

Computability and complexity over the reals

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Introduction

We want to algebraically characterise PTIME and PSPACE...

Working on **real numbers**

Example: The set of computable functions

- constant function 0 ,
- the projection functions π_i^P ,
- the function successor s ,
- and that is closed under composition, minimisation and primitive recursion.

$[0, \pi_i^P, s; \text{composition, primitive recursion, minimisation}]$

Computability and complexity over the reals : how?

Computable real numbers

- What is a **computable real number**?

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Example

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- What is a **computable function over the reals**?
- How do we define **complexity over the reals**?

**The beginning of algebraic
characterisation: a class you know
well**

$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

“Length ODE”?

Definition (Length ODE)

A function \mathbf{f} is “length-ODE” definable (from u , g and h) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

“Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

θ solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$

Motivation behind Length-ODEs

Now “time-complexity” is measured by the **length** of the solution curve of the ODE.

Invariance by rescaling

Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

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Unique solution : $f(x) = 2^{2^{\ell(x)}}$

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- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$

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- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$
- Then, $F(z+1) = F(z) + (z+1 - z)(F(z)F(z) - F(z))$

Characterisation of PTIME (the one you are used to)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

composition, linear length ODE]

Theorem (Bournez & Durand, '19)

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

First step: real sequences

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is an **effective limit** of $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation : \bar{f} computable in polynomial time $\Rightarrow f$ computable in polynomial time.

Algebraic characterisation of PTIME

We consider:

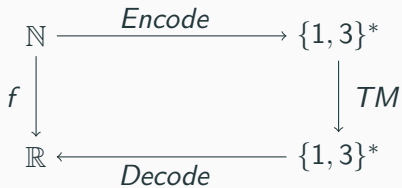
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MCU22)

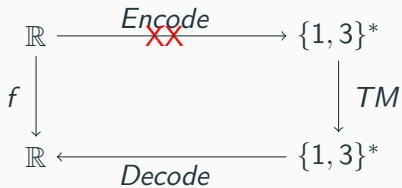
$$\text{FP}\text{TIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

“Proof”



**Second step: functions over the
reals with discrete ODEs**

What we cannot do



We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, linear length ODE, effective limit]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Definition (Robust linear ODE)

A bounded function \mathbf{f} is a robustly linear ODE definable (with \mathbf{u} *essentially linear* in $\mathbf{f}(x, \mathbf{y})$, \mathbf{g} and \mathbf{h}) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) **numerically stable**.

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, robust linear ODE, effective limit]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Third step: Functions over the reals with continuous ODEs

Robust Continuous ODEs (1/3)

A function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ is robustly ODE definable (from initial condition \mathbf{g} , and dynamic \mathbf{u}) if

- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), \mathbf{h}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

Robust Continuous ODEs (2/3)

- $\exists \Delta \in \mathbb{Q}_+^*$, and some polynomial p such that the previous ODE is (polynomially) numerically stable on $[0, \Delta]$.

i.e. for all integer n , considering $\eta(n) = p(n + \ell(\mathbf{x}))$ we can compute $\mathbf{f}(t, \mathbf{x})$ at 2^{-n} by working a precision $2^{-\eta(n)}$.

Robust Continuous ODEs (3/3)

- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $\epsilon(n)$.

i.e. of $\Phi(\mathbf{g}(\mathbf{x}), t/2)$, and then some approximation of $\Phi(\widetilde{\mathbf{f}(t/2, \mathbf{y})}, t/2)$ at 2^{-n} , working at precision $2^{-\eta(n)}$.

Characterisation of PSPACE with continuous ODEs

We consider :

$$\overline{\text{RCD}} = [0, 1, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

composition, robust ODE, ELim]

Theorem (B., Bournez, ICALP24 Track B)

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

Conclusion

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- We now have “natural” algebraic characterisations for functions over the reals computable in FPTIME and FPSPACE.
- Further works:
 - Cover other complexity classes
 - See what we can say for more general functions