

Measuring the robustness of dynamical systems

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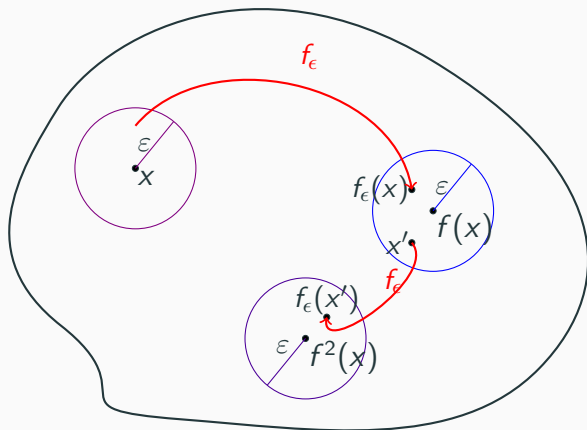
ÉCOLE
POLYTECHNIQUE



Introduction

- Motivated by the field of verification.
- Informal conjecture: undecidability in verification does not happen for robust systems.
- Approach: Asarin and Bouajjani (LICS 01).
- Our goal:
 - go from computability to complexity,
 - quantifying the robustness to characterise FPSPACE.

Introduction

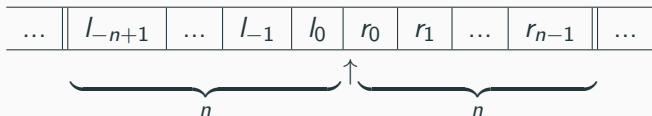


Frame : We want to talk about **discrete- and continuous-time dynamical systems**. We first use the frame of Turing machines (seen as a particular dynamical system).

Space-perturbation of a Dynamical System

n-perturbation of a TM

Let \mathcal{M}_n , the *n-space-perturbed version* of TM \mathcal{M} : the idea is that the *n*-perturbed version of the machine \mathcal{M} is unable to remain correct at a distance more than *n* from the head of the machine.



Some Properties (from Asarin and Bouajjani '01)

Let $L_n(\mathcal{M})$ be the n -perturbed language of \mathcal{M} .

- a word accepted by \mathcal{M} is also recognised by all the \mathcal{M}_n 's.

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- $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \dots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$.

Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)

$$L_\omega(\mathcal{M}) \in \Pi_1^0.$$

Say L is robust if $L = L_\omega$:

L robust $\Rightarrow L$ decidable.

A Characterisation of PSPACE

Definition

For $f : \mathbb{N} \rightarrow \mathbb{N}$, we write $L_{\{f\}}(\mathcal{M})$ the set of words accepted by \mathcal{M} with a "space-perturbation" f :

$$L_{\{f\}}(\mathcal{M}) = \{w \mid w \in L_{f(\ell(w))}(\mathcal{M})\}.$$

Theorem (Polynomial precision robust \Leftrightarrow PSPACE)

$L \in \text{PSPACE}$ iff for some \mathcal{M} and some polynomial p ,
 $L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M})$.

Reachability Relations

Reachability Relation for Dynamical Systems

To each rational discrete time dynamical system \mathcal{P} is associated its reachability relation $R^{\mathcal{P}}(\cdot, \cdot)$ on $\mathbb{Q}^d \times \mathbb{Q}^d$.

→ two rational points \mathbf{x} and \mathbf{y} , $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ holds iff there exists a trajectory of \mathcal{P} from \mathbf{x} to \mathbf{y} .

Reachability Relation With Small Perturbations

- We add small perturbations: consider a discrete-time dynamical system \mathcal{P} with function \mathbf{f} :
For $\varepsilon > 0$ we consider the ε -perturbed system \mathcal{P}_ε .
- Its trajectories are defined as sequences \mathbf{x}_t satisfying $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$ for all t .
- We denote reachability in the system \mathcal{P}_ε by $R_\varepsilon^{\mathcal{P}}(\cdot, \cdot)$.
- $R_\omega^{\mathcal{P}} = \bigcap_\varepsilon R_\varepsilon^{\mathcal{P}}(\cdot, \cdot)$ is co-c.e..
- Say $R^{\mathcal{P}}$ is robust when $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$: $R^{\mathcal{P}}$ robust $\Rightarrow R^{\mathcal{P}}$ computable.

Proposition (Robust \Leftrightarrow reachability true or ϵ -far from being true)

We have $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$ iff for all $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$, either $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ is true or $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ is false and there exists $\epsilon > 0$ such that it is ϵ -far from being true.

\Rightarrow relation with δ -reachability (Gao, Kong, Chen, Clarke '06).

Complexity of reachability relations

Theorem (Polynomially robust to precision \Leftarrow PSPACE)

Any PSPACE language is reducible to PAM's reachability relation:

$$R^{\mathcal{P}} = R^{\mathcal{P}}_{\{p\}}, \text{ for some polynomial } p.$$

Theorem (Polynomially robust to precision \Rightarrow PSPACE)

Take a locally Lipschitz system, with \mathbf{f} poly. time computable, whose domain X is a closed rational box, and that for all rational \mathbf{x} , $R^{\mathcal{P}}(\mathbf{x})$ is closed and $R^{\mathcal{P}}(\mathbf{x}) = R^{\mathcal{P}}_{\{p\}}(\mathbf{x})$ for a polynomial p .

Then the ball decision problem is in PSPACE.

Time-perturbation

Another type of perturbation

Definition

Given $f : \mathbb{N} \rightarrow \mathbb{N}$, we write $L^{\{f\}}(\mathcal{M})$ for the set of words accepted by \mathcal{M} with time perturbation f :

$$L^{\{f\}}(\mathcal{M}) = \{w \mid w \in L^{f(\ell(w))}(\mathcal{M})\}.$$

A Characterisation of PTIME

Theorem (Polynomially robust to time \Leftarrow PTIME)

A language $L \in \text{PTIME}$ iff for some \mathcal{M} and some polynomial p ,
 $L = L(\mathcal{M}) = L^{\{p\}}(\mathcal{M})$. Any PTIME language is reducible to
PAM's reachability: $R^{\mathcal{P}} = L^{\{p\}}(\mathcal{P})$ for some polynomial p .

Theorem (Polynomially length robust \Rightarrow PTIME)

Assume distance d is time metric and $R^{\mathcal{P}} = R^{\mathcal{P},(p)}$ for some
polynomial p . Then $R^{\mathcal{P}} \in \text{PTIME}$.

Alternative view: Drawability

Drawability: geometric properties

But we could also see it as a relation over the reals and use the framework of CA, regarding subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

Definition

A closed subset of \mathbb{R}^d is said *c.e. closed* if we can effectively enumerate the rational open balls intersecting it.

From the statements of CA, the following holds:

Theorem

Consider a computable discrete time system \mathcal{P} whose domain is a computable compact. For all x , $\text{cls}(R^{\mathcal{P}}(x)) \subseteq \mathbb{R}^d$ is a c.e. closed subset.

And With Reachability Properties

Theorem (Perturbed reachability is co-c.e.)

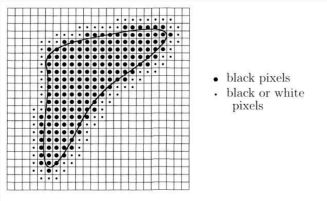
Consider a dynamical system, with \mathbf{f} locally Lipschitz, computable, whose domain is a computable compact, then, for all \mathbf{x} ,

$\text{cls}(R_{\omega}^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$ is a co-c.e. closed subset.

Say $R^{\mathcal{P}}$ is robust when $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$: $R^{\mathcal{P}}$ robust $\Rightarrow R^{\mathcal{P}}$ computable.

Plot twist

From CA, Computable \Leftrightarrow can be plotted.



Theorem

Assume $R^{\mathcal{P}}$ is closed and can be plotted effectively in a name of \mathbf{f} .

Then the system is robust, i.e. $R_{\omega}^{\mathcal{P}} = R^{\mathcal{P}}$.

Conclusion

- Characterisation of FPSPACE with the suitable notion of perturbation.
- Characterisation of PTIME with the suitable notion of perturbation.
- Extension to drawability.