

# A characterisation of functions computable in polynomial time and space over the reals with discrete ordinary differential equations

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# Introduction

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- providing **algebraic characterisations** of polynomial complexity classes, without:
  - explicit bounds (Cobham)
  - several types of arguments (Bellantoni and Cook)
  - tiering (Leivant and Marion)...
- We are in the framework of **implicit complexity**

- Born from the work of Olivier Bournez and Arnaud Durand, using linear length discrete ODEs...

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- ... but here we are interested in functions over the real numbers

# Our framework: Computable Analysis

- Computable real number
- Computable function over the reals
- Discrete derivation :  $f : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$ ,

$$f'(x, y) = \frac{\partial f}{\partial x}(x, y) = f(x + 1, y) - f(x, y).$$

**Previous work: characterising  
computable reals and sequences  
(MCU22)**

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# “Length ODE”?

## Definition (Length ODE)

A function  $\mathbf{f}$  is “length-ODE” definable (from  $u$ ,  $g$  and  $h$ ) if it is a solution of

$$f(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial \ell} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}). \quad (1)$$

Formal synonym for right-hand side of (1):

$$\mathbf{f}(x + 1, \mathbf{y}) = \mathbf{f}(x, \mathbf{y}) + (\ell(x + 1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y})$$



## “Length ODE”?

A derivation with respect to the length corresponds to change of variable in the ODE. Variation when:

$$\ell(x + 1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

## Example of length-ODE

$$f(0) = 2$$

$$\frac{\partial f}{\partial \ell}(x) = f(x) \cdot f(x) - f(x)$$

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$$f(0) = 2$$

$$\frac{\partial f}{\partial \ell}(x) = f(x) \cdot f(x) - f(x)$$

Unique solution :  $f(x) = 2^{2^{\ell(x)}}$

## Definition

$f : \mathbb{R} \rightarrow \mathbb{R}$  is an **effective limit** of  $\bar{f} : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$  if

$$|f(x) - \bar{f}(x, 2^n)| \leq 2^{-n}$$

Key observation :  $\bar{f}$  computable in polynomial time  $\Rightarrow f$  computable in polynomial time.

# Algebraic characterisation of PTIME

We consider:

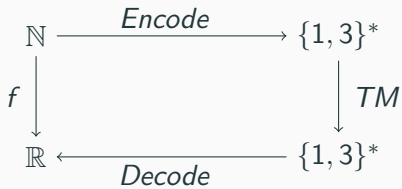
$$\overline{\text{LDL}}^\bullet = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\text{cond}}(x), \frac{x}{2};$$

*composition, linear length ODE, effective limit]*

**Theorem (MCU22)**

$$\text{FPTime} \cap \mathbb{R}^{\mathbb{N}} = \overline{\text{LDL}}^\bullet \cap \mathbb{R}^{\mathbb{N}}$$

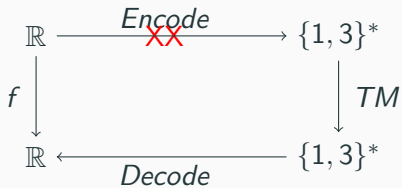
# “Proof”



# Characterising computable functions over the reals

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# What we cannot do





# FPTIME for computable functions over the reals

We consider:

$$\overline{\text{LDL}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, linear length ODE, effective limit]*

**Theorem**

$$\overline{\text{LDL}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

### Definition (Robust linear ODE)

A bounded function  $\mathbf{f}$  is a robustly linear ODE definable (with  $\mathbf{u}$  *essentially linear* in  $\mathbf{f}(x, \mathbf{y})$ ,  $\mathbf{g}$  and  $\mathbf{h}$ ) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \text{and} \quad \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) **numerically stable**.

# FPSPACE for computable functions over the reals

We consider:

$$\overline{\text{RLD}}^\circ = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

*composition, robust linear ODE, effective limit]*

**Theorem**

$$\overline{\text{RLD}}^\circ \cap \mathbb{R}^{\mathbb{R}} = \text{FPSPACE} \cap \mathbb{R}^{\mathbb{R}}$$

## Conclusion

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# Conclusion

- We now have “natural” algebraic characterisations for functions over the reals computable in FPTIME and FPSPACE.
- Further works:
  - Going to **continuous** ODEs
  - Cover **other complexity classes**
  - See what we can say for **more general functions**