

A Recipe for the Semantics of Reversible Programming

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THE UNIVERSITY *of* EDINBURGH
informatics

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Reversible Programming

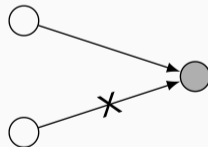
Originally

- Landauer and Bennett, 1961: Reversible Computation and Energy Dissipation.
- Reversible programs: for a program t , there is t^{-1} such that $t; t^{-1} = \text{skip}$.
- Applications to quantum computing.

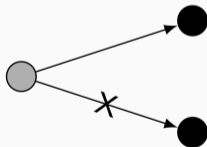
What we do

- Reversibility, but **not totality**.
- Syntax for reversible functions.
- Through the categorical semantics.

Backward determinism



Forward determinism



[Kaarsgaard&Rennela21]

A general framework: dagger categories

Origine: functional analysis where $\langle fx | y \rangle = \langle x | f^\dagger y \rangle$.

Category \mathbf{C} equipped with a functor $(-)^{\dagger} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{C}$, such that:

- On objects, $A^{\dagger} = A$.
- On morphisms:
 - $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$,
 - $f^{\dagger\dagger} = f$.

A very important class of morphisms: *partial \dagger -isomorphism*. $ff^{\dagger}f = f$.

Example with partial injective functions between sets, here $\{0, 1\}$.

$$\text{not: } \begin{array}{ccc} 0 & \begin{array}{c} \searrow \\ \nearrow \end{array} & 0 \\ 1 & \begin{array}{c} \nearrow \\ \searrow \end{array} & 1 \end{array} \quad g: \begin{array}{ccc} 0 & \searrow & 0 \\ 1 & \searrow & 1 \end{array} \quad g^{\dagger}: \begin{array}{ccc} 0 & \nearrow & 0 \\ 1 & \nearrow & 1 \end{array}$$

Examples of relevant dagger categories

Sets and bijections.

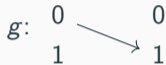


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Sets and bijections.



Sets and partial injections.



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Hilbert spaces and unitary maps. $\begin{array}{ccc} |0\rangle & \longrightarrow & |+\rangle \\ |1\rangle & \longrightarrow & |-\rangle \end{array}$ where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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Hilbert spaces and contractions. $h: \begin{array}{ccc} |0\rangle & \longrightarrow & |+\rangle \\ |1\rangle & \longrightarrow & |-\rangle \end{array}$ $h|1\rangle = 0$.

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Hopefully, there is a way to **cheat**.

Our new functions

We can **cheat** with our partial inverse!

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Given $\Delta \vdash t : A$ $\Delta \vdash t' : B$

We form a function $t \mapsto t' : A \leftrightarrow B$, Whose semantics is

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Together with pattern-matching

With a sum type \oplus :

$$\frac{\Delta \vdash t: A}{\Delta \vdash \text{inj}_l t: A \oplus B}$$

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$$\frac{}{\text{inj}_l t_1 \perp \text{inj}_r t_2} \quad \frac{t_1 \perp t_2}{C[t_1] \perp C[t_2]} \quad \text{with } \begin{cases} \llbracket t_1 \rrbracket^\dagger \circ \llbracket t_1 \rrbracket = \text{id} \\ \llbracket t_1 \rrbracket^\dagger \circ \llbracket t_2 \rrbracket = 0 \end{cases} \quad \text{when } t_1 \perp t_2$$

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Our functions are then:

$$\left\{ \begin{array}{l} t_1 \mapsto t'_1 \\ t_2 \mapsto t'_2 \\ \vdots \\ t_m \mapsto t'_m \end{array} \right\} : A \leftrightarrow B$$

whenever $\Delta_i \vdash t_i: A$ and $t_j \perp t_k$, $\Delta_i \vdash t'_i: B$ and $t'_j \perp t'_k$.

Example and semantics

$$\left\{ \begin{array}{l} \text{inj}_l x \mapsto \text{inj}_r x \\ \text{inj}_r y \mapsto \text{inj}_l y \end{array} \right\} : A \oplus B \leftrightarrow B \oplus A$$

Denotational semantics:

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 - ◆ **Compatible** morphisms on their domain and codomain admit a join.
 - ◆ Sometimes, provides a nice structure on morphisms.

Examples:

- Sets and partial injective functions **Pinj**.
- Hilbert spaces and contractions **Contr**.

The case of inverse categories (such as \mathbf{PInj})

To Infinity and Beyond

Some reading: [Axelsen&Kaarsgaard16] + [Fiore04] + some calculations.

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→ A suitable inverse category \mathbf{C} is **parameterised DCPO-algebraically ω -compact**.

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And we want to parse those infinite types:

$$\text{map}(\omega) = \text{fix } f. \left\{ \begin{array}{l} [] \mapsto [] \\ h :: t \mapsto (\omega h) :: (f t) \end{array} \right\} : [A] \leftrightarrow [B]$$

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Summary of the language (mandatory slide)

(Ground types) $A, B ::= I \mid A \oplus B \mid A \otimes B \mid$

(Function types) $T_1, T_2 ::= A \leftrightarrow B \mid$

(Unit term) $t, t_1, t_2 ::= *$

(Pairing) $\mid t_1 \otimes t_2$

(Injections) $\mid \text{inj}_l t \mid \text{inj}_r t$

(Function application) $\mid \omega t$

(Abstraction) $\omega ::= \{t_1 \mapsto t'_1 \mid \dots \mid t_m \mapsto t'_m\}$

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(Function types)	T_1, T_2	$::=$	$A \leftrightarrow B \mid$
(Unit term)	t, t_1, t_2	$::=$	$*$
(Pairing)			$\mid t_1 \otimes t_2$
(Injections)			$\mid \text{inj}_l t \mid \text{inj}_r t$
(Function application)			$\mid \omega t$
(Inductive terms)			$\mid \text{fold } t$
(Abstraction)	ω	$::=$	$\{t_1 \mapsto t'_1 \mid \dots \mid t_m \mapsto t'_m\}$
(Fixed points)			$\mid f \mid \text{fix } f.\omega$

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(Ground types)	A, B	$::=$	$I \mid A \oplus B \mid A \otimes B \mid X \mid \mu X.A$
(Function types)	T_1, T_2	$::=$	$A \leftrightarrow B \mid T_1 \rightarrow T_2$
(Unit term)	t, t_1, t_2	$::=$	$*$
(Pairing)			$\mid t_1 \otimes t_2$
(Injections)			$\mid \text{inj}_l t \mid \text{inj}_r t$
(Function application)			$\mid \omega t$
(Inductive terms)			$\mid \text{fold } t$
(Abstraction)	ω	$::=$	$\{t_1 \mapsto t'_1 \mid \dots \mid t_m \mapsto t'_m\}$
(Fixed points)			$\mid f \mid \text{fix } f.\omega$
(Higher abstractions)			$\mid \lambda f.\omega \mid \omega_2 \omega_1$

λ -calculus thanks to **DCPO**-enrichment.

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← ask this guy (Kostia Chardonnet, currently works in Nancy)

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———— Roughly ————

- Reversible Turing Machines [Axelsen&Glück11].
 - ◆ Simulate your favourite Turing machines.
- Encode RTMs in our language:
 - ◆ Alphabet & states mapped to $I \oplus \dots \oplus I$.
 - ◆ Tape as lists.
 - ◆ Functions simulating one-step transition of δ .
 - ◆ Iterate until final state.

A variety of diverging functions

Usual

$$\text{fix } f^T . f^T : T$$

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Less usual

$$\text{fix } f^{A \leftrightarrow B} . \{x \mapsto f x\} : A \leftrightarrow B$$

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$$\begin{aligned} (\text{fix } f . \{x \mapsto f x\}) v &\rightarrow \{x \mapsto (\text{fix } f . \{x \mapsto f x\}) x\} v \\ &\rightarrow ((\text{fix } f . \{x \mapsto f x\}) x)[v/x] \\ &= (\text{fix } f . \{x \mapsto f x\}) v \end{aligned}$$

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New challenger

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New challenger

$$\lambda g^{A_1 \leftrightarrow B_1} . \text{fix } f^{A_2 \leftrightarrow B_2} . \left\{ \begin{array}{l} \text{inj}_l x \mapsto \text{inj}_l (g x) \\ \text{inj}_r y \mapsto \text{inj}_r (f y) \end{array} \right\} : A_1 \oplus A_2 \leftrightarrow B_1 \oplus B_2$$

The (pure) quantum case

The quantum troubles

Contr is not enriched in an **interesting** way.

Is there a better category?

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Can we find a way to mimic the story we have for classical reversibility?

A kind of solution with techniques adapted from *guarded recursion*.

A guarded category

Start with a dagger category \mathbf{C} .

Consider the category whose objects are cochains in \mathbf{C} :

$$X(0) \longleftarrow X(1) \longleftarrow X(2) \longleftarrow \dots$$

And morphisms are natural transformations in \mathbf{C} :

$$\begin{array}{ccccccc} X(0) & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y(0) & \longleftarrow & Y(1) & \longleftarrow & Y(2) & \longleftarrow & \dots \end{array}$$

This category is enriched in the topos of trees $\mathbf{Set}^{\mathbb{N}^{\text{op}}}$ (with some sort of fixed point operator).

Computation in the topos of trees

Objects in the topos of trees are cochains in **Set**:

$$X(0) \xleftarrow{r_0} X(1) \xleftarrow{r_1} X(2) \xleftarrow{\quad} \dots$$

There is a functor $L: \mathbf{Set}^{\mathbb{N}^{\text{op}}} \rightarrow \mathbf{Set}^{\mathbb{N}^{\text{op}}}$, such that LX is:

$$1 \xleftarrow{!} X(0) \xleftarrow{r_0} X(1) \xleftarrow{r_1} X(2) \xleftarrow{\quad} \dots$$

and a natural transformation $\nu: \text{id} \Rightarrow L$, such that ν_X is:

$$\begin{array}{ccccccc} X(0) & \xleftarrow{r_0} & X(1) & \xleftarrow{r_1} & X(2) & \xleftarrow{r_2} & X(3) \xleftarrow{\quad} \dots \\ \downarrow! & & \downarrow r_0 & & \downarrow r_1 & & \downarrow r_2 \\ 1 & \xleftarrow{!} & X(0) & \xleftarrow{r_0} & X(1) & \xleftarrow{r_1} & X(2) \xleftarrow{\quad} \dots \end{array}$$

and a family of morphisms $\text{fix}_X: [LX \rightarrow X] \rightarrow X$.

Some mild conditions for guarded recursion

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It still allows for the `map` function.

$$\text{map}(\omega) = \text{fix } f. \left\{ \begin{array}{l} [] \mapsto [] \\ h :: t \mapsto (\omega h) :: (f t) \end{array} \right\} : [A] \leftrightarrow [B]$$

What about the categorical limit?

Once we have this diagram in **Contr**:

$$\begin{array}{ccccccc} X(0) & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y(0) & \longleftarrow & Y(1) & \longleftarrow & Y(2) & \longleftarrow & \dots \end{array}$$

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We can take the limit of both cochains.

$$\begin{array}{cccccccc} X(0) & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \dots & \longleftarrow & X_\infty \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ Y(0) & \longleftarrow & Y(1) & \longleftarrow & Y(2) & \longleftarrow & \dots & \longleftarrow & Y_\infty \end{array}$$

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But we lose the enrichment (and therefore the nice calculus that we could have on top).

Conclusion

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The situation is trickier for (pure) quantum computation.