Semantics of Quantum Loops and Recursion

Louis Lemonnier

University of Edinburgh

THE UNIVERSITY *of* EDINBURGH
informatics

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Programming languages

```
def fact(n):
        if n == 0:
                return(1)
        else:
                return(n*fact(n-1))
```

Programming languages

```
let rec fact (n:int) =
        if n=0 then 1
        else n * fact (n-1)
```

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```
fact:
        movq   $0, %rax
        je     .case_zero
        movq   %rdi, %rax
        pushq  %rax
        subq   $1, %rdi
        call   fact
        popq   %rbx
        multq  %rbx, %rax
        jmp    .else
.case_zero:
        movq   $1, %rax
.else:
        ret
```

Programming languages

$$\lambda m.\lambda g.(\lambda x.g(xx))(\lambda x.g(xx))(\lambda p.\lambda a.\lambda b.pab)$$
$$(\lambda n.n(\lambda x.(\lambda a.\lambda b.b))(\lambda a.\lambda b.a))(\lambda f.\lambda x.fx)$$
$$(m(\lambda n.\lambda f.\lambda x.n(\lambda g.\lambda h.h(gf))(\lambda u.x)(\lambda x.x)))$$

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Strong enough model if relation between operational and denotational:

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- Prove that the language does what it is supposed to do,
- Compile the language and perform optimisations safely,
- Inform on which features can be added to the language.

The point of semantics

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But we are not telling this story.

A gentle walk through classical programming

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$$[\![f]\!] = \begin{cases} \mathbb{N} \cup \{\perp\} & \to & \mathbb{N} \cup \{\perp\} \\ n & \mapsto & \perp \end{cases}$$

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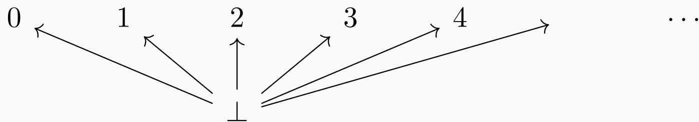

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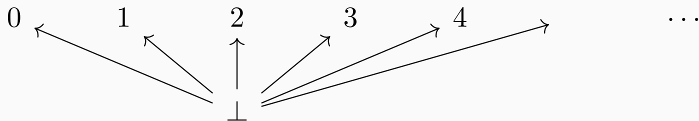

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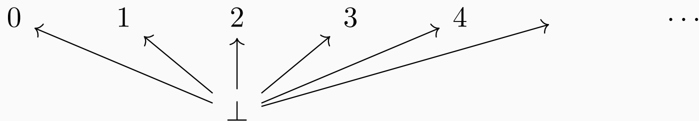

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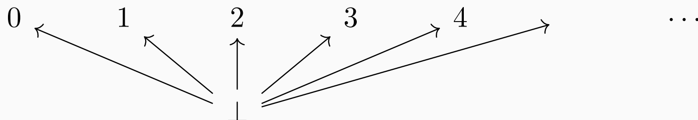

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How do we use it?

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \to \mathbb{N}_\perp] \to [\mathbb{N}_\perp \to \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \to \mathbb{N}_\perp$ the constant function with output $\perp$.

$\perp : \mathbb{N}_\perp \to \mathbb{N}_\perp$ is:

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Interpretation of recursion

Define $F : [[\mathbb{N}_\bot \to \mathbb{N}_\bot] \to [\mathbb{N}_\bot \to \mathbb{N}_\bot]]$ as:

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Write $\bot : \mathbb{N}_\bot \to \mathbb{N}_\bot$ the constant function with output $\bot$.

$F(F(\bot)) : \mathbb{N}_\bot \to \mathbb{N}_\bot$ is:

$$1 \quad 1 \quad \bot \quad \bot \quad \bot \quad \bot \quad \bot \quad \cdots$$

$F(F(F(\bot))) : \mathbb{N}_\bot \to \mathbb{N}_\bot$ is:

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A notion of limit linked to the order is necessary.

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The undefined function is interpreted as the constant map 0 (no need to add a $\perp$).

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Is it possible to interpret loop or recursion in this setting?

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What about its quantum counterpart?

The Quantum Case

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$$|+\rangle\langle 0| + |-\rangle\langle 1| = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad |+\rangle\langle 0| = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad |-\rangle\langle 1| = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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A disappointment

However, this order does not interact well with composition:

$$
\begin{aligned}
& & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &\leq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
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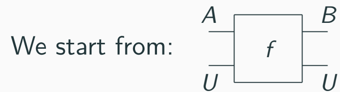
Theorem (No go theorem)

There is no such order on subunitary maps between Hilbert spaces.

Recursion through a Trace

We start from:



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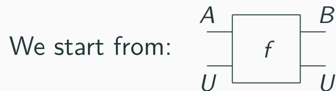
We start from: 

The recurring example: $g\colon \begin{cases} \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) & \to & \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) \\ \texttt{left}\ n & \mapsto & \texttt{right}\ (n, 1) \\ \texttt{right}\ (n+1, m) & \mapsto & \texttt{right}\ (n, n \times m) \\ \texttt{right}\ (0, m) & \mapsto & \texttt{left}\ m \end{cases}$

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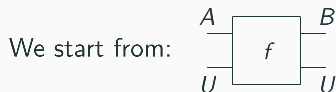

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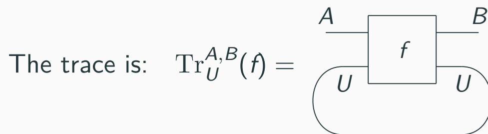

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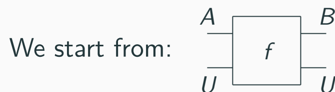
We start from: 

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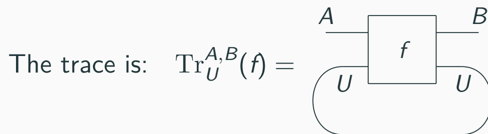
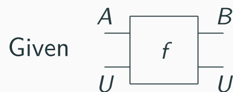
The trace is: $\mathrm{Tr}_U^{A,B}(f) =$ 

And $\mathrm{Tr}_{\mathbb{N} \uplus \mathbb{N}}^{\mathbb{N},\mathbb{N}}(g)(n) = n!$.

The Quantum Case



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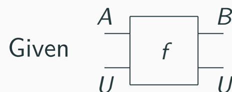

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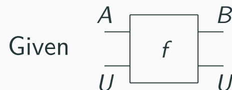
Given 

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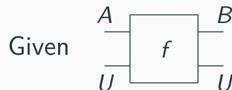
Given 

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Lemma (Towards no go)

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My take:

Conjecture: No go

There is no suitable trace for (sub)unitaries between infinite dimension Hilbert spaces.

Until we meet again

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If you ask people in the community:

Until we meet again

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A technique called guarded recursion.

Guarded Quantum Recursion

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However, the expressivity is limited.

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Thank you!