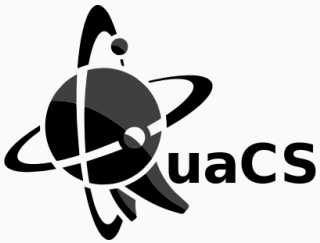
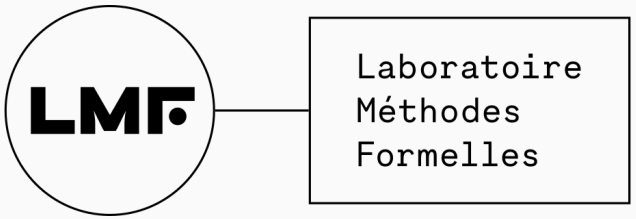
The Syntax and Semantics of Simply-typed Quantum Control

Louis Lemonnier

Acknowledgements to: Kostia Chardonnet, Robin Kaarsgaard, Benoît Valiron, Vladimir Zamdzhiev





Équipe MOCQUA. 30th May 2024

What is Quantum Control?

- Flow of the program controlled by pure quantum data.
- No use of measurement to control the flow.

```
input:  |xy>

if x:
        |xy>
else:
        |x(1−y)>
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        |y>
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if x:                          if meas(x):
        |xy>                           |y>
else:                          else:
        |x(Uy)>                        |Uy>
```

(My) Quantum Control: work with reversible quantum operations.

They are more known as unitaries.

Is this approach new?

The syntax

Types:

$$\mathtt{I} \qquad A \oplus B \qquad A \otimes B \qquad \mathtt{Nat}$$

Terms and semantics as isometries:

The syntax

Types:

$$\boxed{\mathtt{I} \qquad A \oplus B} \qquad A \otimes B \qquad \mathtt{Nat}$$

Terms and semantics as isometries:

Syntax	$\mathbb{C}^n$	
$*\colon \mathtt{I}$	$\begin{pmatrix} 1 \end{pmatrix}$	1
$\mathtt{left}\ *\colon \mathtt{I} \oplus \mathtt{I}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\lvert 0 \rangle$
$\mathtt{right}\ *\colon \mathtt{I} \oplus \mathtt{I}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\lvert 1 \rangle$
$(\alpha \cdot \mathtt{left}\ *) + (\beta \cdot \mathtt{right}\ *)\colon \mathtt{I} \oplus \mathtt{I}$	$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$	$\alpha \lvert 0 \rangle + \beta \lvert 1 \rangle$

The syntax

Types:

$$\mathtt{I} \qquad A \oplus B \qquad \boxed{A \otimes B} \qquad \mathtt{Nat}$$

Terms and semantics as isometries:

	Syntax	$\mathbb{C}^n$	
t_i	$t_i \colon \mathtt{I} \oplus \mathtt{I}$	$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$	$\lvert\phi_i\rangle$
$t_1 \otimes t_2$	$t_1 \otimes t_2 \colon (\mathtt{I} \oplus \mathtt{I}) \otimes (\mathtt{I} \oplus \mathtt{I})$	$\begin{pmatrix} \alpha_1\alpha_2 \\ \beta_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\beta_2 \end{pmatrix}$	$\lvert\phi_1\rangle \otimes \lvert\phi_2\rangle$

The syntax

Types:

$$\mathtt{I} \qquad A \oplus B \qquad A \otimes B \qquad \boxed{\mathtt{Nat}}$$

Terms and semantics as isometries:

Syntax	$\ell^2(\mathbb{N})$	
$\mathtt{zero} : \mathtt{Nat}$	e_0	$\lvert\underline{0}\rangle$
$t : \mathtt{Nat}$	e_n	$\lvert\underline{n}\rangle$
$\mathtt{S}\ t : \mathtt{Nat}$	e_{n+1}	$\lvert\underline{n+1}\rangle$

The syntax

Types:

$$\mathtt{I} \qquad A \oplus B \qquad A \otimes B \qquad \mathtt{Nat} \qquad \boxed{\mu X.A}$$

Terms and semantics as isometries:

Syntax	$\bigoplus_{n=0}^{\infty} H^{\otimes n}$
$t\colon A[\mu X.A/X]$	$\star$
$\mathtt{fold}\ t\colon \mu X.A$	$\star$

Unitaries

Can be seen as λ-abstractions with pattern-matching.

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$$\left\{ \begin{array}{lll} (\texttt{left}\ *) \otimes x & \mapsto & (\texttt{left}\ *) \otimes x \\ (\texttt{right}\ *) \otimes (\texttt{left}\ *) & \mapsto & (\texttt{right}\ *) \otimes (\texttt{right}\ *) \\ (\texttt{right}\ *) \otimes (\texttt{right}\ *) & \mapsto & (\texttt{right}\ *) \otimes (\texttt{left}\ *) \end{array} \right\}$$

Semantics of unitaries

Operationally: $\mathit{CNOT} = \left\{ \begin{array}{lcl} (\texttt{left}\ *) \otimes x & \mapsto & (\texttt{left}\ *) \otimes x \\ (\texttt{right}\ *) \otimes (\texttt{left}\ *) & \mapsto & (\texttt{right}\ *) \otimes (\texttt{right}\ *) \\ (\texttt{right}\ *) \otimes (\texttt{right}\ *) & \mapsto & (\texttt{right}\ *) \otimes (\texttt{left}\ *) \end{array} \right\}$

$$\mathit{CNOT}((\tfrac{1}{\sqrt{2}}\texttt{left}\ * + \tfrac{1}{\sqrt{2}}\texttt{right}\ *) \otimes (\texttt{left}\ *))$$

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$$\begin{array}{rl} & \mathit{CNOT}((\frac{1}{\sqrt{2}}\texttt{left}\ * + \frac{1}{\sqrt{2}}\texttt{right}\ *) \otimes (\texttt{left}\ *)) \\ \rightarrow & \mathit{CNOT}(\frac{1}{\sqrt{2}}(\texttt{left}\ *) \otimes (\texttt{left}\ *) + \frac{1}{\sqrt{2}}(\texttt{right}\ *) \otimes (\texttt{left}\ *)) \end{array}$$

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$$

Mathematically:

$$
\left[\!\!\left[\vdash \left\{ \begin{array}{ccc} t_1 & \mapsto & t'_1 \\ & \vdots & \\ t_m & \mapsto & t'_m \end{array} \right\} : A \leftrightarrow B \right]\!\!\right] = \sum_{i=1}^{m} [\![\Delta \vdash t'_i : B]\!] \circ [\![\Delta \vdash t_i : A]\!]^\dagger
$$

How do we ensure unitarity?

A syntactic notion of orthonormal basis (ONB).

In linear algebra, a unitary transforms an ONB into an ONB.

$$\left\{ \begin{array}{ccc} t_1 & \mapsto & t'_1 \\ & \vdots & \\ t_m & \mapsto & t'_m \end{array} \right\}$$

This is a unitary if both $\{t_1\}_i$ and $\{t'_j\}_j$ are ONBs.

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$$ONB_A\{x\} \qquad ONB_{A\otimes B}\{x \otimes y\}$$

$$ONB_{\mathrm{I}\oplus\mathrm{I}}\{\texttt{left}\ *, \texttt{right}\ *\}$$

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$$ONB_{\mathtt{Nat}}\{\mathtt{zero}, \mathtt{S}\ \mathtt{zero}, \mathtt{S}\ \mathtt{S}\ x\}$$

$$ONB_{\mathtt{I}\oplus\mathtt{I}}\{\tfrac{1}{\sqrt{2}}\mathtt{zero} + \tfrac{1}{\sqrt{2}}\mathtt{S}\ \mathtt{zero}, \quad \tfrac{1}{\sqrt{2}}\mathtt{zero} - \tfrac{1}{\sqrt{2}}\mathtt{S}\ \mathtt{zero}, \quad \mathtt{S}\ \mathtt{S}\ x\}$$

Properties

With a suitable equational theory (cf. my thesis).

Theorem

If $\Delta \vdash t\colon A$ is well-typed and $ONB_A(\{t_i\})$ then $\Delta \vdash t = \sum_i \alpha_i t_i\colon A$.

It implies that there is a unique normal form. And thus:

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Theorem (Completeness)

$$\cdot \vdash t_1 = t_2 : A \quad \text{iff} \quad [\![\cdot \vdash t_1 : A]\!] = [\![\cdot \vdash t_2 : A]\!].$$

With a denotational semantics in Hilbert spaces and isometries.

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With a denotational semantics in Hilbert spaces and isometries.

Other cool result: our syntactic ONBs are realisations of the identity.

If $ONB_A(S)$, then $\sum_{t \in S} [\![t]\!] \circ [\![t]\!]^\dagger = \mathrm{id}$.

What about Higher-order?

Comparison with the λ-calculus is limited.

In a unitary $\{t_i \mapsto t'_i\}$, the components t_i and t'_i are ground terms.

Cannot take a unitary as an input and cannot output a unitary.

Need to add a new form of (higher-order) functions.

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We have a λ-calculus at the level of unitaries.

What does Higher-order achieve?

A general control operation.

$$\lambda\phi.\left\{\begin{array}{lcl} (\texttt{left}\ *) \otimes x & \mapsto & (\texttt{left}\ *) \otimes x \\ (\texttt{right}\ *) \otimes x & \mapsto & \texttt{let}\, y = \phi\ x\ \texttt{in} \\ & & (\texttt{right}\ *) \otimes y \end{array}\right\}$$

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The quantum switch.

$$\lambda\phi.\lambda\psi.\left\{\begin{array}{lcl} (\texttt{left}\ *)\otimes x & \mapsto & \texttt{let}\, y = \phi\ x\ \texttt{in} \\ & & \texttt{let}\, z = \psi\ y\ \texttt{in} \\ & & (\texttt{left}\ *)\otimes z \\ (\texttt{right}\ *)\otimes x & \mapsto & \texttt{let}\, y = \psi\ x\ \texttt{in} \\ & & \texttt{let}\, z = \phi\ y\ \texttt{in} \\ & & (\texttt{right}\ *)\otimes z \end{array}\right\}$$

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Conjecture

Completeness still holds with higher-order.

What is next?

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What is next?

Recursion!

With a fixed point combinator.

$$\texttt{fix}\ \phi.\,\{\ldots\}$$

Quick stop: Recursion in Classical Reversibility

Suppose we have a syntax with:

- Only reversible functions,
- Inductive types,
- A fixpoint operator,
- A CbV operational semantics.

It is Turing-complete! (proven by Kostia Chardonnet)

Sound and adequate denotational semantics in partial injective functions.

Given two sets A and B, the set $[A \rightharpoonup B]$ has nice properties for fixpoints.

Read about it: [CLV23].

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(The category of partial injective functions between sets is enriched in pointed dcpos.)

Read about it: [CLV23].

Until we meet again

Reasons to believe that *general* quantum recursion does not hold.

In Hilbert spaces: no enrichment (some details?), no trace (conjecture).

Many questions remain. Is quantum recursion not possible?

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Thank you!

Come to my defence on 19th June, 2pm.

Domain theory (for babies)

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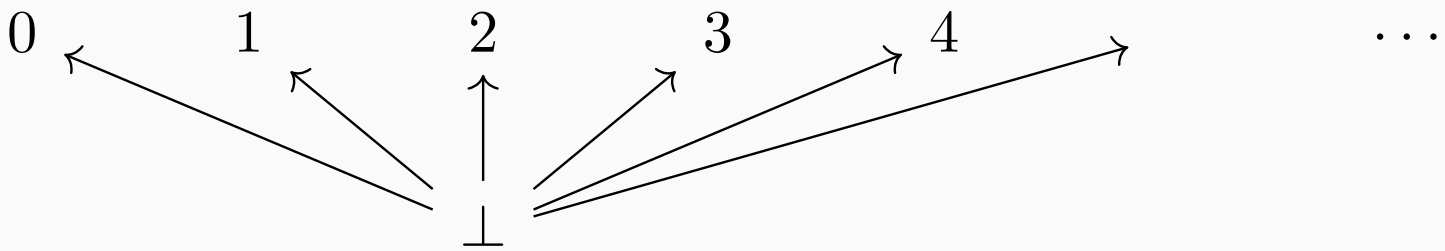
$\bot$ means undefined.



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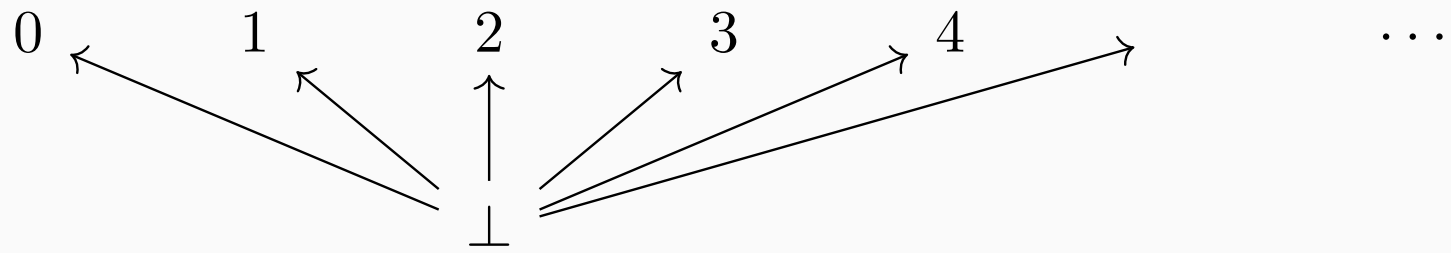

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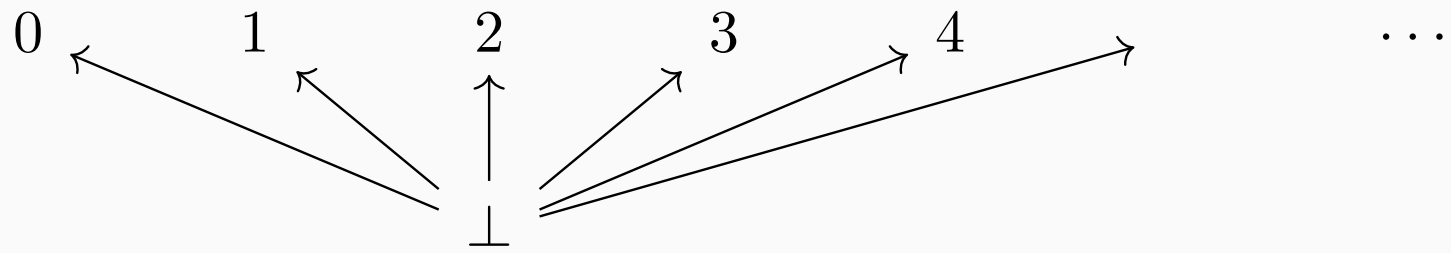

The set $[D \to D']$ of monotone functions also has nice properties.

How do we use it?

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \to \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \to \mathbb{N}_\perp$ the constant function with output $\perp$.
$\perp : \mathbb{N}_\perp \to \mathbb{N}_\perp$ is:

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Interpretation of recursion

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A notion of limit linked to the order is necessary.

Quick stop: Recursion in Classical Reversibility

Read about it: [CLV23].

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Suppose we have a syntax with:

- Only reversible functions,
- Inductive types,
- A fixpoint operator,
- A CbV operational semantics.

It is Turing-complete! (proven by Kostia Chardonnet)

Sound and adequate denotational semantics in partial injective functions.

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(The category of partial injective functions between sets is enriched in pointed dcpos.)

Read about it: [CLV23].

The Quantum Case

A unitary operator f is such that the inverse of f is its adjoint.

Dimension 2 for simple examples.

Reversible quantum operations are subunitaries such as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is less defined than $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which is less defined than $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A disappointment

However, this order does not interact well with composition:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

thus $$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \leq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

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Theorem (No go theorem)

There is no such order on subunitary maps between Hilbert spaces.