

# Computability and complexity over the reals

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# Introduction

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We want to algebraically characterise PTIME and PSPACE...

Working on **real numbers**

## Example: The set of computable functions

- constant function  $0$ ,
- the projection functions  $\pi_i^P$ ,
- the function successor  $s$ ,
- and that is closed under composition, minimisation and primitive recursion.

$[0, \pi_i^P, s; \text{composition, primitive recursion, minimisation}]$

## **Computability and complexity over the reals : how?**

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# Computable real numbers

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- What is a **computable function over the reals**?
- How do we define **complexity over the reals**?

**The beginning of algebraic  
characterisation: a class you know  
well**

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$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

# “Length ODE”?

## Definition (Length ODE)

A function  $\mathbf{f}$  is “length-ODE” definable (from  $u$ ,  $g$  and  $h$ ) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$

## “Length ODE”?

Derivation with respect to the length = change of variable.

Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

# Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$\theta$  solution of:

$$y' = f(y)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$\phi$  solution of:

$$z = z'$$

$$y' = f(y)z$$

# Motivation behind Length-ODEs



Fig. 5. A continuous system before and after an exponential speed-up.

$$\text{Re-scaling: } \phi_1(t) = \theta(e^t)$$

## Motivation behind Length-ODEs

Now “time-complexity” is measured by the **length** of the solution curve of the ODE.

Invariance by rescaling



## Example of length-ODE

$$f(0) = 2$$

$$\frac{\delta f}{\delta \ell}(x) = f(x) \cdot f(x) - f(x)$$

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Unique solution :  $f(x) = 2^{2^{\ell(x)}}$

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- We have  $f(x) = F(z)$
- $F(z+1) = 2^{2^z+2^z} = F(z)F(z)$
- Then,  $F(z+1) = F(z) + (z+1 - z)(F(z)F(z) - F(z))$



# Characterisation of PTIME (the one you know)

We consider

$$\text{LDL} = [0, 1, \pi_i^k, \ell(x), +, -, \times, \text{cond}(x);$$

*composition, linear length ODE]*

**Theorem (Bournez & Durand, '19)**

$$\text{LDL} \cap \mathbb{N}^{\mathbb{N}} = \text{PTIME} \cap \mathbb{N}^{\mathbb{N}}$$

## **First step: real sequences**

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## Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$  is an **effective limit** of  $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$  if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation :  $\bar{f}$  computable in polynomial time  $\Rightarrow f$  computable in polynomial time.

# Algebraic characterisation of PTIME

We consider:

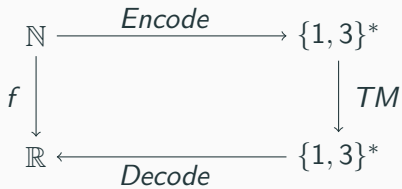
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

*composition, linear length ODE, effective limit]*

**Theorem (B., Bournez, MCU22)**

$$\text{FPTime} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

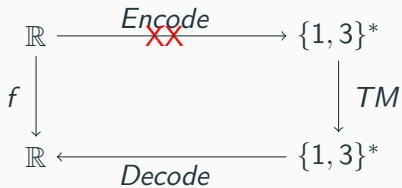
# “Proof”



## **Second step: functions over the reals with discrete ODEs**

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# What we cannot do



We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, linear length ODE, effective limit]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$



### Definition (Robust linear ODE)

A bounded function  $\mathbf{f}$  is a robustly linear ODE definable (with  $\mathbf{u}$  *essentially linear* in  $\mathbf{f}(x, \mathbf{y})$ ,  $\mathbf{g}$  and  $\mathbf{h}$ ) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) **numerically stable**.

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, robust linear ODE, effective limit]*

**Theorem (B., Bournez, MFCS23)**

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

## **Third step: Functions over the reals with continuous ODEs**

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## Robust Continuous ODEs

A function  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$  is robustly ODE definable (from initial condition  $\mathbf{g}$ , and dynamic  $\mathbf{u}$ ) if

- it corresponds to the solution of the following continuous ODE:

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), \mathbf{h}(t, \mathbf{x}), t, \mathbf{x}), \quad (2)$$

## Robust Continuous ODEs

- $\exists \Delta \in \mathbb{Q}_+^*$ , and some polynomial  $p$  such that the previous ODE is (polynomially) numerically stable on  $[0, \Delta]$ .
- For  $t \geq \Delta$ , we can compute  $\mathbf{f}(t, \mathbf{x})$  by computing some approximation  $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$  of  $\mathbf{f}(t/2, \mathbf{x})$  at precision  $\epsilon(n)$ .

We consider :

$$\overline{\text{RCD}} = [0, 1, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

*composition, robust ODE, ELim]*

**Theorem (B., Bournez, submitted)**

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

## Conclusion

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- We have (nice) characterisations of polynomial complexity classes.
- We can have other points of view on them (another part of my PhD).