

Hypergraph simplification: Linking the path-sum approach to the ZH-calculus

Louis Lemonnier, John van de Wetering, Aleks Kissinger

June 6, 2020

QPL 2020

Path-sum approach

Path-sum¹ definition

Definition

The associated operator of a path-sum is:

$$\lambda \sum_{\mathbf{x}} e^{2\pi i \phi(\mathbf{x})} |\mathbf{x}_0\rangle \langle \mathbf{x}_1|$$

¹Matthew Amy, QPL 2018

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- Very easy to compose, very easy to \otimes
- Terms can algebraically simplify \rightarrow verification
- drawback: nearly impossible to get a circuit back

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$$\text{---} \boxed{H} \text{---} :: \frac{1}{\sqrt{2}} \sum_{x, y \in \mathbb{B}} e^{\pi i xy} |y\rangle \langle x|$$

¹Matthew Amy, QPL 2018

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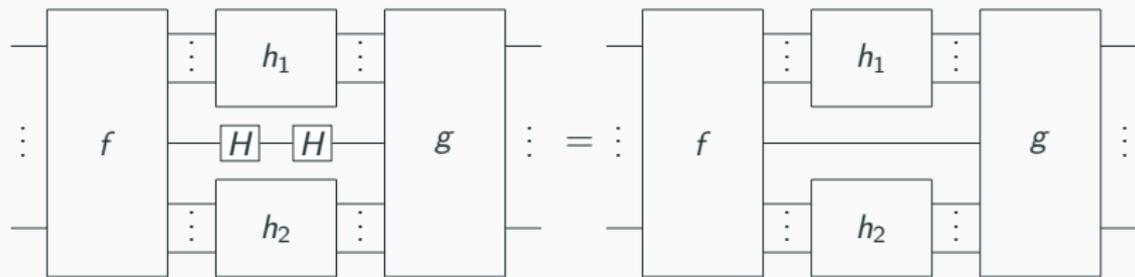
$$\lambda \sum_{\mathbf{x}} e^{2\pi i \phi(\mathbf{x})} |\mathbf{x}_0\rangle \langle \mathbf{x}_1|$$

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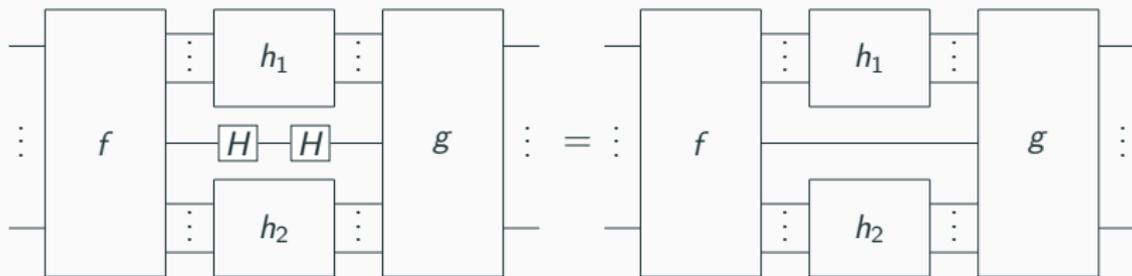
$$\text{---} \boxed{H} \boxed{H} \text{---} :: \frac{1}{2} \sum_{x,y,z \in \mathbb{B}} e^{\pi i(xy+yz)} |z\rangle \langle x|$$

¹Matthew Amy, QPL 2018

[HH] rule



[HH] rule



Resulting rule

$$\lambda \sum_{y_0, y_1, x} e^{2\pi i \left(\frac{1}{2} y_0 (y_1 + Q(x)) + R(y_1, x) \right)} |x_0\rangle \langle x_1|$$
$$\rightarrow 2\lambda \sum_x e^{2\pi i (R[y_1 \leftarrow \bar{Q}])(x)} |x_0\rangle \langle x_1|$$

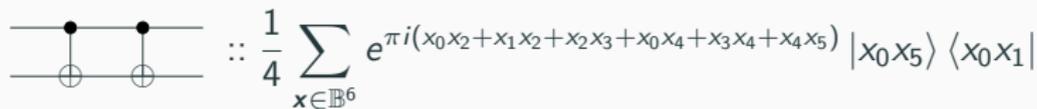
Example with [HH]

$$\begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \oplus \text{---} \end{array} \quad :: \quad \frac{1}{2} \sum_{\mathbf{x} \in \mathbb{B}^4} e^{\pi i (x_0 x_2 + x_1 x_2 + x_2 x_3)} |x_0 x_3\rangle \langle x_0 x_1|$$

Example with [HH]

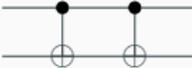

$$\therefore \frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

Example with [HH]


$$\text{Diagram} \quad :: \frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i (x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$\frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i (x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

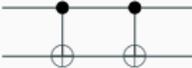
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$$\frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$\longrightarrow \frac{1}{2} \sum_{x_0, x_1, x_4, x_5} e^{\pi i(x_0 x_4 + x_0 x_4 + x_1 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

Example with [HH]


$$\text{Circuit} \quad :: \quad \frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$\frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

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$$= \frac{1}{2} \sum_{x_0, x_1, x_4, x_5} e^{\pi i(x_1 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

Example with [HH]

$$\begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} \bullet \\ \oplus \end{array} \quad :: \frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 x_2 + x_1 x_2 + x_2 x_3 + x_0 x_4 + x_3 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$\frac{1}{4} \sum_{\mathbf{x} \in \mathbb{B}^6} e^{\pi i(x_0 \color{red}{x_2} + x_1 \color{red}{x_2} + \color{red}{x_2} \color{red}{x_3} + x_0 x_4 + \color{red}{x_3} x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$\longrightarrow \frac{1}{2} \sum_{x_0, x_1, x_4, x_5} e^{\pi i(x_0 x_4 + x_0 x_4 + x_1 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1|$$

$$= \frac{1}{2} \sum_{x_0, x_1, x_4, x_5} e^{\pi i(x_1 x_4 + x_4 x_5)} |x_0 x_5\rangle \langle x_0 x_1| \longrightarrow \text{Id}$$

The other rules



The other rules

$[\omega]$ rule

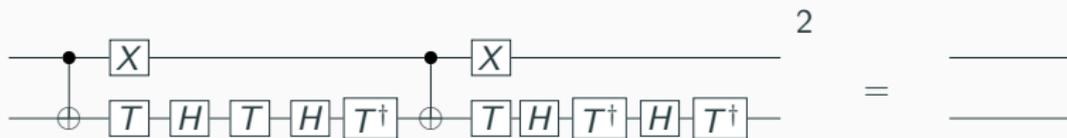
$$\lambda \sum_{y_0, \mathbf{x}} e^{2\pi i \left(\frac{1}{4} y_0 + \frac{1}{2} y_0 Q(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$
$$\longrightarrow \sqrt{2} \lambda \sum_{\mathbf{x}} e^{2\pi i \left(\frac{1}{8} - \frac{1}{4} \bar{Q}(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

The other rules

$[\omega]$ rule

$$\lambda \sum_{y_0, \mathbf{x}} e^{2\pi i \left(\frac{1}{4} y_0 + \frac{1}{2} y_0 Q(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

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The other rules

$[\omega]$ rule

$$\begin{aligned} & \lambda \sum_{y_0, \mathbf{x}} e^{2\pi i \left(\frac{1}{4} y_0 + \frac{1}{2} y_0 Q(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \\ & \longrightarrow \sqrt{2} \lambda \sum_{\mathbf{x}} e^{2\pi i \left(\frac{1}{8} - \frac{1}{4} \bar{Q}(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \end{aligned}$$

[Case] rule

$$\begin{aligned} & \lambda \sum_{y_0, y_1, \mathbf{x}} e^{2\pi i \left(\alpha y_0 X(\mathbf{x}) + \frac{1}{2} y_0 Q(\mathbf{x}) + \frac{1}{2} y_0 y_1 + \frac{1}{2} y_1 Q'(\mathbf{x}) + \beta y_1 (1 - X(\mathbf{x})) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \\ & \longrightarrow 2\lambda \sum_{\mathbf{x}} e^{2\pi i \left(\frac{1}{2} Q(\mathbf{x}) Q'(\mathbf{x}) + \alpha X(\mathbf{x}) \bar{Q}'(\mathbf{x}) + \beta (1 - X(\mathbf{x})) \bar{Q}(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \end{aligned}$$

ZH-calculus

Z-spider


$$:= |0 \dots 0\rangle \langle 0 \dots 0| + |1 \dots 1\rangle \langle 1 \dots 1|$$

²Miriam Backens, Aleks Kissinger, QPL 2018

Z-spider

$$\begin{array}{c} \overbrace{\quad\quad\quad}^n \\ \cdots \\ \text{---} \\ \text{---} \\ \cdots \\ \underbrace{\quad\quad\quad}_m \end{array} := |0 \dots 0\rangle \langle 0 \dots 0| + |1 \dots 1\rangle \langle 1 \dots 1|$$

H-spider

$$\begin{array}{c} \overbrace{\quad\quad\quad}^n \\ \cdots \\ \boxed{a} \\ \cdots \\ \underbrace{\quad\quad\quad}_m \end{array} := \sum a^{i_1 \dots i_m j_1 \dots j_n} |j_1 \dots j_n\rangle \langle i_1 \dots i_m|, \quad a \in \mathbb{C}$$

²Miriam Backens, Aleks Kissinger, QPL 2018

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H-spider

$$\begin{array}{c} \overbrace{\quad\quad\quad}^n \\ \cdots \\ \boxed{a} \\ \cdots \\ \underbrace{\quad\quad\quad}_m \end{array} := \sum a^{i_1 \dots i_m j_1 \dots j_n} |j_1 \dots j_n\rangle \langle i_1 \dots i_m|, \quad a \in \mathbb{C}$$

$$\begin{array}{c} \cdots \\ \text{---} \\ \text{---} \\ \cdots \end{array} := \begin{array}{c} \cdots \\ \text{---} \\ \boxed{-1} \\ \text{---} \\ \cdots \end{array}$$

²Miriam Backens, Aleks Kissinger, QPL 2018

Introduction to ZH-calculus²

Z-spider

$$\begin{array}{c} n \\ \dots \\ \text{---} \\ \text{---} \\ \dots \\ m \end{array} := |0 \dots 0\rangle \langle 0 \dots 0| + |1 \dots 1\rangle \langle 1 \dots 1|$$

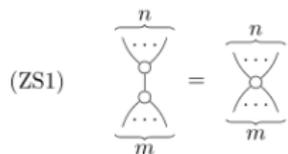
H-spider

$$\begin{array}{c} n \\ \dots \\ \boxed{a} \\ \dots \\ m \end{array} := \sum a^{i_1 \dots i_m j_1 \dots j_n} |j_1 \dots j_n\rangle \langle i_1 \dots i_m|, \quad a \in \mathbb{C}$$

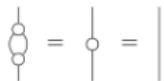
$$\begin{array}{c} \dots \\ \text{---} \\ \square \\ \text{---} \\ \dots \end{array} := \begin{array}{c} \dots \\ \text{---} \\ \boxed{-1} \\ \text{---} \\ \dots \end{array} \quad \begin{array}{c} \dots \\ \text{---} \\ \boxed{\alpha} \\ \text{---} \\ \dots \end{array} := \begin{array}{c} \dots \\ \text{---} \\ \boxed{e^{i\alpha}} \\ \text{---} \\ \dots \end{array}$$

²Miriam Backens, Aleks Kissinger, QPL 2018

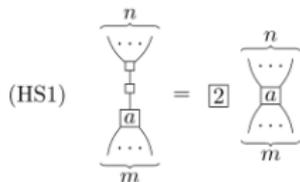
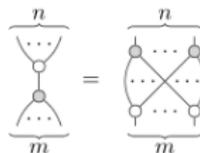
Set of rules



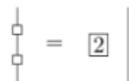
(ZS2)



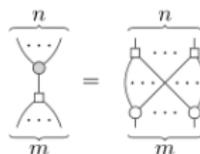
(BA1)



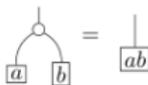
(HS2)



(BA2)



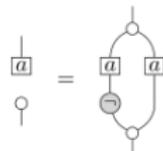
(M)



(U)



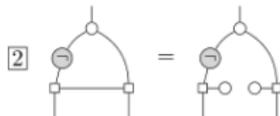
(I)



(A)



(O)



Translating

Why ZH-calculus?

The phase in a path-sum term is:

$$e^{2\pi i \phi(\mathbf{x})} =$$

Why ZH-calculus?

The phase in a path-sum term is:

$$e^{2\pi i \phi(\mathbf{x})} = e^{2\pi i \sum \alpha_j m_j(\mathbf{x})}$$

Why ZH-calculus?

The phase in a path-sum term is:

$$e^{2\pi i \phi(\mathbf{x})} = e^{2\pi i \sum \alpha_j m_j(\mathbf{x})} = \prod e^{2\pi i \alpha_j m_j(\mathbf{x})}$$

Why ZH-calculus?

The phase in a path-sum term is:

$$e^{2\pi i \phi(\mathbf{x})} = e^{2\pi i \sum \alpha_j m_j(\mathbf{x})} = \prod e^{2\pi i \alpha_j m_j(\mathbf{x})} = \prod (e^{2\pi i \alpha_j})^{m_j(\mathbf{x})}$$

Why ZH-calculus?

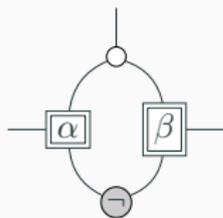
The phase in a path-sum term is:

$$e^{2\pi i \phi(\mathbf{x})} = e^{2\pi i \sum \alpha_j m_j(\mathbf{x})} = \prod e^{2\pi i \alpha_j m_j(\mathbf{x})} = \prod (e^{2\pi i \alpha_j})^{m_j(\mathbf{x})}$$

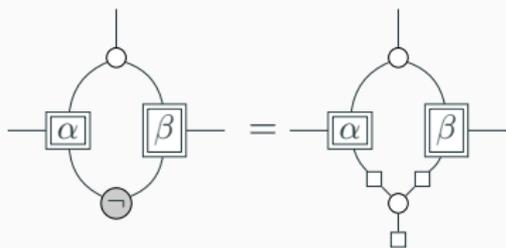
And remember that:

$$\underbrace{\underbrace{\underbrace{\alpha}_{\square}}_{\dots}}_{\dots}^{\dots} := \sum (e^{i\alpha})^{i_1 \dots i_m j_1 \dots j_n} |j_1 \dots j_n\rangle \langle i_1 \dots i_m|$$

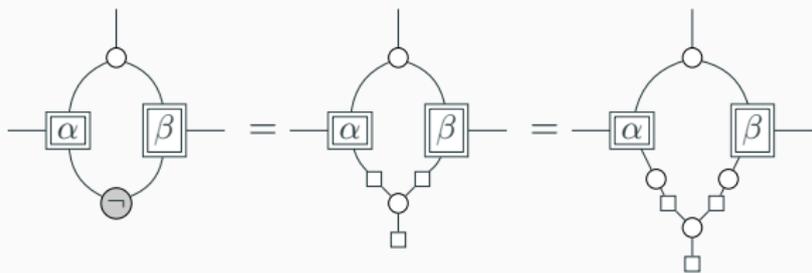
ZH into path-sum: How is it done?



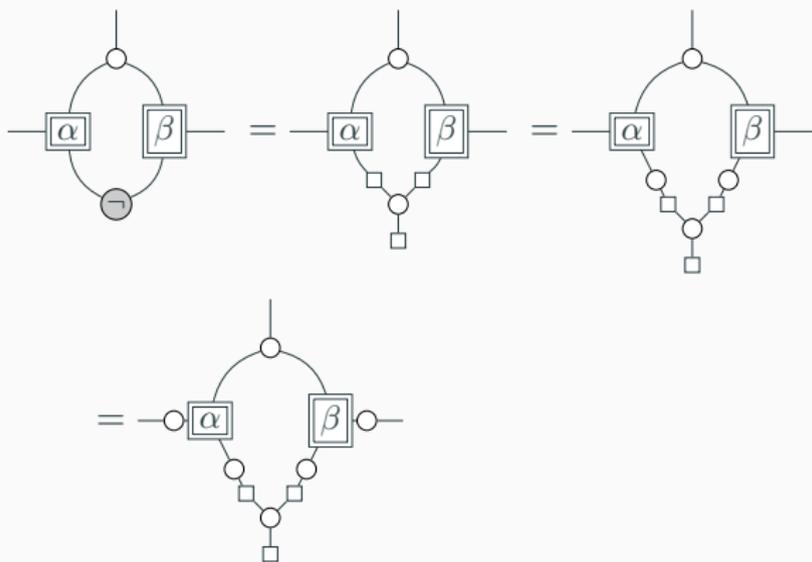
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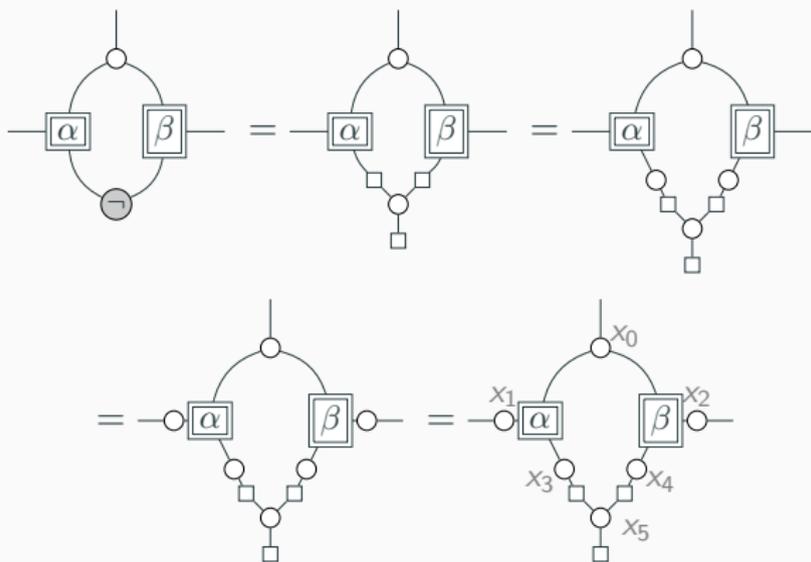


ZH into path-sum: How is it done?

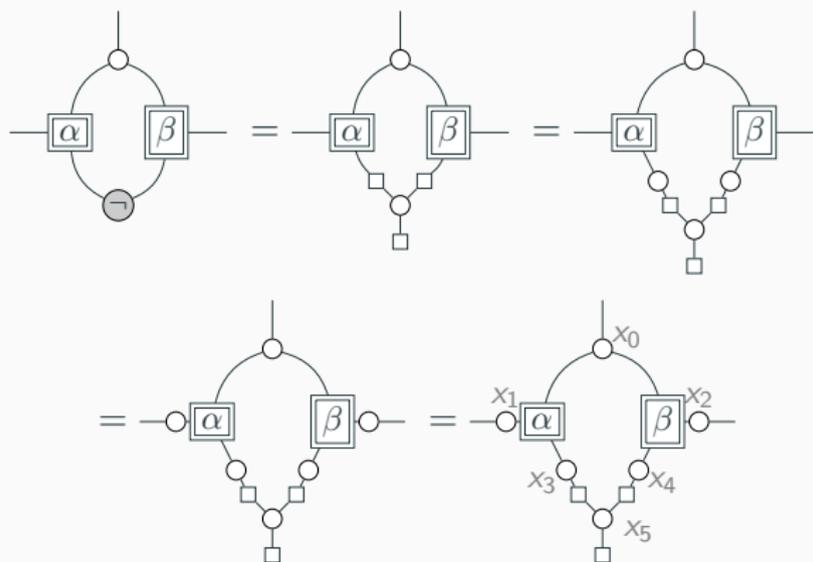


Hypergraph-like!

ZH into path-sum: How is it done?



ZH into path-sum: How is it done?



Corresponding path-sum term:

$$\frac{1}{\sqrt{32}} \sum_{\mathbf{x}} e^{\pi i(x_3 x_5 + x_4 x_5 + x_5)} e^{i\alpha x_0 x_1 x_3} e^{i\beta x_0 x_2 x_4} |x_0\rangle \langle x_1 x_2|$$

Bijection

- Each **hypergraph-like** diagram has a corresponding path-sum term

Bijection

- Each **hypergraph-like** diagram has a corresponding path-sum term
- Also the other way around

Bijection

- Each **hypergraph-like** diagram has a corresponding path-sum term
- Also the other way around
- Corresponding diagram and path-sum represent the same matrix operator

The power of [HH]

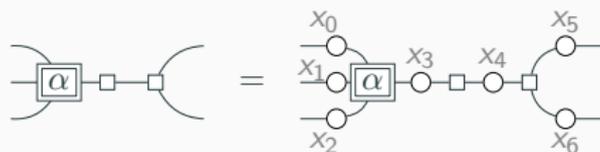


The power of [HH]

$$\text{---}\square\text{---}\square\text{---} = \text{---}\overset{x_0}{\circ}\text{---}\square\text{---}\overset{x_1}{\circ}\text{---}\square\text{---}\overset{x_2}{\circ}\text{---} :: \frac{1}{2} \sum_x e^{\pi i(x_0 x_1 + x_1 x_2)} |x_2\rangle \langle x_0| \longrightarrow \text{Id}$$

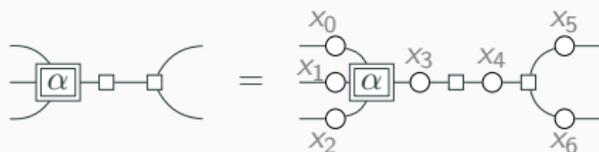
The power of [HH]

$$\text{---} \square \text{---} \square \text{---} = \text{---} \overset{x_0}{\circ} \text{---} \square \text{---} \overset{x_1}{\circ} \text{---} \square \text{---} \overset{x_2}{\circ} \text{---} \quad :: \quad \frac{1}{2} \sum_{\mathbf{x}} e^{\pi i(x_0 x_1 + x_1 x_2)} |x_2\rangle \langle x_0| \longrightarrow \text{Id}$$



The power of [HH]

$$\text{---} \square \text{---} \square \text{---} = \text{---} \overset{x_0}{\circ} \text{---} \square \text{---} \overset{x_1}{\circ} \text{---} \square \text{---} \overset{x_2}{\circ} \text{---} \quad \text{::} \quad \frac{1}{2} \sum_{\mathbf{x}} e^{\pi i(x_0 x_1 + x_1 x_2)} |x_2\rangle \langle x_0| \longrightarrow \text{Id}$$

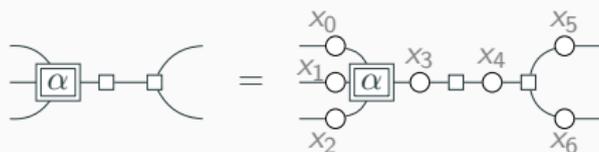


Corresponding path-sum term:

$$\frac{1}{\sqrt{8}} \sum_{\mathbf{x}} e^{\pi i(x_3 x_4 + x_4 x_5 x_6)} e^{i\alpha x_0 x_1 x_2 x_3} |x_5 x_6\rangle \langle x_0 x_1 x_2|$$

The power of [HH]

$$\text{---} \square \text{---} \square \text{---} = \text{---} \circ_{x_0} \square \circ_{x_1} \square \circ_{x_2} \text{---} \quad :: \quad \frac{1}{2} \sum_{\mathbf{x}} e^{\pi i(x_0 x_1 + x_1 x_2)} |x_2\rangle \langle x_0| \longrightarrow \text{Id}$$



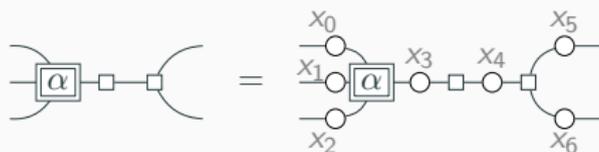
Corresponding path-sum term:

$$\frac{1}{\sqrt{8}} \sum_{\mathbf{x}} e^{\pi i(x_3 x_4 + x_4 x_5 x_6)} e^{i\alpha x_0 x_1 x_2 x_3} |x_5 x_6\rangle \langle x_0 x_1 x_2|$$

$$\longrightarrow \frac{1}{\sqrt{2}} \sum_{\mathbf{x}} e^{i\alpha x_0 x_1 x_2 x_3 x_5 x_6} |x_5 x_6\rangle \langle x_0 x_1 x_2|$$

The power of [HH]

$$\text{---} \square \text{---} \square \text{---} = \text{---} \circ_{x_0} \text{---} \square \text{---} \circ_{x_1} \text{---} \square \text{---} \circ_{x_2} \text{---} \quad \text{::} \quad \frac{1}{2} \sum_{\mathbf{x}} e^{\pi i(x_0 x_1 + x_1 x_2)} |x_2\rangle \langle x_0| \longrightarrow \text{Id}$$



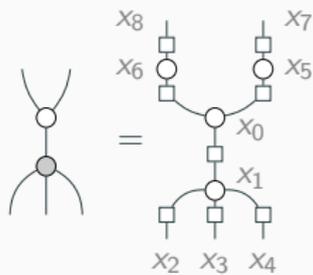
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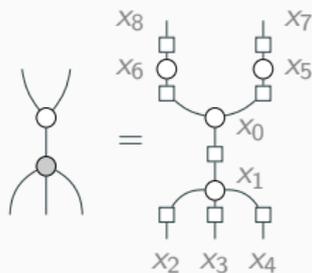
$$\longrightarrow \frac{1}{\sqrt{2}} \sum_{\mathbf{x}} e^{i\alpha x_0 x_1 x_2 x_3 x_5 x_6} |x_5 x_6\rangle \langle x_0 x_1 x_2| \quad \text{::} \quad \text{---} \square \text{---}$$

The diagram shows a box labeled α with four external lines extending outwards, forming a cross shape. This represents the final term in the path-sum expansion.

The power of [HH]

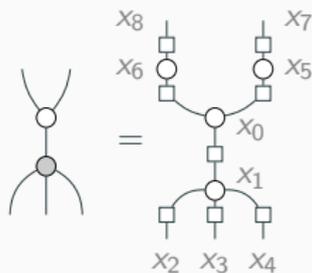


The power of [HH]



$$\frac{1}{16} \sum_{\mathbf{x}} e^{\pi i (x_0(x_1+x_5+x_6)+x_1(x_2+x_3+x_4)+x_5x_7+x_6x_8)} |x_7x_8\rangle \langle x_2x_3x_4|$$

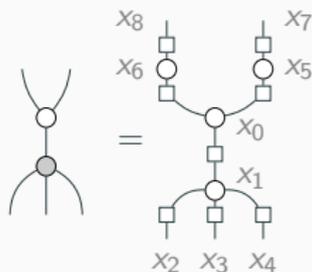
The power of [HH]



$$\frac{1}{16} \sum_{\mathbf{x}} e^{\pi i (x_0(x_1 + x_5 + x_6) + x_1(x_2 + x_3 + x_4) + x_5 x_7 + x_6 x_8)} |x_7 x_8\rangle \langle x_2 x_3 x_4|$$

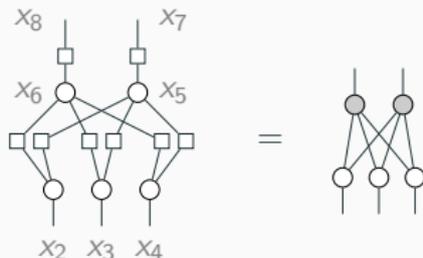
$$\longrightarrow \frac{1}{8} \sum_{\mathbf{x}} e^{\pi i (x_5(x_2 + x_3 + x_4 + x_7) + x_6(x_2 + x_3 + x_4 + x_8))} |x_7 x_8\rangle \langle x_2 x_3 x_4|$$

The power of [HH]



$$\frac{1}{16} \sum_{\mathbf{x}} e^{\pi i (x_0(x_1+x_5+x_6)+x_1(x_2+x_3+x_4)+x_5x_7+x_6x_8)} |x_7x_8\rangle \langle x_2x_3x_4|$$

$$\rightarrow \frac{1}{8} \sum_{\mathbf{x}} e^{\pi i (x_5(x_2+x_3+x_4+x_7)+x_6(x_2+x_3+x_4+x_8))} |x_7x_8\rangle \langle x_2x_3x_4|$$



The power of [HH] summarised

- H-spider cancellation

The power of [HH] summarised

- H-spider cancellation
- H-spider fusion

The power of [HH] summarised

- H-spider cancellation
- H-spider fusion
- ZX bialgebra \rightarrow pivot

The power of [HH] summarised

- H-spider cancellation
- H-spider fusion
- ZX bialgebra \longrightarrow pivot
- ZH bialgebra \longrightarrow hyper-pivot?

Path-sum rules into ZH

Definition



Definition



Annotated and overlapped

Definition



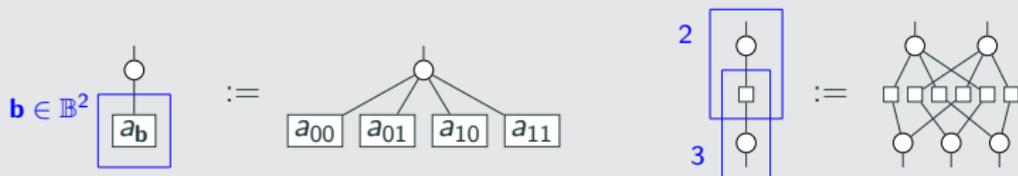
Annotated and overlapped



Definition



Annotated and overlapped



Picturing the [HH] rule: Hyper pivot

$$\lambda \sum_{y_0, y_1, \mathbf{x}} e^{\pi i (y_0 (y_1 + Q(\mathbf{x})) + y_1 R(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \longrightarrow 2\lambda \sum_{\mathbf{x}} e^{\pi i Q(\mathbf{x}) R(\mathbf{x})} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

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With $R(\mathbf{x}) = \bigoplus_{i=1}^m m_i(\mathbf{x})$ and $Q(\mathbf{x}) = \bigoplus_{j=1}^n m'_j(\mathbf{x})$

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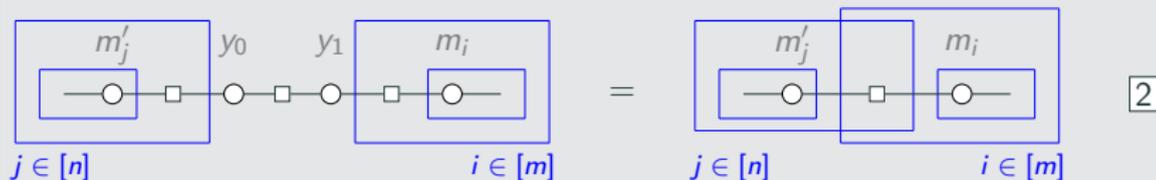


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Theorem

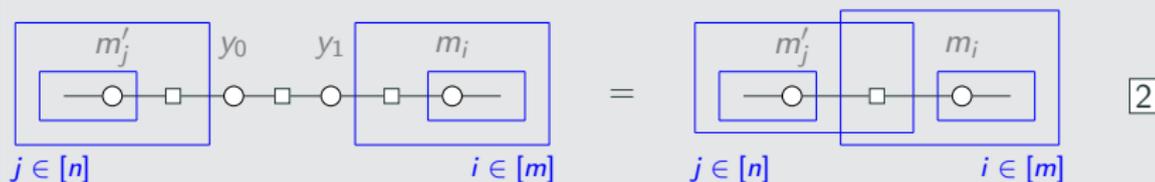


Picturing the [HH] rule: Hyper pivot

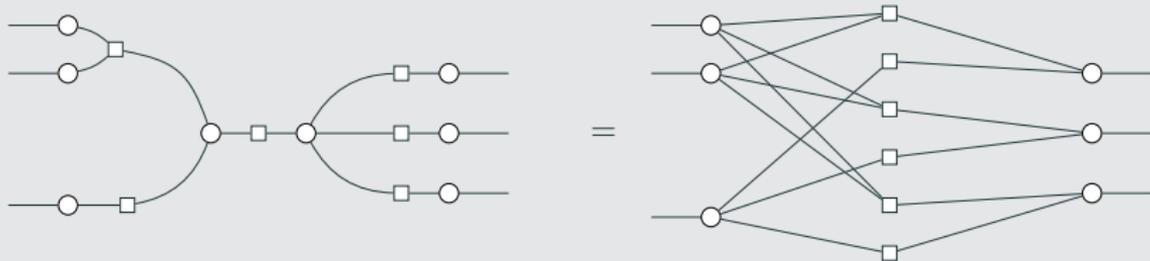
$$\lambda \sum_{y_0, y_1, \mathbf{x}} e^{\pi i (y_0 (y_1 + Q(\mathbf{x})) + y_1 R(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \longrightarrow 2\lambda \sum_{\mathbf{x}} e^{\pi i Q(\mathbf{x}) R(\mathbf{x})} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

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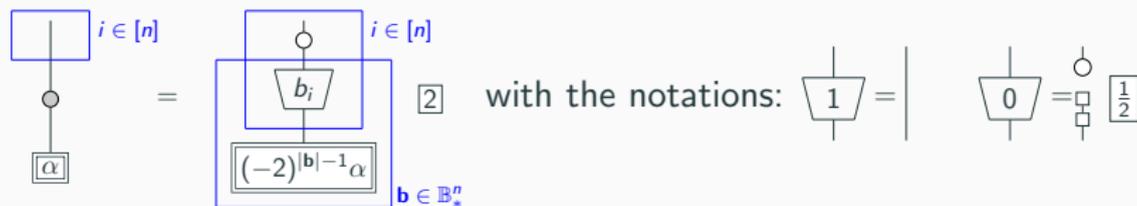


Example



Some useful tools

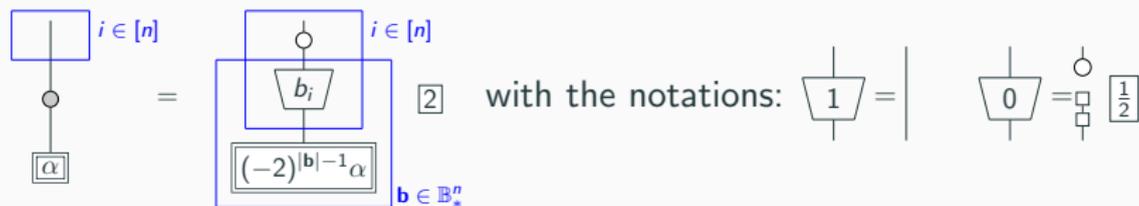
Graphical Fourier transform³:



³Stach Kuijpers, John van de Wetering, Aleks Kissinger, arXiv:1904.07551

Some useful tools

Graphical Fourier transform³:

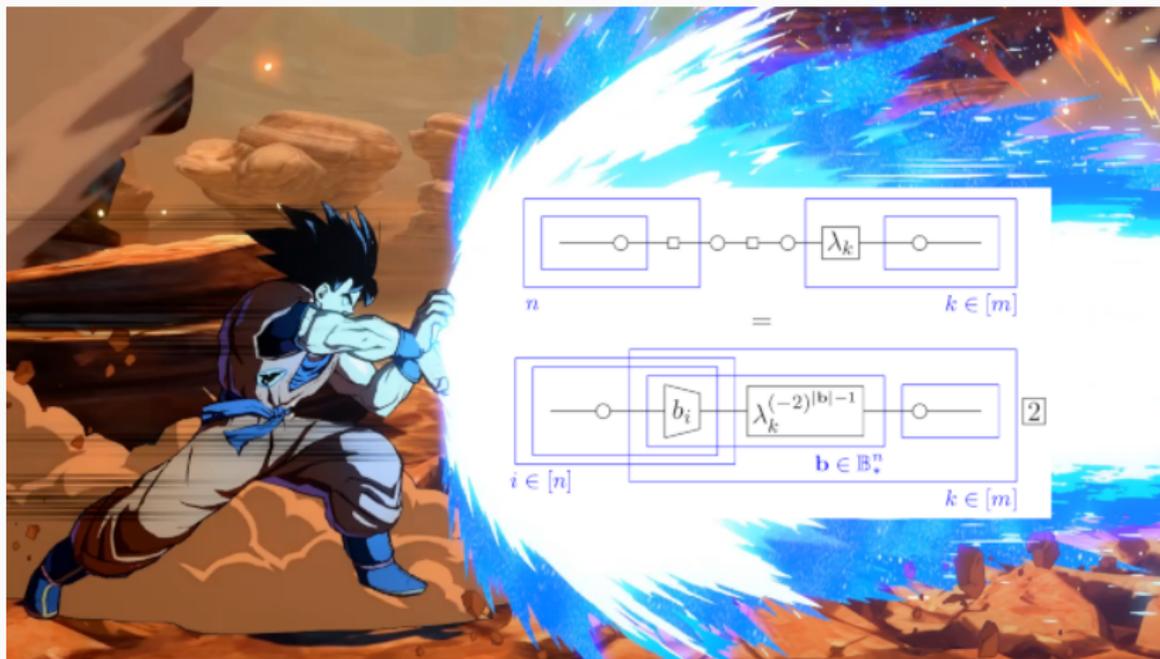


Featuring:



³Stach Kuijpers, John van de Wetering, Aleks Kissinger, arXiv:1904.07551

Fourier Hyper pivot



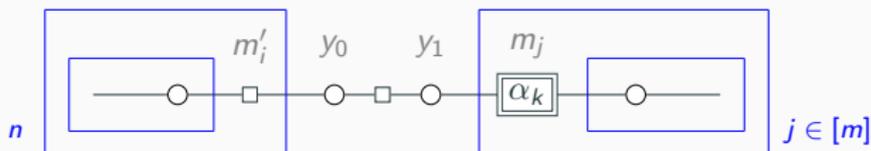
Picturing the [HH] rule: Fourier hyper pivot

$$\lambda \sum_{y_0, y_1, \mathbf{x}} e^{\pi i(y_0(y_1+Q(\mathbf{x}))+S(\mathbf{x})y_1+T(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$
$$\longrightarrow 2\lambda \sum_{\mathbf{x}} e^{2\pi i(S(\mathbf{x})\bar{Q}(\mathbf{x})+T(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

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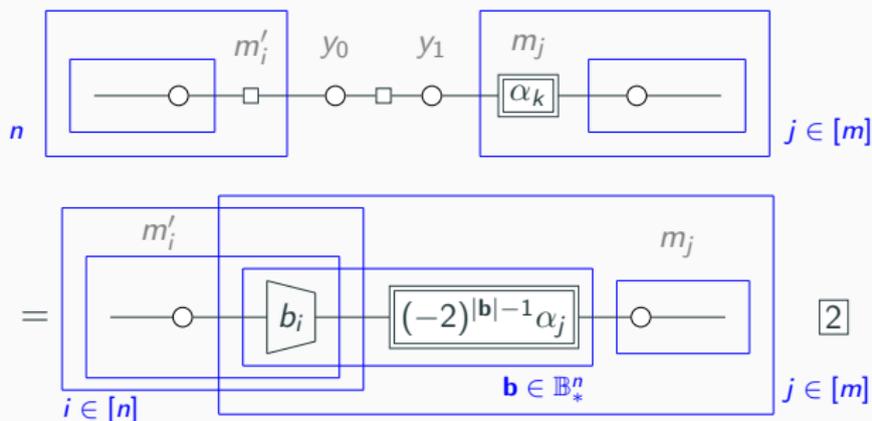
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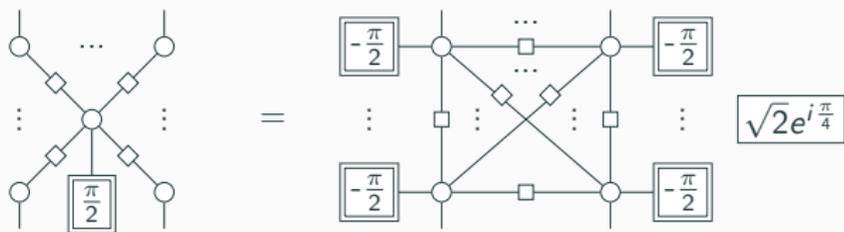


Picturing the $[\omega]$ rule: Hyper local complementation

$$\lambda \sum_{y_0, \mathbf{x}} e^{2\pi i \left(\frac{1}{4} y_0 + \frac{1}{2} y_0 Q(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i| \longrightarrow \sqrt{2} \lambda \sum_{\mathbf{x}} e^{2\pi i \left(\frac{1}{8} - \frac{1}{4} \bar{Q}(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

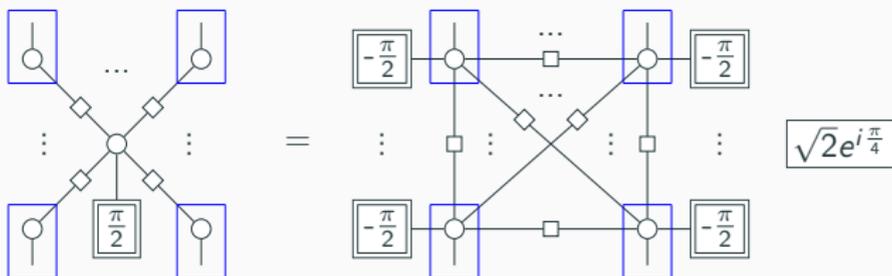
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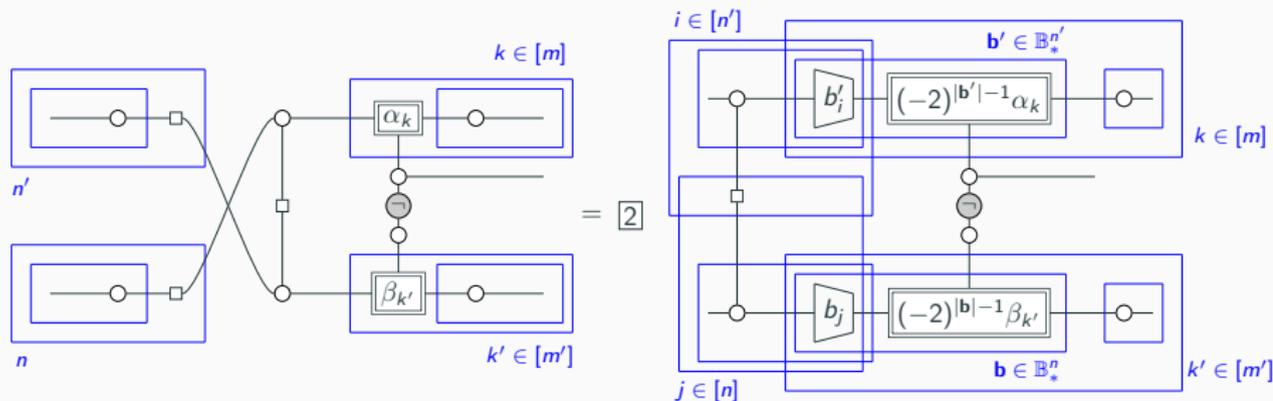
Case hyper pivot

$$\lambda \sum_{y_0, y_1, \mathbf{x}} e^{2\pi i \left(\alpha y_0 X(\mathbf{x}) + \frac{1}{2} y_0 Q(\mathbf{x}) + \frac{1}{2} y_0 y_1 + \frac{1}{2} y_1 Q'(\mathbf{x}) + \beta y_1 (1 - X(\mathbf{x})) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$
$$\longrightarrow 2\lambda \sum_{\mathbf{x}} e^{2\pi i \left(\frac{1}{2} Q(\mathbf{x}) Q'(\mathbf{x}) + \alpha X(\mathbf{x}) \overline{Q}'(\mathbf{x}) + \beta (1 - X(\mathbf{x})) \overline{Q}(\mathbf{x}) + R(\mathbf{x}) \right)} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

Case hyper pivot

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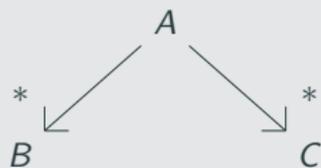
$$\longrightarrow 2\lambda \sum_{\mathbf{x}} e^{2\pi i(\frac{1}{2} Q(\mathbf{x}) Q'(\mathbf{x}) + \alpha X(\mathbf{x}) \overline{Q'}(\mathbf{x}) + \beta (1 - X(\mathbf{x})) \overline{Q}(\mathbf{x}) + R(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$



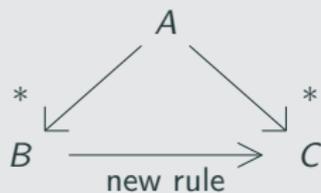
Conclusion and future work

In search of new rules

Critical pairs (such as [Case])

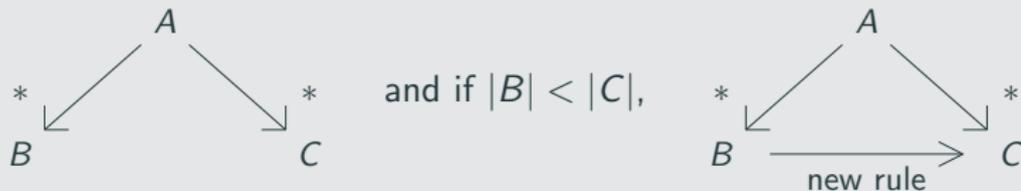


and if $|B| < |C|$,

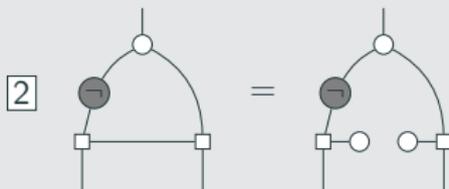


In search of new rules

Critical pairs (such as [Case])



Ortho rule



$$\lambda \sum_{y_0, y_1, \mathbf{x}} e^{2\pi i (y_0 X(\mathbf{x})R(\mathbf{x}) + y_1 (1-X(\mathbf{x}))R'(\mathbf{x}) + R''(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$

$$\longrightarrow 2\lambda \sum_{y_0, \mathbf{x}} e^{2\pi i (y_0 X(\mathbf{x})R(\mathbf{x}) + y_0 (1-X(\mathbf{x}))R'(\mathbf{x}) + R''(\mathbf{x}))} |\mathbf{x}_0\rangle \langle \mathbf{x}_i|$$