

Characterisations of polynomial-space complexity classes over the reals: algebraic characterisations of classes

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Introduction

- Why algebraic characterisations?

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- We already do it for computability classes
- Computation over the reals : Framework of computable analysis

- Framework of **implicit complexity** (Cobham, Bellantoni, Cook, Levant, Marion, de Naurois...)
- **Time = Length:**
 - Continuous ODEs : Bournez, Graça, Pouly (ICALP 2017)
 - Discrete ODEs : Bournez, Durand (MFCS 2019)

Discrete derivation

$$\frac{\delta f(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x}} = f(\mathbf{x} + 1, \mathbf{y}) - f(\mathbf{x}, \mathbf{y})$$

Some basic definitions of computable analysis

Definitions (from computable analysis)

- What is a **computable real number** x ?
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- What is a **computable function** f over the reals?
→ on an input $x \in \mathbb{R}$, we produce any digit of $f(x)$.
- How do we define **complexity over the reals**?

Functions over the reals with discrete ODEs

PSPACE with Discrete ODEs

Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$ is an **effective limit** of $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation : \bar{f} computable in polynomial time $\Rightarrow f$ computable in polynomial time.

Definition (Robust linear ODE)

A bounded function \mathbf{f} is a robustly linear ODE definable from g and u (with \mathbf{u} essentially linear in $\mathbf{f}(x, \mathbf{y})$: $\mathbf{u} = \mathbf{A}(\dots)\mathbf{f}(x) + \mathbf{B}(\dots)$) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\delta \mathbf{f}(x, \mathbf{y})}{\delta x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(n), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

composition, robust linear ODE, ELim]

Theorem (B., Bournez, MFCS23)

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

By double inclusion:

- All the functions in $\overline{\text{RLD}}^\circ$ are computable in FPSPACE;
- We encode the execution of the Turing machine into $\overline{\text{RLD}}^\circ$

Functions over the reals with continuous ODEs

Why it is not easy to solve ODEs

$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}),$$

- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)

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- Non-computable in the **general case**, for \mathbf{u} computable (Pour-El, Richards)
- Computable for \mathbf{u} **computable** and unique solution (Collins, Graça, Ruohonen)
- PSPACE-completeness on a **bounded domain** for \mathbf{u} computable in polynomial-time (Kawamura, Ko)

Robust Continuous ODEs (1/2)

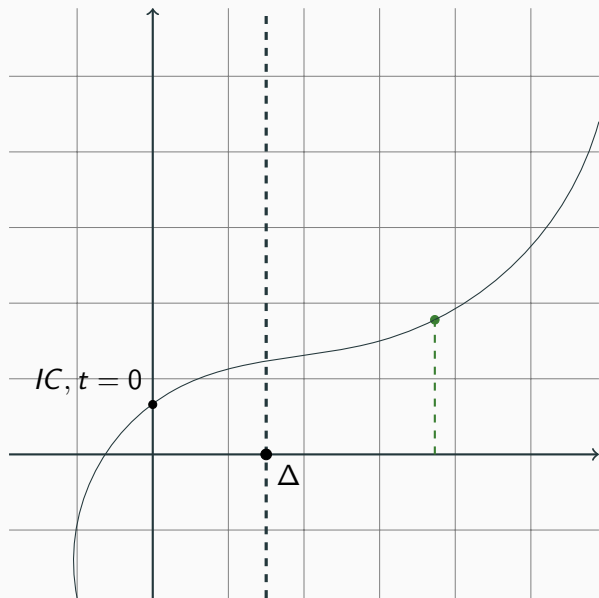
A function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ is robustly ODE definable (from initial condition \mathbf{g} , and dynamic \mathbf{u}) if

- it corresponds to the solution of the following continuous ODE:

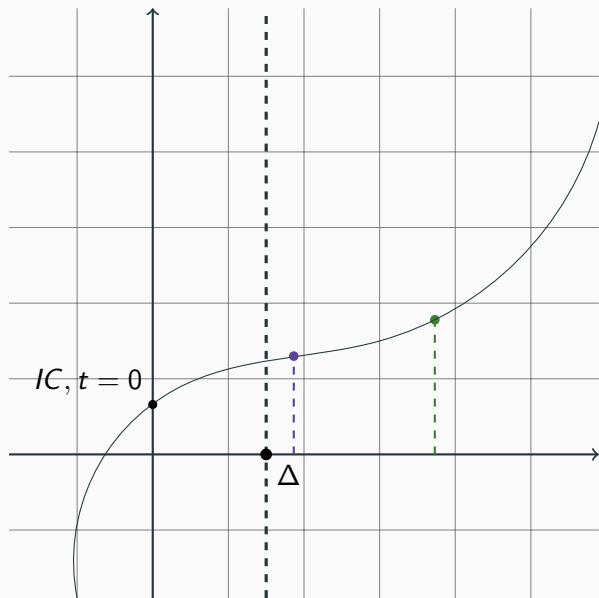
$$\mathbf{f}(0, \mathbf{x}) = \mathbf{g}(\mathbf{x}) \quad \text{and} \quad \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial t} = \mathbf{u}(\mathbf{f}(t, \mathbf{x}), t, \mathbf{x}), \quad (1)$$

- And polynomially numerically stable (to be defined : next slides)
→ we allow computation by dichotomy.

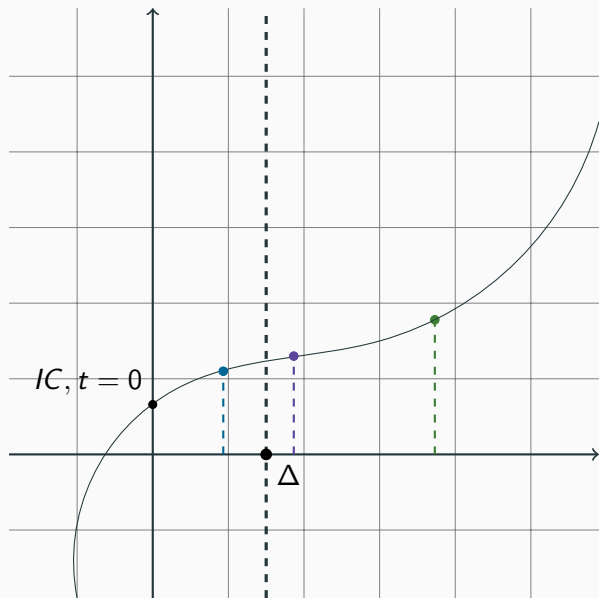
An unusual way of solving an ODE



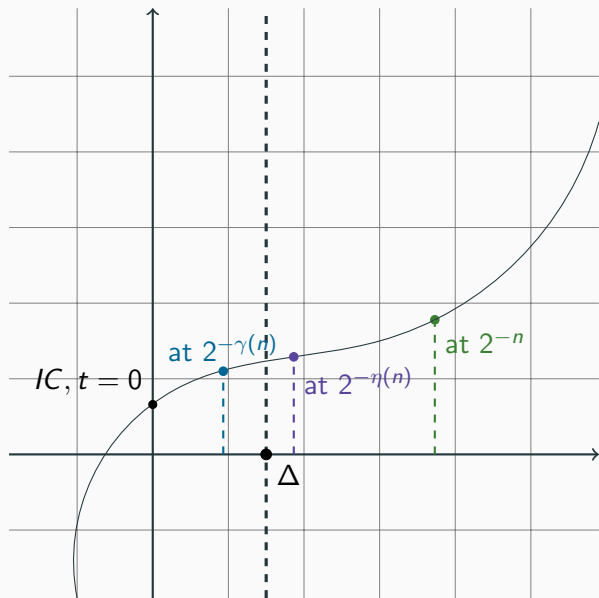
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Robust Continuous ODEs (2/2)

More “formally” :

- $\exists \Delta \in \mathbb{Q}_+^*$, such that the previous ODE is (polynomially spaced) **solvable** on $[0, \Delta]$.
- For $t \geq \Delta$, we can compute $\mathbf{f}(t, \mathbf{x})$ at 2^{-n} by computing some approximation $\widetilde{\mathbf{f}(t/2, \mathbf{x})}$ of $\mathbf{f}(t/2, \mathbf{x})$ at precision $2^{-\eta(n)}$.

We consider :

$$\overline{\text{RCD}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, +, -, \times, \tanh, \cos, \pi, \frac{x}{2}, \frac{x}{3};$$

composition, robust ODE, ELim]

Theorem (B., Bournez, ICALP24)

$$\overline{\text{RCD}} \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

By double inclusion, using $\overline{\text{RLD}^\circ}$:

- Discrete to continuous : adaptation of the “Branicky trick”
- Continuous to discrete : we took the proper notion of robustness for our ODEs

Conclusion

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- We have an algebraic characterisations of FPSPACE over the reals using *discrete* and *continuous ODEs*.
- The last algebra might not be minimal
- Further work :
 - talking about NP (non-determinism)?
 - having completeness results