

# Characterisations of polynomial-time and -space complexity classes over the reals: algebraic characterisations of classes

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# Introduction

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- Why algebraic characterisations?

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- We already do it for computability classes
- Computation over the reals : Framework of computable analysis

- Framework of **implicit complexity** (Cobham, Bellantoni, Cook, Levant, Marion, de Naurois...)
- **Time = Length:**
  - Continuous ODEs : Bournez, Graça, Pouly (ICALP 2017)
  - Discrete ODEs : Bournez, Durand (MFCS 2019)

**The beginning of algebraic  
characterisation: a class you know  
well**

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$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

# “Length ODE”?

## Definition (Length ODE)

A function  $\mathbf{f}$  is “length-ODE” definable (from  $u$ ,  $g$  and  $h$ ) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$



## “Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

# Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$\theta$  solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$\phi$  solution of:

$$z = z'$$

$$y' = f(y)z$$

# Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$

# Motivation behind Length-ODEs

Now “time-complexity” is measured by the **length** of the solution curve of the ODE.

Invariance by rescaling

## Example of length-ODE

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Unique solution :  $f(x) = 2^{2^{\ell(x)}}$

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- We have  $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$
- Then,  $F(z+1) = F(z) + (z+1 - z)(F(z)F(z) - F(z))$

# Characterisation of PTIME (the one you are used to)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

*composition, linear length ODE]*

**Theorem (Bournez & Durand, '19)**

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

## **Some basic definitions of computable analysis**

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## Definitions (from computable analysis)

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- What is a **computable function**  $f$  over the reals?  
→ on an input  $x \in \mathbb{R}$ , we produce any digit of  $f(x)$ .
- How do we define **complexity over the reals**?

## PTIME with Discrete ODEs

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## **First step: real sequences**

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## Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$  is an **effective limit** of  $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$  if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation :  $\bar{f}$  computable in polynomial time  $\Rightarrow f$  computable in polynomial time.

# Algebraic characterisation of PTIME

We consider:

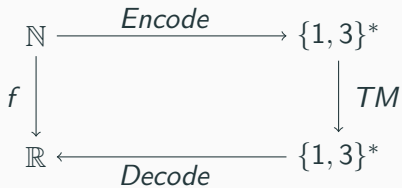
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

*composition, linear length ODE, effective limit]*

**Theorem (B., Bournez, MCU22)**

$$\text{FPTime} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

# “Proof”

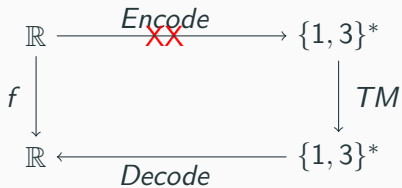


## **Second step: functions over the reals with discrete ODEs**

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# What we cannot do



We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, linear length ODE, effective limit]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

## PSPACE with Discrete ODEs

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### Definition (Robust linear ODE)

A bounded function  $\mathbf{f}$  is a robustly linear ODE definable from  $g$  and  $u$  (with  $\mathbf{u}$  essentially linear in  $\mathbf{f}(x, \mathbf{y})$ :  $u = A(\dots)\mathbf{f}(x) + B(\dots)$ ) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

# FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, robust linear ODE, ELim]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

# Functions over the reals with continuous ODEs

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## Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for  $\mathbf{u}$  computable (Pour-El, Richards)

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- Computable for  $\mathbf{u}$  **computable** and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a **bounded domain** for  $\mathbf{u}$  computable in polynomial-time (Kawamura, Ko)

## Robust Continuous ODEs (1/2)

A function  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$  is robustly ODE definable (from initial condition  $\mathbf{g}$ , and dynamic  $\mathbf{u}$ ) if

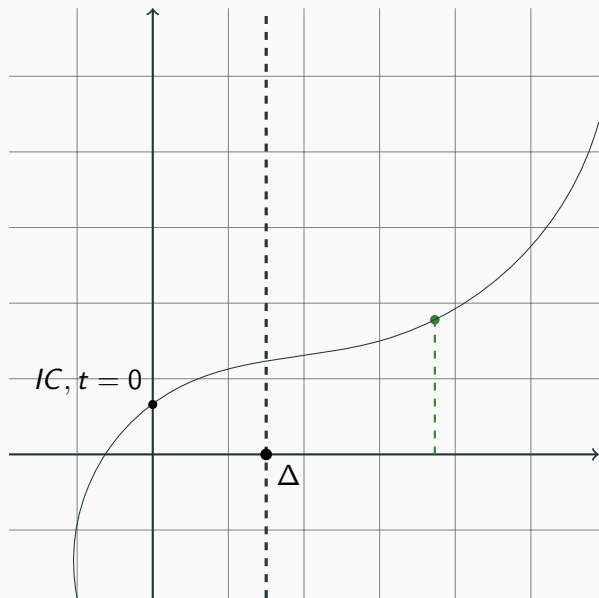
- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

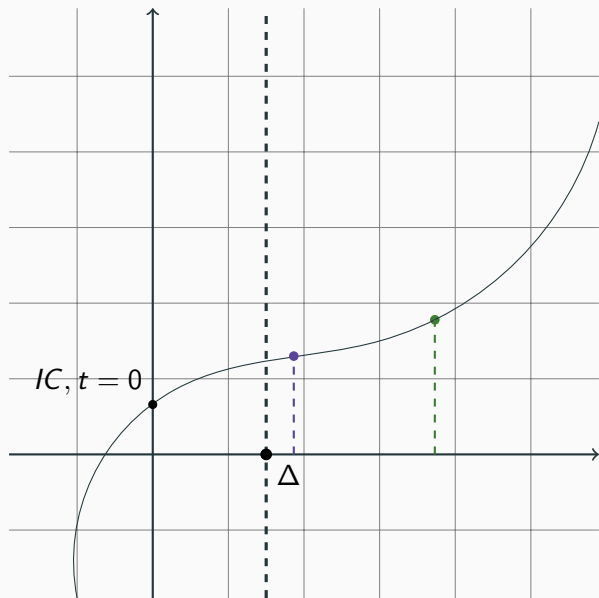
- And polynomially numerically stable (to be defined : next slides)

→ we allow computation by **dichotomy**.

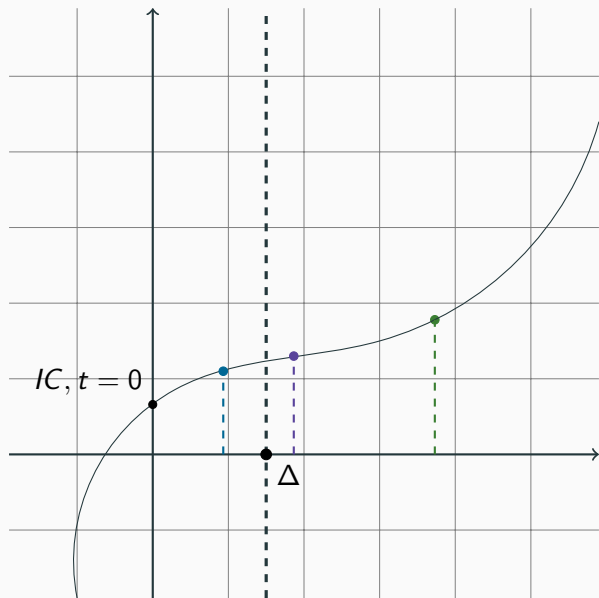
## An unusual way of solving an ODE



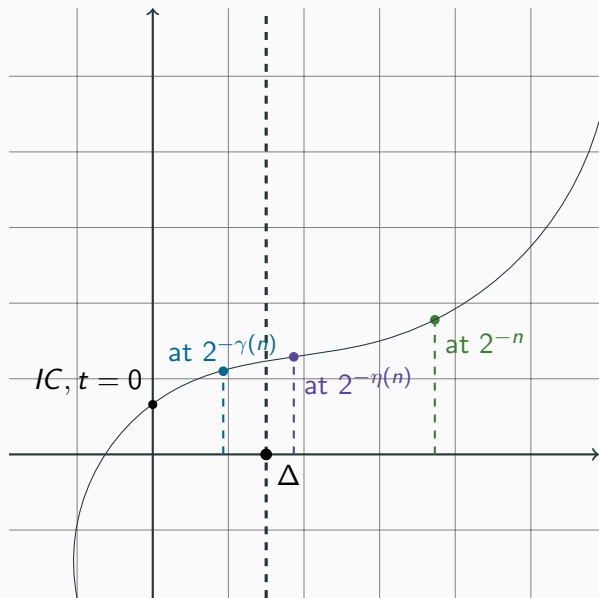
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## Robust Continuous ODEs (2/2)

More “formally” :

- $\exists \Delta \in \mathbb{Q}_+^*$ , such that the previous ODE is (polynomially spaced) **solvable** on  $[0, \Delta]$ .
- For  $t \geq \Delta$ , we can compute  $\mathbf{f}(t, \mathbf{x})$  at  $2^{-n}$  by computing some approximation  $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$  of  $\mathbf{f}(t/2, \mathbf{x})$  at precision  $2^{-\eta(n)}$ .

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

*composition, robust ODE, ELim]*

**Theorem (B., Bournez, ICALP24)**

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$



## Conclusion

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# Conclusion

- We have an algebraic characterisations of PTIME and PSPACE over the reals using *discrete* and *continuous ODEs*.
- The last algebra might not be minimal
- Further work :
  - talking about NP (non-determinism)?
  - having completeness results