Non-Cartesian Guarded Recursion with Daggers

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Is guarded recursion artificial recursion?

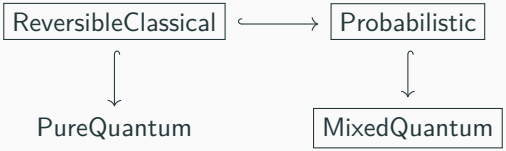
Background

- [Nakano00] defines a modality $\blacktriangleright$, representing the delay between recursive calls.
- Helps for a better formalisation of streams, with notions of productivity.
- [Birkedal&al12] model of guarded recursion in the topos of trees.

What we do and how

- Obtain (artificial) recursion from any setting.
  - by mimicking the structure of guarded recursion.
- Apply this structure to pure quantum programming (hence the dagger).

Motivation: The quantum troubles with recursion



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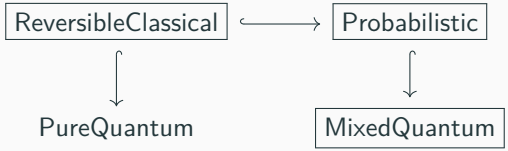

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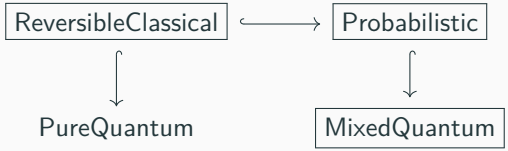

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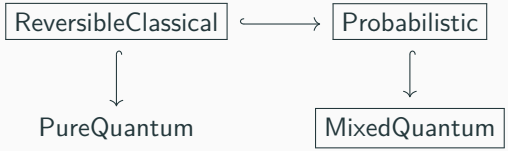

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We propose a solution with techniques adapted from guarded recursion.

Computation in the topos of trees

Objects in the topos of trees $\mathbf{S} = \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ are cochains in **Set**:

$$X(0) \xleftarrow[r_0]{} X(1) \xleftarrow[r_1]{} X(2) \longleftarrow \cdots$$

There is a functor $L\colon \mathbf{S} \to \mathbf{S}$, such that LX is:

$$1 \xleftarrow[!]{} X(0) \xleftarrow[r_0]{} X(1) \xleftarrow[r_1]{} X(2) \longleftarrow \cdots$$

and a natural transformation $\nu\colon \mathrm{id} \Rightarrow L$, such that ν_X is:

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and a family of morphisms $\mathrm{fix}_X\colon [LX \to X] \to X$.

A guarded category

$\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$ is enriched in the topos of trees $\mathbf{S}$, with hom-objects $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X, Y)$:

- $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X, Y)(n)$ is the set of truncated natural transformations $(\alpha_0, \ldots, \alpha_n)$;
- arrows $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X, Y)(n{+}1) \to \mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X, Y)(n)$ forget the last component.

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If $\mathbf{C}$ has a terminal object 1, there is a functor $L^{\mathbf{C}} \colon \mathbf{C}^{\mathbb{N}^{\mathrm{op}}} \to \mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$, such that LX is:

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Some results

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We also have:

$$L^{\mathbf{Set}}\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X, Y) \cong \mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(L^{\mathbf{C}}X, L^{\mathbf{C}}Y)$$

$$\nu^{\mathbf{Set}}_{\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}(X,Y)} \cong L^{\mathbf{C}}_{X,Y}$$

What about programs?

Consider $\mathbf{C}$ is a model of a typed language:

- types A interpreted as objects $[\![A]\!]$;
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We could stop there, but

Reversibility

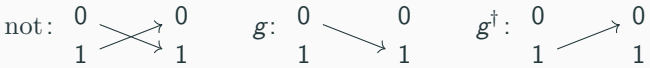
Dagger categories

Origin: functional analysis where $\langle fx \mid y \rangle = \langle x \mid f^\dagger y \rangle$.

Category $\mathbf{C}$ equipped with a functor $(-)^\dagger \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, such that:

- On objects, $A^\dagger = A$.
- On morphisms: $f^{\dagger\dagger} = f$.

Example with partial injective functions between sets, here $\{0, 1\}$.



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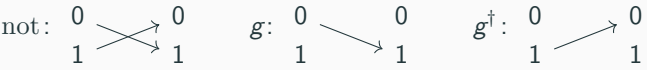

If $g^\dagger = g^{-1}$, we say that g is unitary.

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$$|0\rangle \longrightarrow |+\rangle$$
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where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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Hilbert spaces and contractions. h:

$$\begin{array}{ccc} |0\rangle & \longrightarrow & |+\rangle \\ |1\rangle & & |-\rangle \end{array}$$

$h|1\rangle = 0.$

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Can we form (interesting) subcategories of $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$ that admit a dagger?

Daggerable maps

Definition

$f\colon X \to Y$ *admits a dagger* or is *daggerable* if the family $\{f_n^\dagger\}_{n\in\mathbb{N}}$ verifies:

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$$\nu_Y^{\mathbf{C}} \circ f \circ \left(\nu_X^{\mathbf{C}}\right)^\dagger$$

when f daggerable.

Let's work out some syntax

Reversible syntax: examples and semantics

We mix functions and pattern-matching in a single entity.

- $x \colon A \vdash t \colon C$
- $y \colon B \vdash t' \colon C$
- $t \perp t'$

$$\left\{ \begin{array}{lcl} \mathtt{inj}_l\, x & \mapsto & t \\ \mathtt{inj}_r\, y & \mapsto & t' \end{array} \right\} : A \oplus B \leftrightarrow C$$

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$$[\![\{t_1 \mapsto t'_1 \mid \cdots \mid t_m \mapsto t'_m\}]\!] = \sum_i [\![t'_i]\!]\, [\![t_i]\!]^\dagger$$

$$[\![A]\!] \xrightarrow{[\![t_i]\!]^\dagger} [\![\Delta_i]\!] \xrightarrow{[\![t'_i]\!]} [\![B]\!]$$

Classical symmetric pattern-matching

$$
\begin{array}{lllll}
\text{(Ground types)} & A, B & ::= & \mathtt{I} \mid A \oplus B \mid A \otimes B \mid \\
\text{(Function types)} & T_1, T_2 & ::= & A \leftrightarrow B \mid \\
 & & & \\
\text{(Unit term)} & t, t_1, t_2 & ::= & * \\
\text{(Pairing)} & & & \mid t_1 \otimes t_2 \\
\text{(Injections)} & & & \mid \mathtt{inj}_l\ t \mid \mathtt{inj}_r\ t \\
\text{(Function application)} & & & \mid \omega\ t \\
 & & & \\
\text{(Abstraction)} & \omega & ::= & \{t_1 \mapsto t_1' \mid \cdots \mid t_m \mapsto t_m'\}
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\text{(Higher functions)} & & & \mid \lambda f.\omega \mid \omega_2 \omega_1
\end{array}
$$

λ-calculus with fixed points thanks to **DCPO**-enrichment of **PInj**.

And now, guarded

Reversible Guarded Recursion

(Ground types)	A, B	$::=$	$\mathtt{I} \mid A \oplus B \mid A \otimes B \mid \blacktriangleright A \mid$
(Function types)	T_1, T_2	$::=$	$A \leftrightarrow B \mid \blacktriangleright T \mid T_1 \to T_2$
(Unit term)	t, t_1, t_2	$::=$	$*$
(Pairing)			$\mid t_1 \otimes t_2$
(Injections)			$\mid \mathtt{inj}_l\ t \mid \mathtt{inj}_r\ t$
(Function application)			$\mid \omega\ t$
(Later modality)			$\mid \mathtt{next}\ t$
(Inductive terms)			$\mid \mathtt{fold}\ t$
(Abstraction)	ω	$::=$	$\{t_1 \mapsto t_1' \mid \cdots \mid t_m \mapsto t_m'\}$
(Higher modality)			$\mid \mathtt{next}\ \omega$
(Fixed points)			$\mid f \mid \mathbf{fix}\ f.\omega$
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The depth condition still allows for the map function.

$$\texttt{map}(\omega) = \textbf{fix}\ \mathrm{f}.\ \left\{ \begin{array}{ll} [\,] & \mapsto [\,] \\ h :: t & \mapsto (\omega\ h) :: (\ \mathrm{f}\ \ t) \end{array} \right\} : [A] \leftrightarrow [B]$$

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But no infinite loops allowed.

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Thank you!