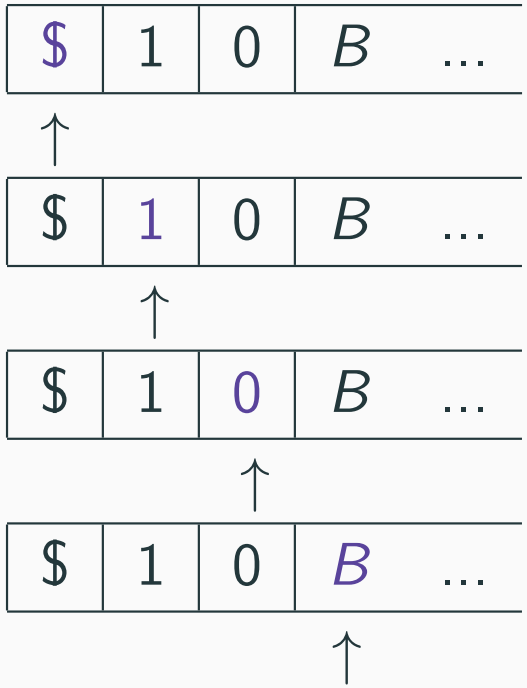

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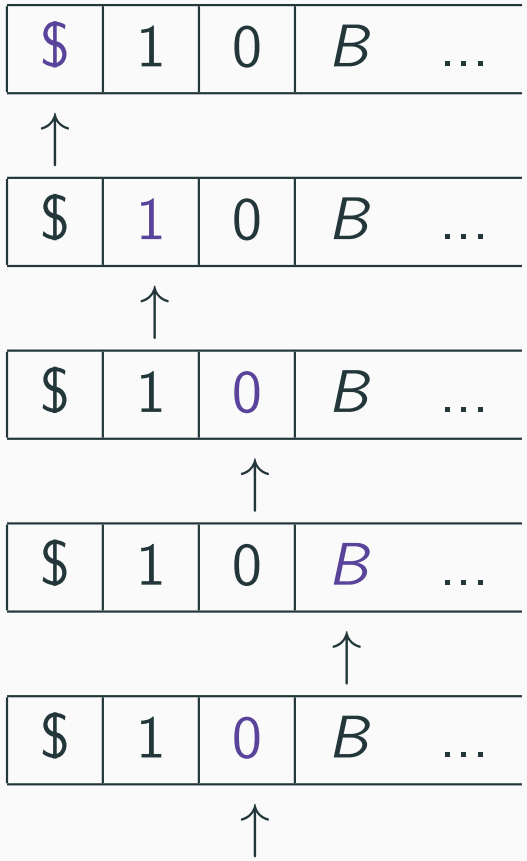

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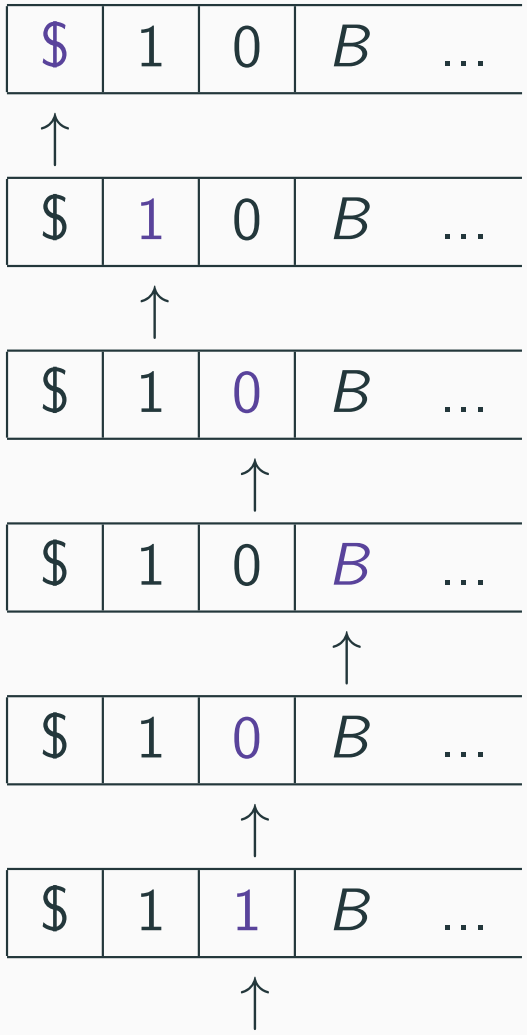

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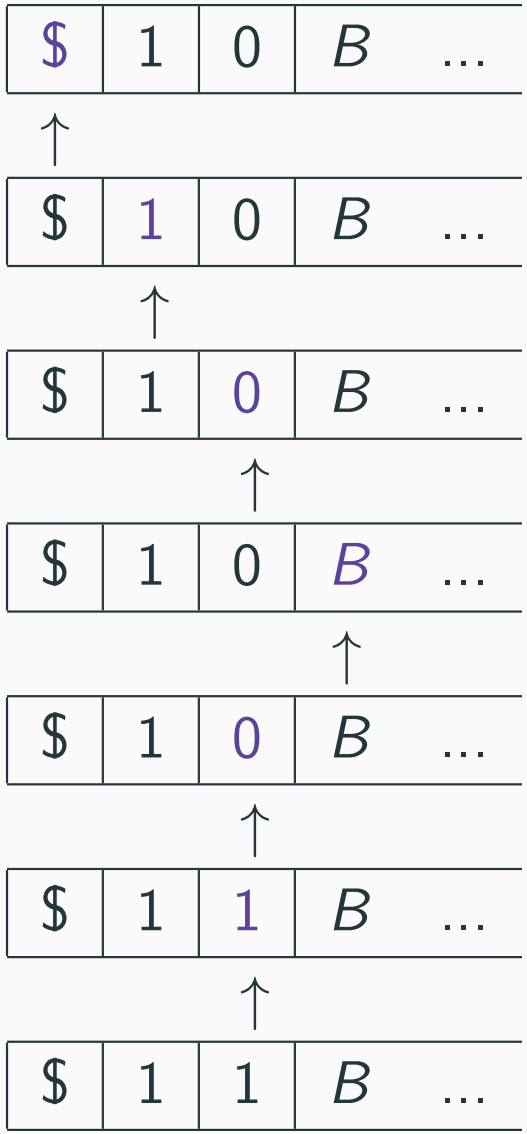


A bit more formally

(we assume the tape is non-empty)

	\$	0	1	B
q_{init}	$q_{init}, \$, \rightarrow$	$q_{init}, 0, \rightarrow$	$q_{init}, 1, \rightarrow$	$q_1, B, \leftarrow$
q_1		$accept, 1$	$q_2, 0, \leftarrow$	
q_2	$q_3, \$, \rightarrow$	$accept, 1$	$q_2, 1, \leftarrow$	
q_3		$q_4^0, 1, \rightarrow$		
q_4^0		$q_4^0, 0, \rightarrow$		$accept, 0$

A notion of cost for TM

- Time = number of steps

A notion of cost for TM

- Time = number of steps
- Space = number of needed cells

On our example

Time : 6 steps $\rightarrow O(n)$, n the size of the input

Space : 2 cells $\rightarrow O(n)$, n the size of the input

Both are linear (in the size of the input), so *polynomial* in time and space.

Robust correspondance

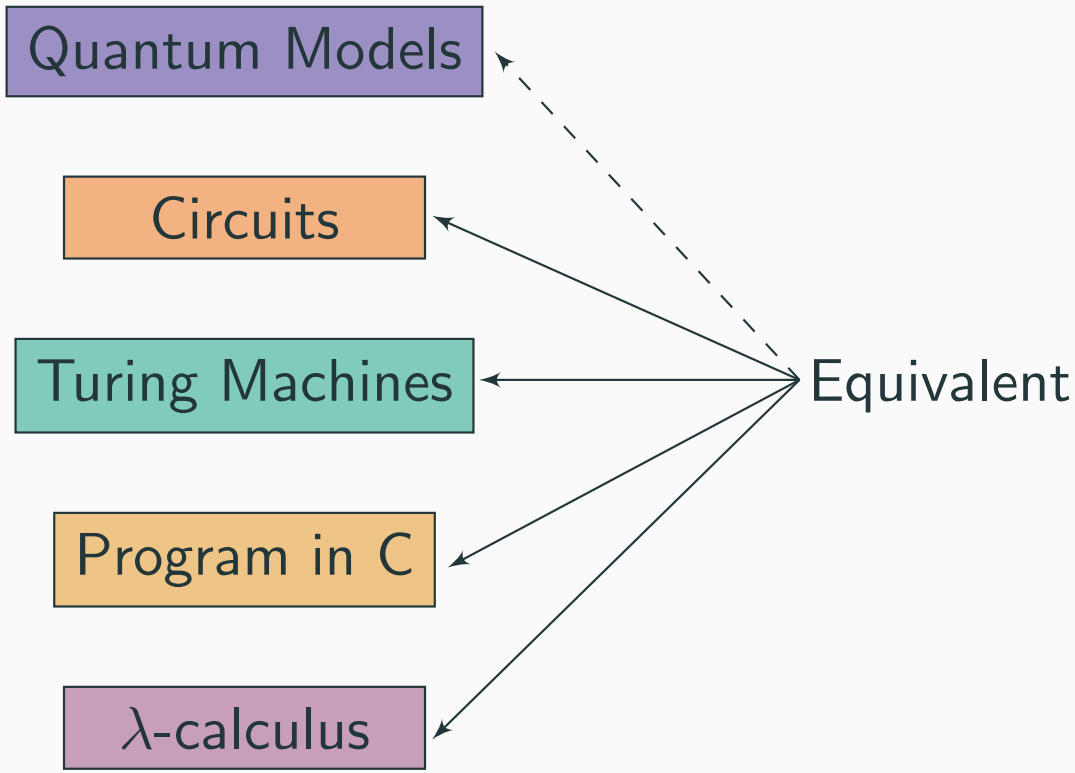
- Time of a TM $\sim$ Time on the laptop...

$\rightarrow$ Notions of time and space does not depend on the model

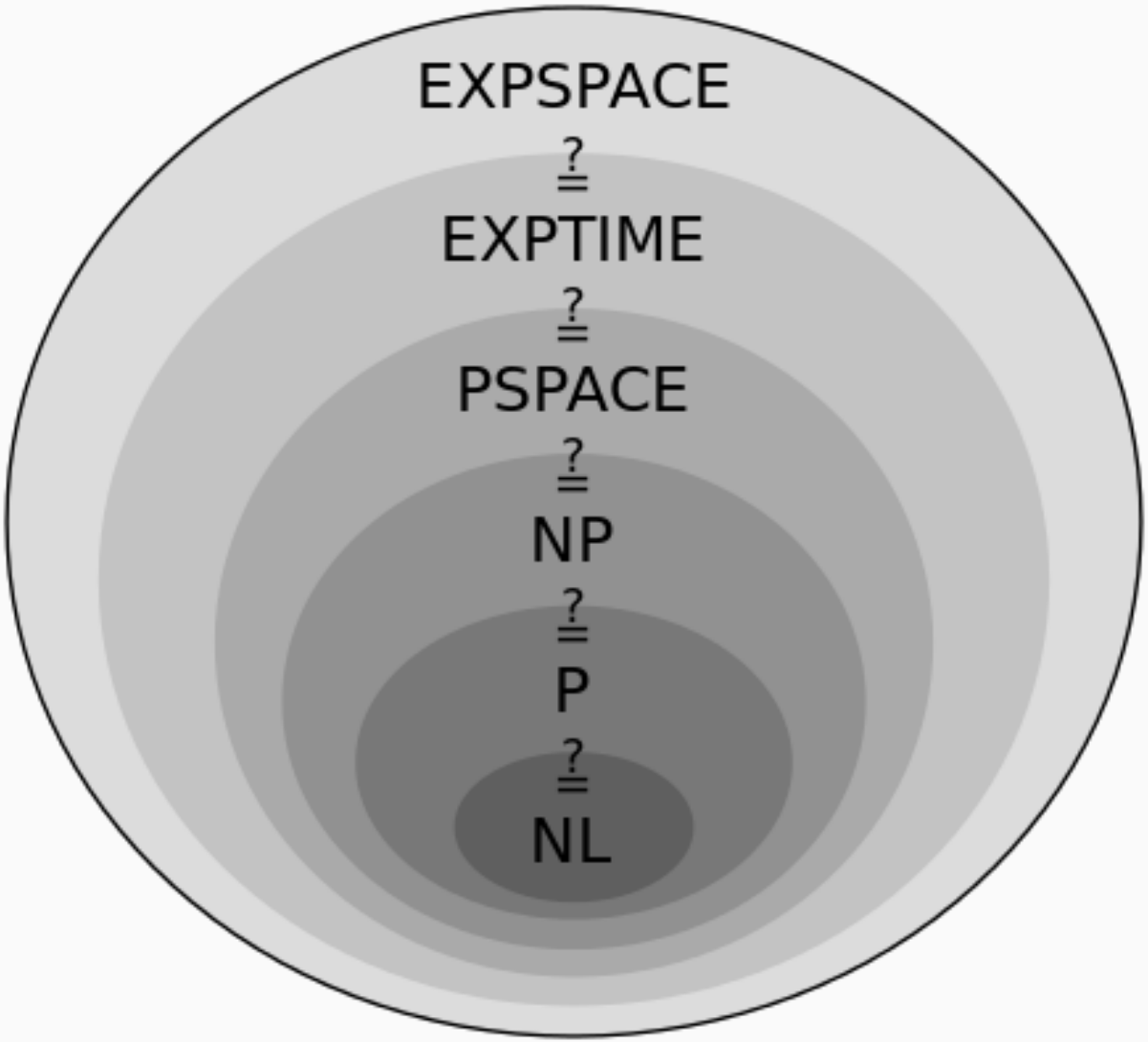
Computational Power

Church-Turing thesis: All discrete sufficiently powerful and reasonable models of computation have the same computational power as a Turing machine.

Power: boxes


Quantum Models
Circuits
Turing Machines
Program in C
λ-calculus
Equivalent

Putting boxes into boxes (but they are very nice boxes)



What an analogue world

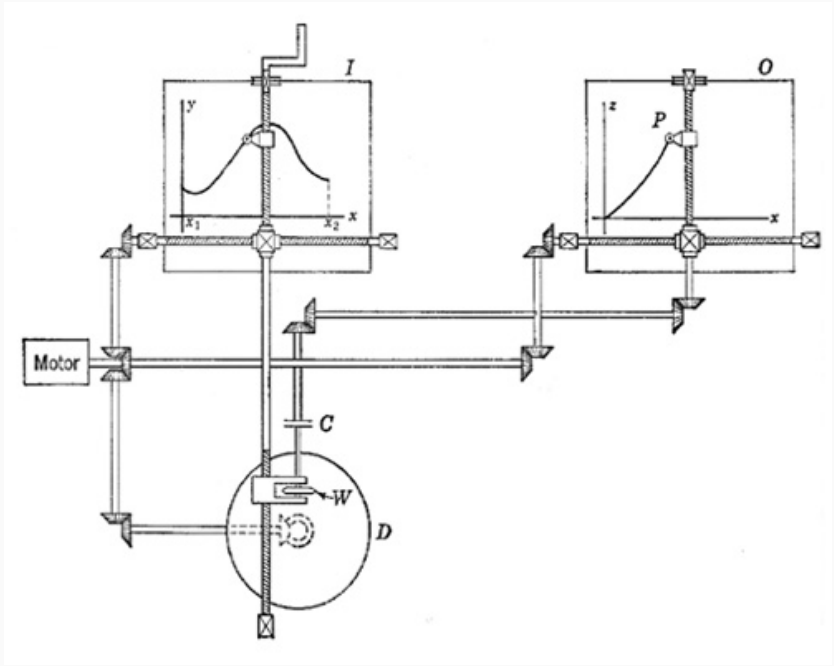
First

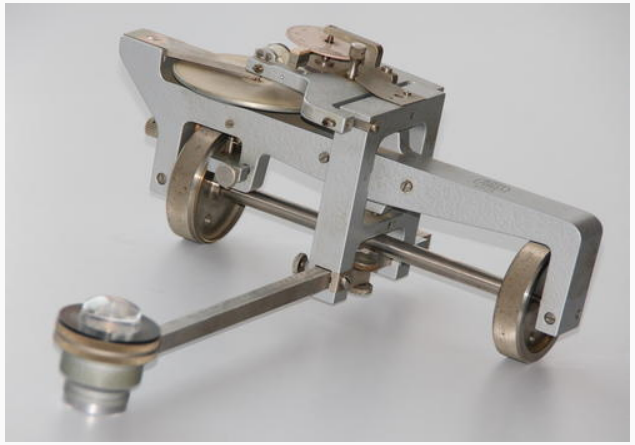
Historicaly, they are the first computational machines and certainely the most accurate

$\rightarrow$ Nothing is lost in translation

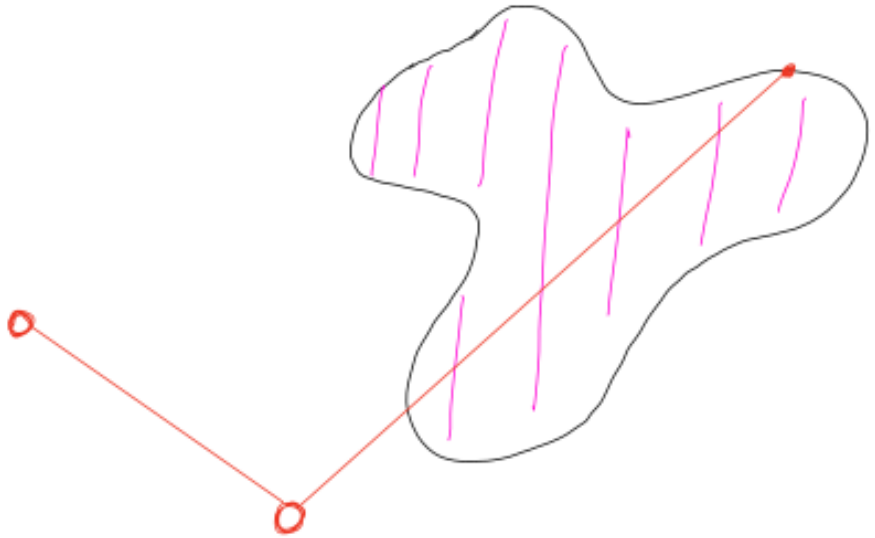
Well, actually... An example of analogue model

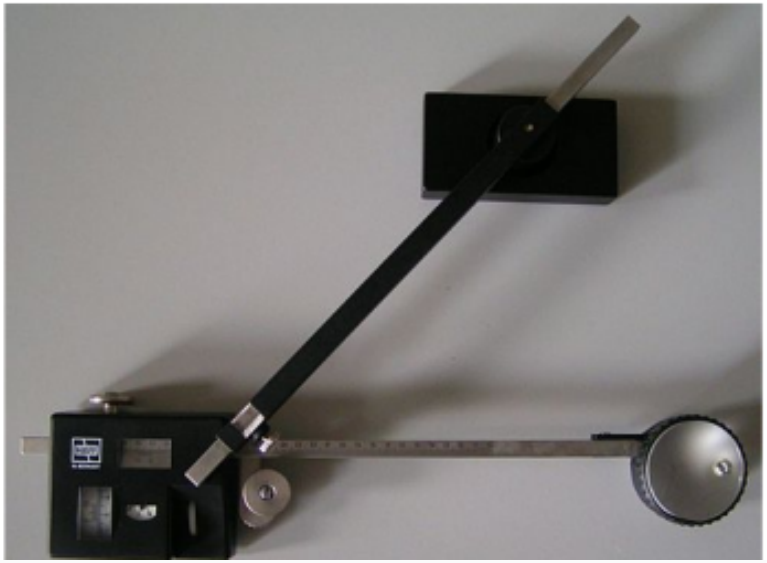
We live in an analogue world

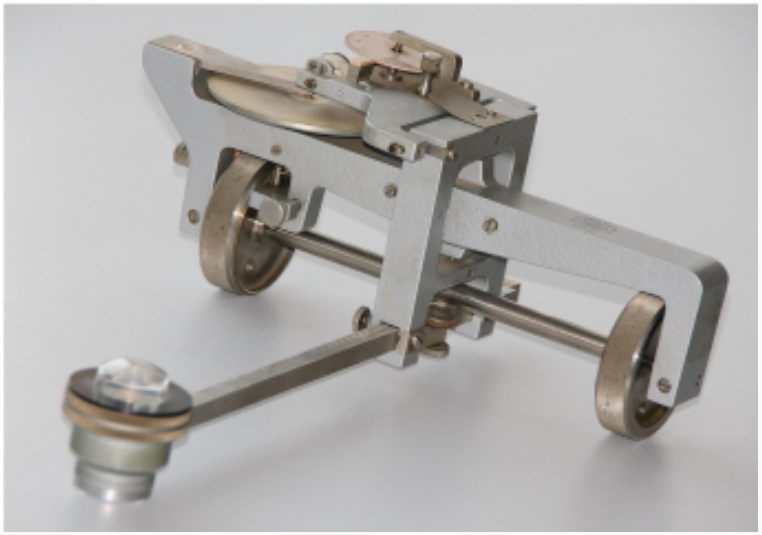




We live in an analogue world: planimeter







We live in an analogue world:

How does it work?

- Making: convert the sound into a print on a support in a spiraling groove;

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- Playing:
  - A head is placed on the groove;

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  - It transmit the deviations of the groove to an electromagnetic transductor;

How does it work?

- Making: convert the sound into a print on a support in a spiraling groove;

- Playing:
  - A head is placed on the groove;
  - It transmit the deviations of the groove to an electromagnetic transductor;
  - An audio power amplifier allows un to actually hear the sound

Better or worse?

Advantages:

- We capture the real sound
- Less mixing needed

Drawbacks:

- Worn out faster
- More fragile
- It is big

Too specific use

One device for one use: can we construct an "equivalent" of the physical computer, but for analogue model?

$\rightarrow$ is there a *general purpose* device?

The GPAC

Where are we now?

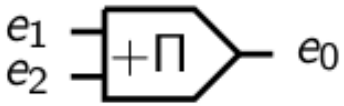
Physical World	Mathematical Model
Laptop	Turing machines
Differential Analyzer	?

The General Purpose Analog Computer (GPAC)

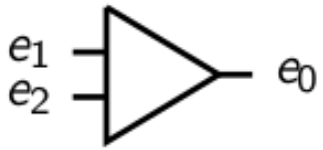
- Introduced by Claude Shannon in 1941;
- Mathematical model of the *Differential Analizer*;
- Model based on circuits, with several interconnected units doing basic operations

GPAC

constante: $e_0 = ke_1$

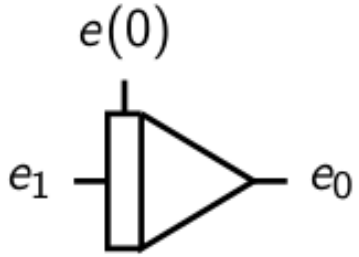


produit: $e_0 = e_1 e_2$



additionneur:

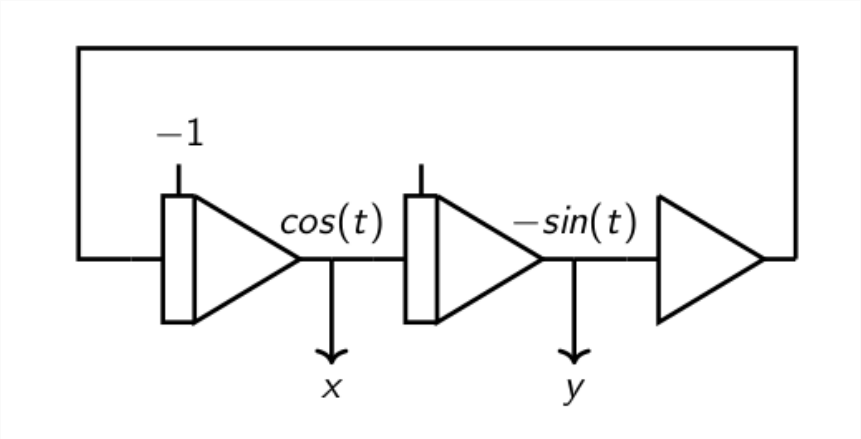
$e_0 = -(e_1 + e_2)$



intégrateur:

$e_0 = e(0) - \int_0^t e_1(u)du$

GPAC: how does it work?



$$\begin{cases} x'(t) &= y(t) \\ y'(t) &= -x(t) \\ x(0) &= -1 \\ y(0) &= 0 \end{cases} \Rightarrow \begin{cases} x(t) &= \cos(t) \\ y(t) &= -\sin(t) \end{cases}$$

Computational Power

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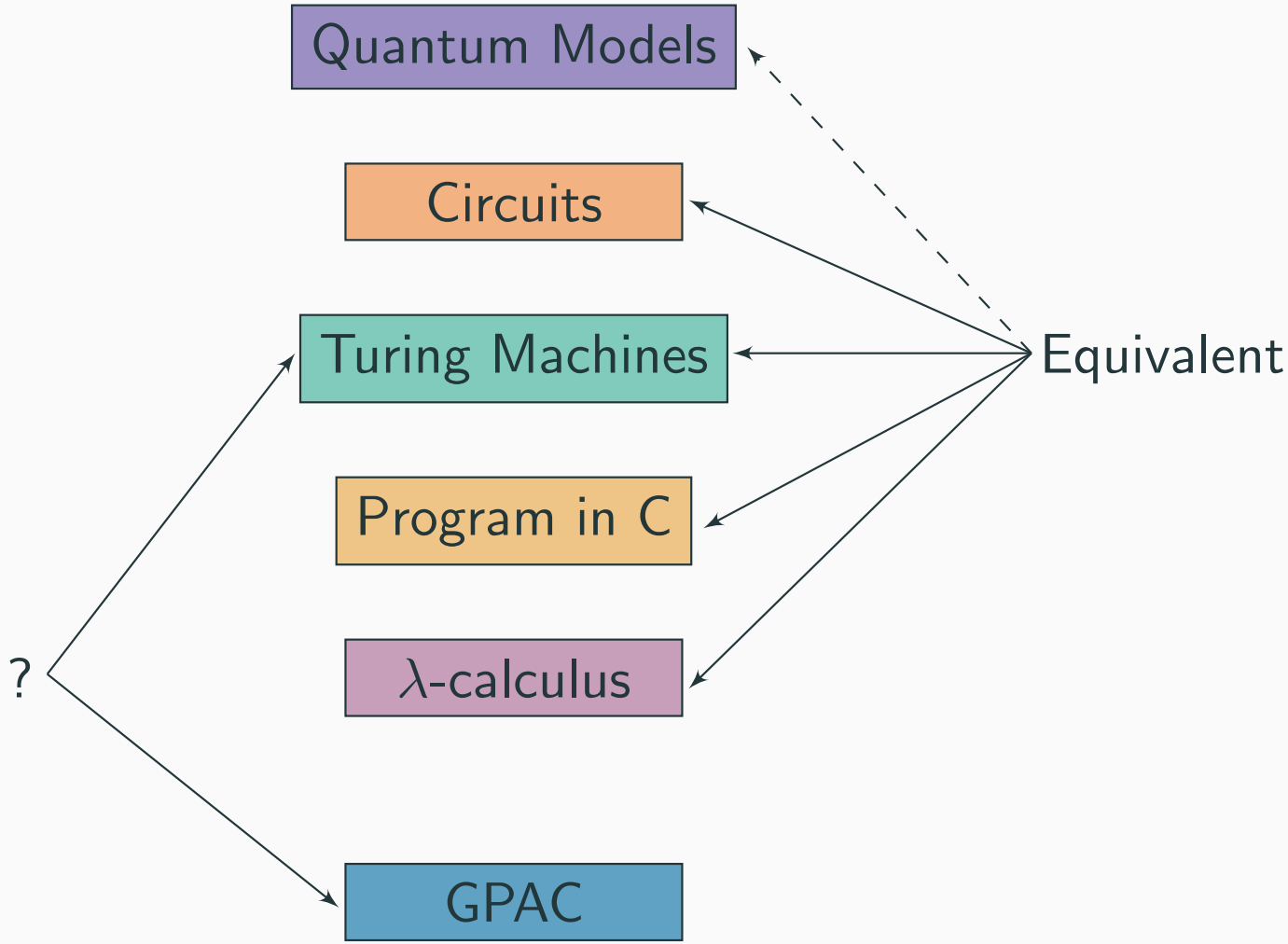
Is GPAC a sufficiently powerful?

Is GPAC reasonable?

Powerful: Other boxes



Computational power: is GPAC weaker?

- Shannon: GPAC $\leq$ Turing Machines: A GPAC corresponds to (a projection of) a *polynomial* differential equation, which can be computed by a TM.

- Shannon: GPAC $<$ Turing Machines: There exist computable functions, which are not projections of differential equations: e.g. Γ, ζ.

Well, he was wrong

We can actually program with polynomial ODE (Bournez, Graça, Pouly, Campagnolo, Hainry)

Little exercise

Transform this system of ODEs:

$$\begin{cases} y_1' &= \sin^2 y_2 \\ y_2' &= y_1 \cos y_2 - e^{e^{y_1}+t} \end{cases} \qquad \begin{cases} y_1(0) &= 0 \\ y_2(0) &= 0 \end{cases}$$

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into a *polynomial* ODE:

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considering $y_3 = \sin y_2$

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into a *polynomial* ODE:

$$\begin{cases} y_1' &= {y_3}^2 \\ y_2' &= y_1 y_4 - y_5 \\ & \\ & \\ & \\ & \end{cases} \qquad \begin{cases} y_1(0) &= 0 \\ y_2(0) &= 0 \\ & \\ & \\ & \\ & \end{cases}$$

considering $y_3 = \sin y_2, y_4 = \cos y_2$, $y_5 = e^{e^{y_1}+t}$

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into a *polynomial* ODE:

$$\begin{cases} y_1' &= {y_3}^2 \\ y_2' &= y_1 y_4 - y_5 \\ y_3' &= y_4(y_1 y_4 - y_5) \\ & \\ & \\ & \end{cases} \qquad \begin{cases} y_1(0) &= 0 \\ y_2(0) &= 0 \\ y_3(0) &= 0 \\ & \\ & \\ & \end{cases}$$

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into a *polynomial* ODE:

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considering $y_3 = \sin y_2, y_4 = \cos y_2$, $y_5 = e^{e^{y_1}+t}, y_6 = e^{y_1}$

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considering $y_3 = \sin y_2, y_4 = \cos y_2$, $y_5 = e^{e^{y_1}+t}, y_6 = e^{y_1}$

The big theorem: GPAC is sufficiently powerful

Theorem
Every computable functions can be computed by polynomial ODE, and conversely

Thus : Shannon: GPAC $\not\leq$ Turing Machines

Turing Machines $\leq$ GPAC

Is GPAC more powerful? [1]

[1] Fun fact: that's a part of my PhD subject 😅

How much does it cost?

How to redefine the cost: Time

Fig. 5. A continuous system before and after an exponential speed-up.

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θ solution of:

$$y' = f(y)$$

with $f : \mathbb{R} \to \mathbb{R}$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

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Re-scaling: $\phi_1(t) = \theta(e^t)$

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With this definition of time, all computation can cost 1 time unit...

How to redefine the cost: Time

Fig. 5. A continuous system before and after an exponential speed-up.

Robust notion of time $\rightarrow$ a quantity invariant by rescaling

The length remains the same

Time = Length of the curve

Now space: Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the general case, for $\mathbf{u}$ computable (Pour-El, Richards)

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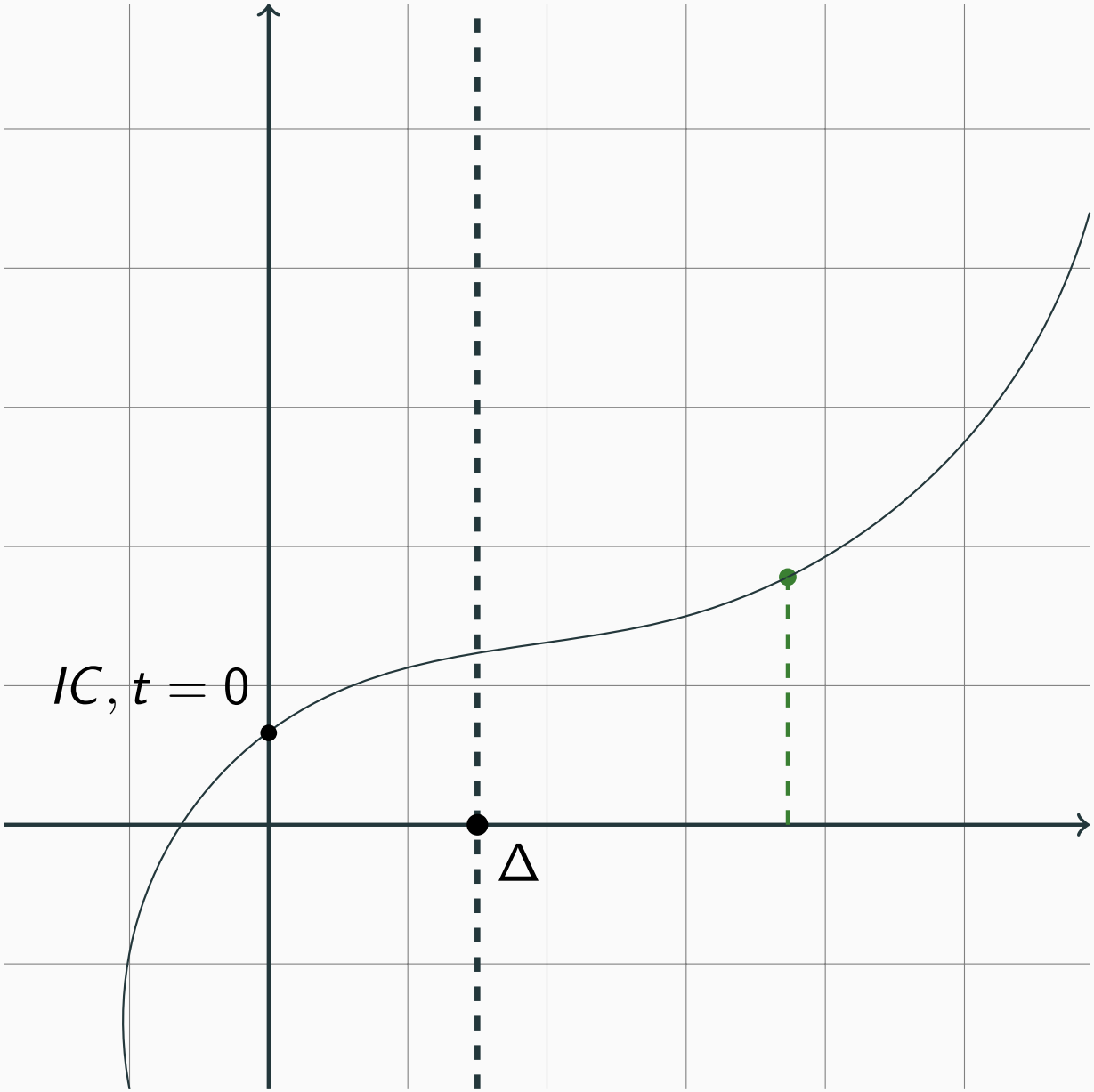
- Non-computable in the general case, for **u** computable (Pour-El, Richards)
- Computable for **u** computable and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a bounded domain for **u** computable in polynomial-time (Kawamura, Ko)

An unusual way of solving an ODE



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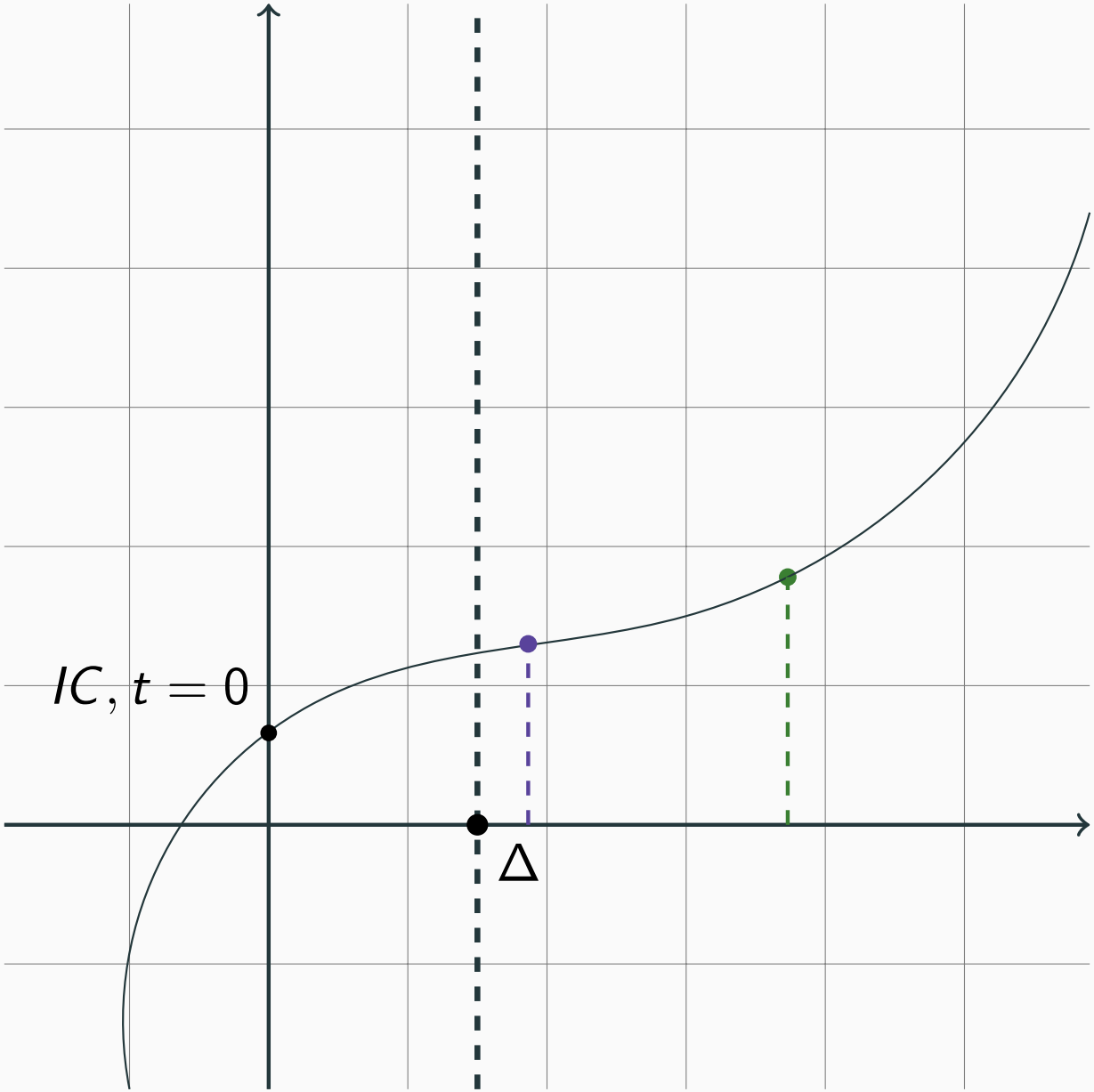

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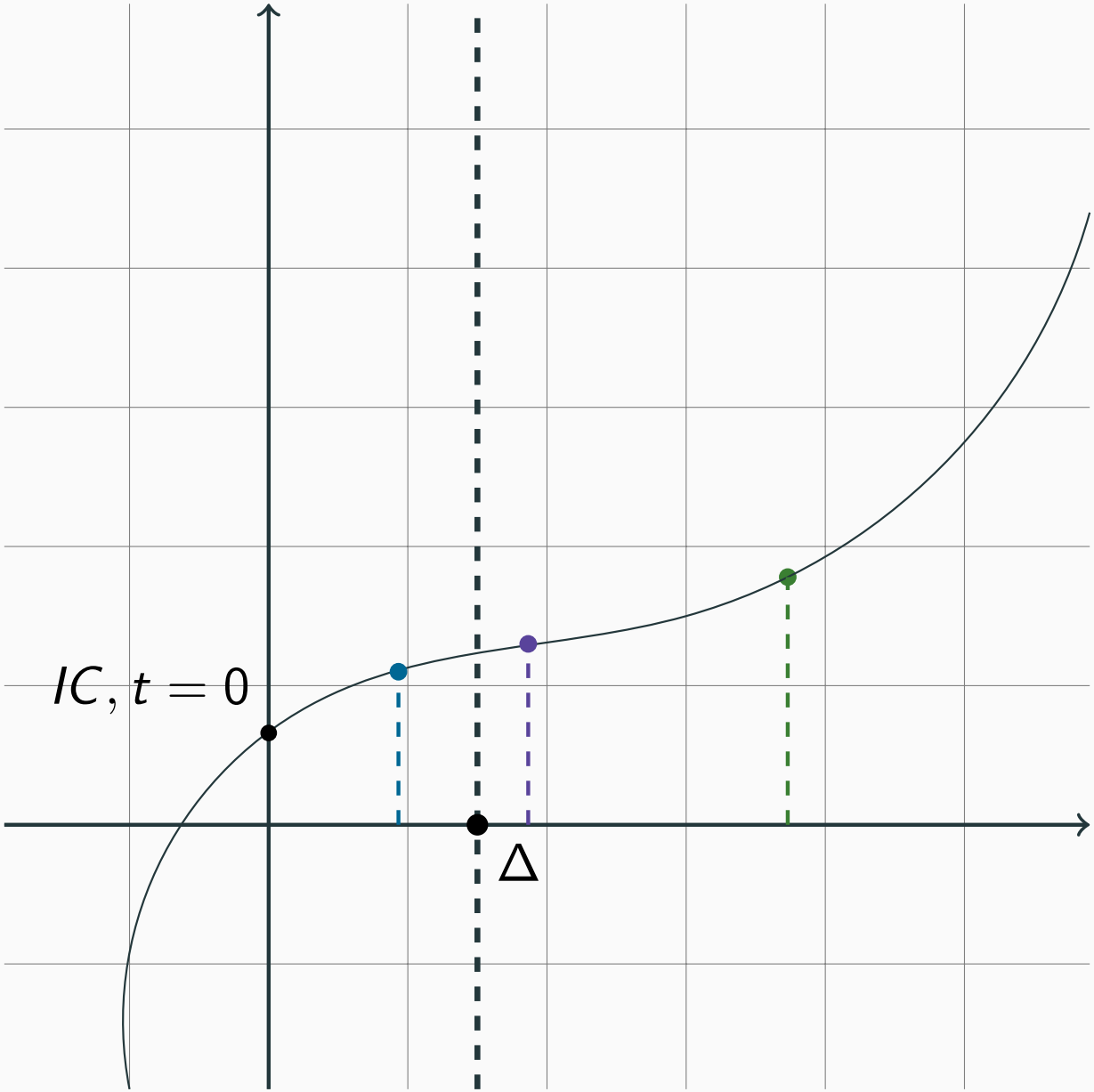

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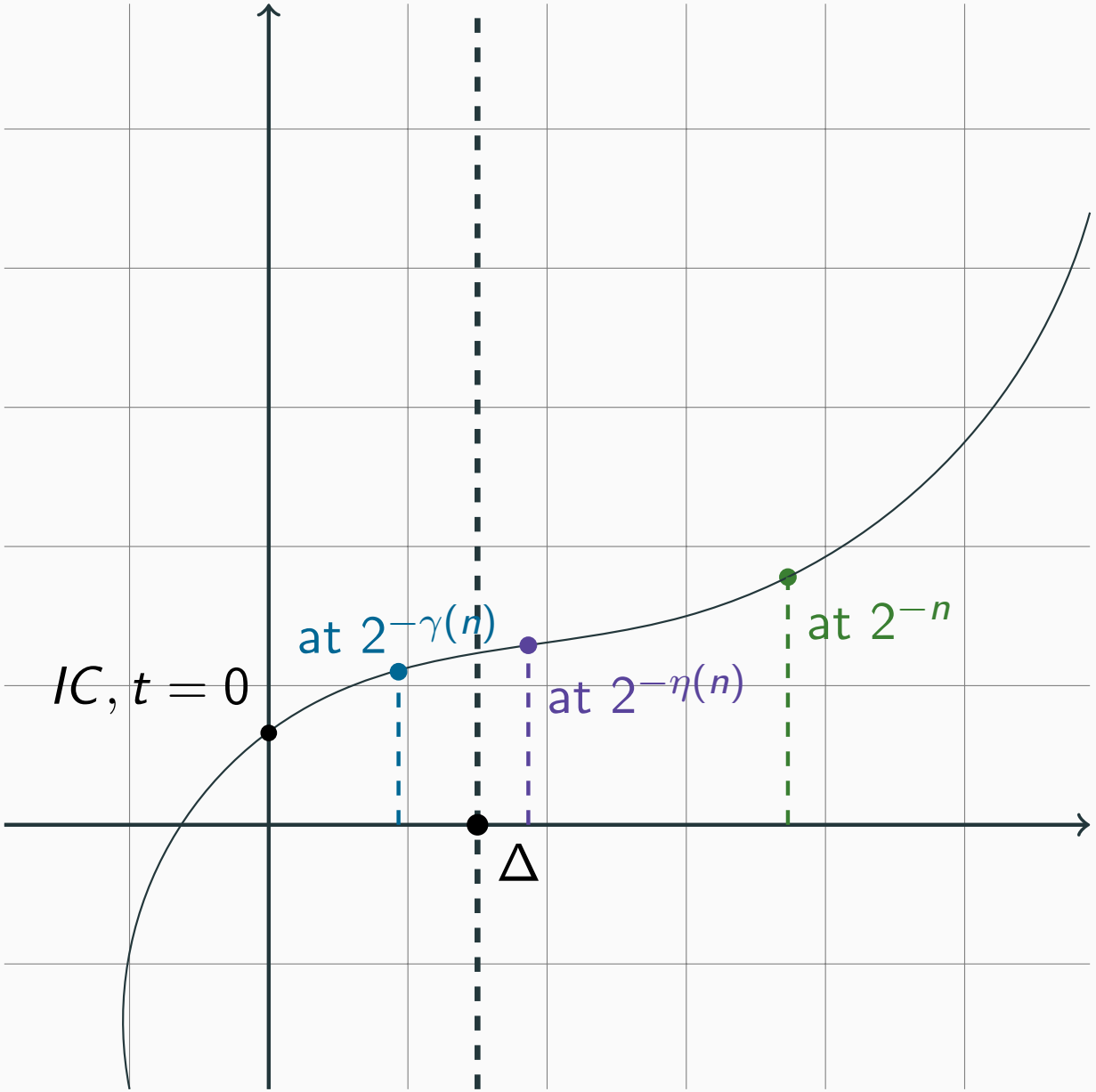


Space=Precision

- Computation at $t = x$ at precision 2^{-n}, with initial condition $t = 0$
- Computation at $t = t_0 + \frac{x}{2}$ at precision $2^{-\eta(n)}$, with initial condition $t_0 = 0$ and $t_0 = \frac{x}{2}$
- Computation at $t = t_0 + \frac{x}{4}$ at precision $2^{-\gamma(n)}$, with initial condition $t_0 = 0$ and $t_0 = \frac{x}{4}$ and $t_0 = \frac{x}{2}$ and $t_0 = \frac{3x}{4}$

Space = Precision of the computation

Why computing with ODEs is interesting

Representations

- Differential analizers might use less energy
- Better representation of "real" systems: ODEs in mechanic, biology, chemistry...

Biochemistry: context

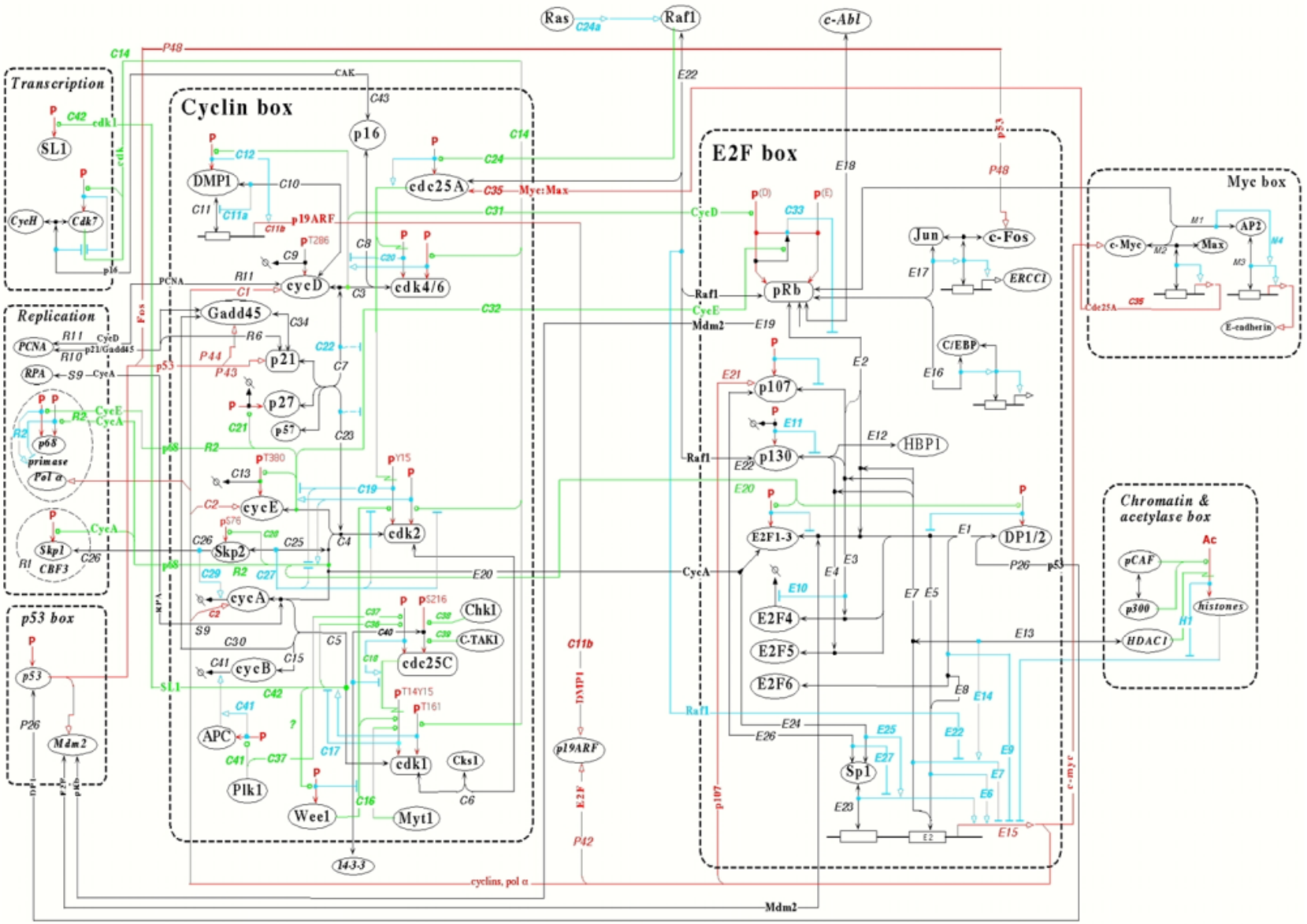
Your own cells do computation: treating signal, regulation of the metabolism

- Division
- Differentiation
- Migration

A major part of those are *analogue computations* with proteins

Once again, we live in an analogue world

Biochemistry: Kohn's map of the mammalian cell cycle



Biochemistry: into a GPAC

Assuming at most binary reactions with mass action law kinetics:

Theorem (Fages, Le Guludec, Bournez, Pouly)

- *Any polynomial ODE can be encoded in a chemically feasible positive system of double dimension.*
- *I.e. The systems of elementary biochemical reactions on finite universes of molecules are Turing-complete in differential semantics.*

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Furthermore: If the initial GPAC has a polynomial computational *complexity*, then the new system will also have a polynomial *complexity*.

So?

- Can you compute faster with GPACs than with TMs?

So?

- Can you compute faster with GPACs than with TMs? YES

So?

- Can you compute faster with GPACs than with TMs? YES
- Can you compute faster with Differential Analizer than with computers?

So?

- Can you compute faster with GPACs than with TMs? YES
- Can you compute faster with Differential Analizer than with computers? MAYBE

So?

- Can you compute faster with GPACs than with TMs? YES
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- Are all ODEs representable by a GPAC?

So?

- Can you compute faster with GPACs than with TMs? YES
- Can you compute faster with Differential Analizer than with computers? MAYBE
- Are all ODEs representable by a GPAC? YES IF POLYNOMIAL

So?

- Can you compute faster with GPACs than with TMs? YES
- Can you compute faster with Differential Analizer than with computers? MAYBE
- Are all ODEs representable by a GPAC? YES IF POLYNOMIAL
- Are all (poly) ODEs corresponding to a Differential Analizer?

So?

- Can you compute faster with GPACs than with TMs? YES
- Can you compute faster with Differential Analizer than with computers? MAYBE
- Are all ODEs representable by a GPAC? YES IF POLYNOMIAL
- Are all (poly) ODEs corresponding to a Differential Analizer? DON'T KNOW

Conclusion

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- We saw and define analogue models of computation
- We saw a mathematical description of those models
- We linked the notion of time and memory costs for TMs with that framework