

# The complexity of computing in continuous time: space complexity is precision

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ÉCOLE  
POLYTECHNIQUE



# Introduction

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- Why algebraic characterisations?

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- We already do it for computability classes
- Computation over the reals : Framework of computable analysis

- Framework of **implicit complexity** (Cobham, Bellantoni, Cook, Levant, Marion, ...)
- **Time = Length:**
  - Continuous ODEs : Bournez, Graça, Pouly (ICALP 2017)
  - Discrete ODEs : Bournez, Durand (MFCS 2019)

## **Some basic definitions of computable analysis**

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- What is a **computable function**  $f$  over the reals?  
→ on an input  $x \in \mathbb{R}$ , we produce any digit of  $f(x)$ .
- How do we define **complexity over the reals**?

## PSPACE with Discrete ODEs

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$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

## Definition (Robust linear ODE)

A bounded function  $\mathbf{f}$  is a robustly linear ODE definable from  $g$  and  $u$  (with  $\mathbf{u}$  essentially linear in  $\mathbf{f}(x, \mathbf{y})$ :  $\mathbf{u} = \mathbf{A}(\dots)\mathbf{f}(x) + \mathbf{B}(\dots)$ ) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

# FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, robust linear ODE, ELim]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

# Functions over the reals with continuous ODEs

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## Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for  $\mathbf{u}$  computable (Pour-El, Richards)
- Computable for  $\mathbf{u}$  **computable** and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a **bounded domain** for  $\mathbf{u}$  computable in polynomial-time (Kawamura)



## Robust Continuous ODEs (1/2)

A function  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$  is robustly ODE definable (from initial condition  $\mathbf{g}$ , and dynamic  $\mathbf{u}$ ) if

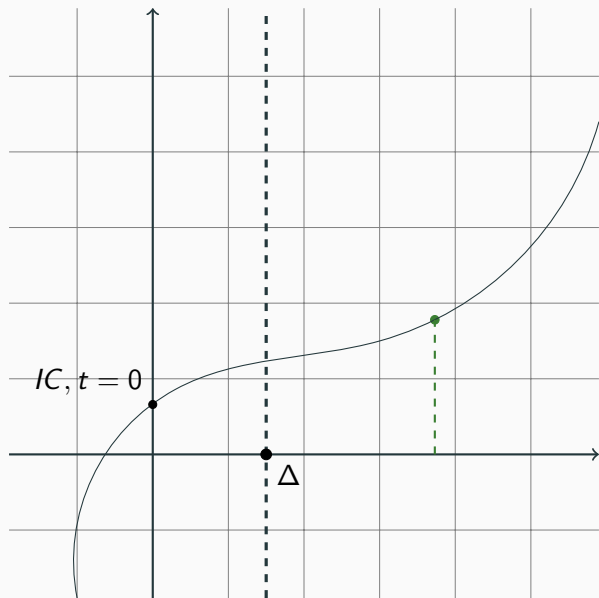
- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (1)$$

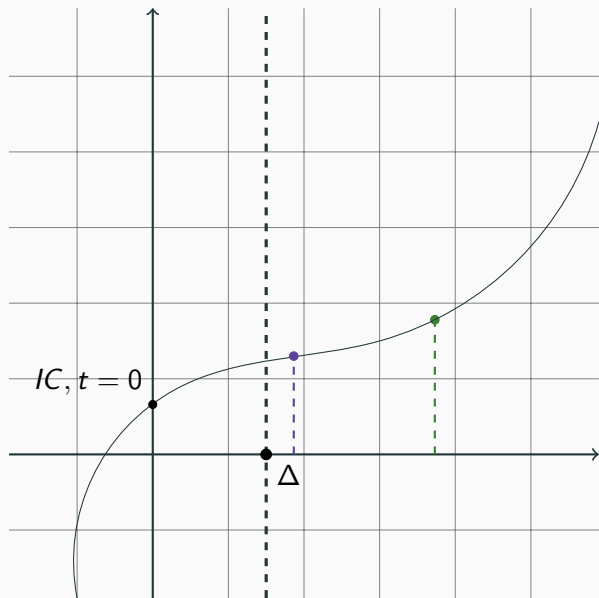
- And polynomially numerically stable (to be defined : next slides)

→ we allow computation by **dichotomy**.

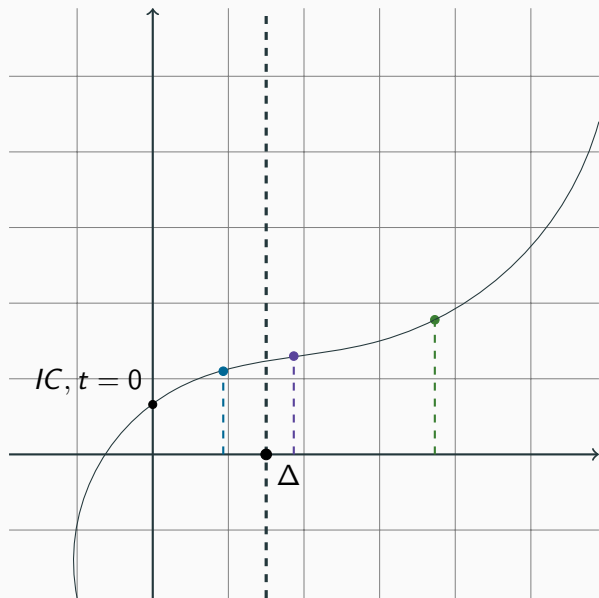
## An unusual way of solving an ODE



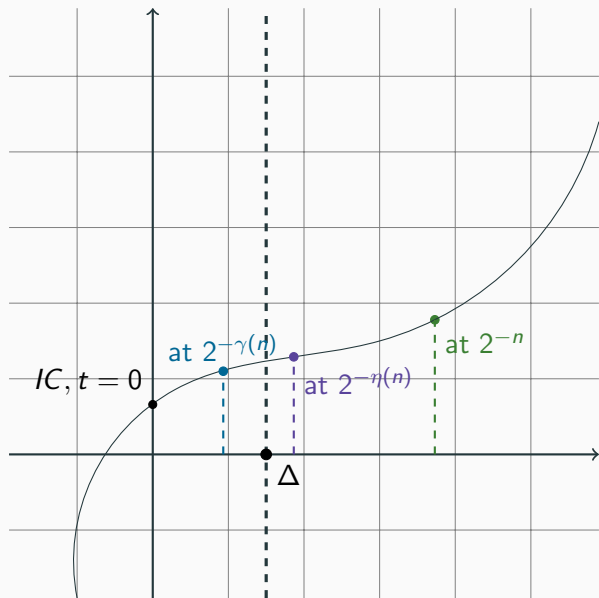
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## Robust Continuous ODEs (2/2)

More formally :

- $\exists \Delta \in \mathbb{Q}_+^*$ , such that the previous ODE is (polynomially spaced) **solvable** on  $[0, \Delta]$ .
- For  $t \geq \Delta$ , we can compute  $\mathbf{f}(t, \mathbf{x})$  at  $2^{-n}$  by computing some approximation  $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$  of  $\mathbf{f}(t/2, \mathbf{x})$  at precision  $2^{-\eta(n)}$ .

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

*composition, robust ODE, ELim]*

**Theorem (Main theorem)**

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

## Conclusion

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- We have an algebraic characterisation of PSPACE over the reals using *continuous ODE*.
- The algebra is not minimal
- Further work : talking about NP (non-determinism)?