

Semantics of Recursion in Quantum Control

Louis Lemonnier

Titouan Carette, Kostia Chardonnet, Robin Kaarsgaard, JS Lemay, Benoît Valiron, Vladimir Zamdzhiev



Laboratoire
Méthodes
Formelles



école —————
normale —————
supérieure —————
paris-saclay —————

GT Scalp. 29th November 2023

```
def fact(n):  
    if n == 0:  
        return 1  
    else:  
        return (n*fact(n-1))
```

```
let rec fact (n:int) =  
    if n=0 then 1  
    else n * fact (n-1)
```

Programming languages

```
fact :  
    movq    $0 , %rax  
    je     .case_zero  
    movq   %rdi , %rax  
    pushq  %rax  
    subq   $1 , %rdi  
    call   fact  
    popq   %rbx  
    multq  %rbx , %rax  
    jmp    .else  
.case_zero :  
    movq   $1 , %rax  
.else :  
    ret
```

$\lambda m. \lambda g. (\lambda x. g(xx)) (\lambda x. g(xx)) (\lambda p. \lambda a. \lambda b. pab)$
 $(\lambda n. n(\lambda x. (\lambda a. \lambda b. b)) (\lambda a. \lambda b. a)) (\lambda f. \lambda x. fx)$
 $(m(\lambda n. \lambda f. \lambda x. n(\lambda g. \lambda h. h(gf)) (\lambda u. x) (\lambda x. x)))$

Syntax: terms t .

Operational semantics: $t \rightarrow t'$.

Interpreted as a mathematical object: $\llbracket t \rrbracket$.

Wish: Relation between an operational view and a denotational view:

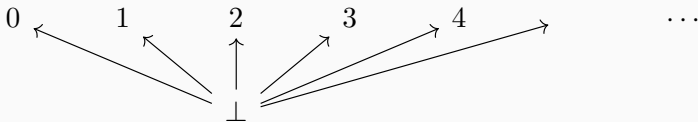
$$t \simeq t' \text{ iff } \llbracket t \rrbracket = \llbracket t' \rrbracket .$$

Domain theory (for babies)

Domain theory (for babies)

$\mathbb{N}_\perp \stackrel{\text{def}}{=} \mathbb{N} \cup \{\perp\}$ is a partial-ordered set with **nice** properties.

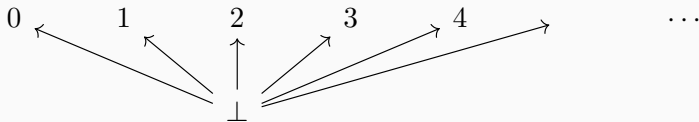
\perp means **undefined**.



Domain theory (for babies)

$\mathbb{N}_\perp \stackrel{\text{def}}{=} \mathbb{N} \cup \{\perp\}$ is a partial-ordered set with **nice** properties.

\perp means **undefined**.

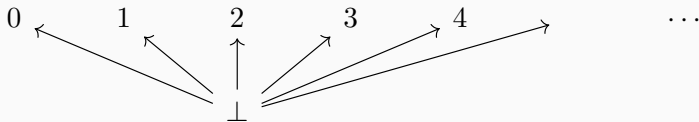


The set $[D \rightarrow D']$ of **monotone** functions also has **nice** properties.

Domain theory (for babies)

$\mathbb{N}_\perp \stackrel{\text{def}}{=} \mathbb{N} \cup \{\perp\}$ is a partial-ordered set with **nice** properties.

\perp means **undefined**.



The set $[D \rightarrow D']$ of **monotone** functions also has **nice** properties.

How do we use it?

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$\perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \dots$

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$\perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \dots$

$F(\perp) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$1 \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \perp \quad \dots$

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(\perp) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 \perp \perp \perp \perp \perp \perp \dots

$F(F(\perp)) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 \perp \perp \perp \perp \perp \dots

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(\perp)) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 \perp \perp \perp \perp \perp ...

$F(F(F(\perp))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 \perp \perp \perp \perp ...

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(F(\perp))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 \perp \perp \perp \perp ...

$F(F(F(F(\perp)))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 6 \perp \perp \perp ...

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(F(\perp))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 6 \perp \perp \perp ...

$F(F(F(F(\perp)))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 6 24 \perp \perp ...

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(F(F(F(\perp)))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 6 24 \perp \perp ...

$F(F(F(F(F(F(\perp))))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

1 1 2 6 24 120 \perp ...

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(F(F(F(\perp)))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$$1 \quad 1 \quad 2 \quad 6 \quad 24 \quad \perp \quad \perp \quad \dots$$

$F(F(F(F(F(F(F(\perp))))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$$1 \quad 1 \quad 2 \quad 6 \quad 24 \quad 120 \quad \perp \quad \dots$$

The interpretation of the factorial is then the **limit** of $(F^n(\perp))_n$.

Interpretation of recursion

Define $F : [[\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp] \rightarrow [\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp]]$ as:

$$F(g)(n) = \begin{cases} 1 & \text{if } n = 0, \\ n \times g(n-1) & \text{else.} \end{cases}$$

Write $\perp : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ the constant function with output \perp .

$F(F(F(F(F(\perp)))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$$1 \quad 1 \quad 2 \quad 6 \quad 24 \quad \perp \quad \perp \quad \dots$$

$F(F(F(F(F(F(F(\perp))))))) : \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is:

$$1 \quad 1 \quad 2 \quad 6 \quad 24 \quad 120 \quad \perp \quad \dots$$

The interpretation of the factorial is then the **limit** of $(F^n(\perp))_n$.

A notion of **limit** linked to the order is necessary.

What is quantum control?

Quantum computation seen as an effect:

- Non-deterministic: $\text{True} + \text{False}$.
- Probabilistic: $\frac{1}{2}\text{True} + \frac{1}{2}\text{False}$.
- Quantum: $\frac{1}{\sqrt{2}}\text{True} + (-\frac{1}{\sqrt{2}})\text{False}$.

What is quantum control?

Quantum computation seen as an effect:

- Non-deterministic: $\text{True} + \text{False}$.
- Probabilistic: $\frac{1}{2}\text{True} + \frac{1}{2}\text{False}$.
- Quantum: $\frac{1}{\sqrt{2}}\text{True} + (-\frac{1}{\sqrt{2}})\text{False}$. (not a monad! [Heunen16])

What is quantum control?

Quantum computation seen as an effect:

- Non-deterministic: $\text{True} + \text{False}$.
- Probabilistic: $\frac{1}{2}\text{True} + \frac{1}{2}\text{False}$.
- Quantum: $\frac{1}{\sqrt{2}}\text{True} + (-\frac{1}{\sqrt{2}})\text{False}$. (not a monad! [Heunen16])

Physics says that apart from **measurement**, operations must be **reversible**.

What is quantum control?

Quantum computation seen as an effect:

- Non-deterministic: True + False.
- Probabilistic: $\frac{1}{2}\text{True} + \frac{1}{2}\text{False}$.
- Quantum: $\frac{1}{\sqrt{2}}\text{True} + (-\frac{1}{\sqrt{2}})\text{False}$. (not a monad! [Heunen16])

Physics says that apart from **measurement**, operations must be **reversible**.

Quantum control:

- Not only superposition of states,
- Superposition of **processes**.

E.g. if x then P_1 else P_2 , where x is in superposition.

Quick stop: Recursion in Classical Reversibility

Suppose we have a syntax with:

- Only reversible functions,
- Inductive types,
- A fixpoint operator,
- A CbV operational semantics.

It is Turing-complete!

Sound and adequate denotational semantics in **partial injective** functions.

Given two sets A and B , the set $[A \multimap B]$ has **nice** properties for fixpoints.

Read about it: [CLV23].

Quick stop: Recursion in Classical Reversibility

Suppose we have a syntax with:

- Only reversible functions,
- Inductive types,
- A fixpoint operator,
- A CbV operational semantics.

It is Turing-complete!

Sound and adequate denotational semantics in **partial injective** functions.

Given two sets A and B , the set $[A \rightarrow B]$ has **nice** properties for fixpoints.

(The category of partial injective functions between sets is enriched in pointed dcpos.)

Read about it: [CLV23].

The Quantum Case

A unitary operator f is such that the inverse of f is its adjoint.

Dimension 2 for simple examples.

Reversible quantum operations are **subunitaries** such as:

$$\begin{array}{ccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}$$

The Quantum Case

A unitary operator f is such that the inverse of f is its adjoint.

Dimension 2 for simple examples.

Reversible quantum operations are **subunitaries** such as:

$$\begin{array}{ccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}$$

Arguably, **undefined** (\perp) in a Hilbert space is the **zero** vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The Quantum Case

A unitary operator f is such that the inverse of f is its adjoint.

Dimension 2 for simple examples.

Reversible quantum operations are **subunitaries** such as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Arguably, **undefined** (\perp) in a Hilbert space is the **zero** vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is less defined than $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which is less defined than $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A disappointment

However, this order does not interact well with composition:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \text{thus} \quad & \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \leq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \\ \text{i.e.} \quad & 1/2 \leq 0. \end{aligned}$$

A disappointment

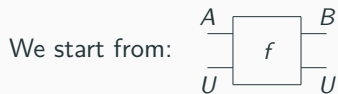
However, this order does not interact well with composition:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \text{thus} \quad & \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \leq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \\ \text{i.e.} \quad & 1/2 \leq 0. \end{aligned}$$

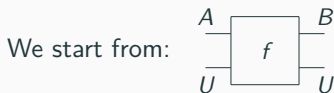
Theorem (No go theorem)

There is no such order on contractive maps between Hilbert spaces.

Recursion through a Trace



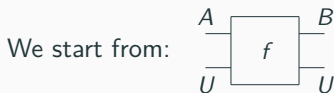
Recursion through a Trace



The recurring example: g :

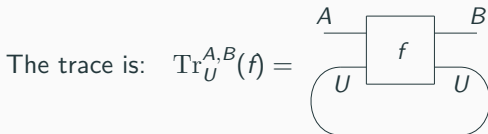
$$\left\{ \begin{array}{ll} \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) & \rightarrow \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) \\ \text{left } n & \mapsto \text{right } (n, 1) \\ \text{right } (n + 1, m) & \mapsto \text{right } (n, n \times m) \\ \text{right } (0, m) & \mapsto \text{left } m \end{array} \right.$$

Recursion through a Trace

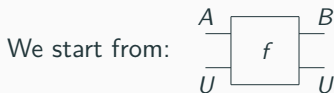


The recurring example: g :

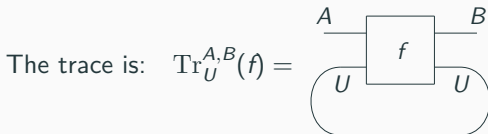
$$\left\{ \begin{array}{ll} \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) & \rightarrow \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) \\ \text{left } n & \mapsto \text{right } (n, 1) \\ \text{right } (n + 1, m) & \mapsto \text{right } (n, n \times m) \\ \text{right } (0, m) & \mapsto \text{left } m \end{array} \right.$$



Recursion through a Trace

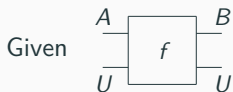


The recurring example: $g: \begin{cases} \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) & \rightarrow & \mathbb{N} \uplus (\mathbb{N} \times \mathbb{N}) \\ \text{left } n & \mapsto & \text{right } (n, 1) \\ \text{right } (n + 1, m) & \mapsto & \text{right } (n, n \times m) \\ \text{right } (0, m) & \mapsto & \text{left } m \end{cases}$



And $\text{Tr}_{\mathbb{N} \uplus \mathbb{N}}^{\mathbb{N}, \mathbb{N}}(g)(n) = n!$.

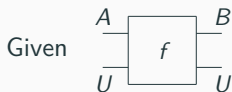
The Quantum Case



The canonical trace formula is:

$$\text{Tr}(f) = f_{AB} + \sum_n f_{UB} f_{UU}^n f_{AU}$$

The Quantum Case



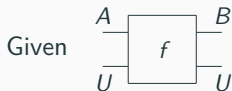
The canonical trace formula is:

$$\text{Tr}(f) = f_{AB} + \sum_n f_{UB} f_{UU}^n f_{AU}$$

This is a trace for unitaries between **finite dimension** Hilbert space.

However:

The Quantum Case



The canonical trace formula is:

$$\text{Tr}(f) = f_{AB} + \sum_n f_{UB} f_{UU}^n f_{AU}$$

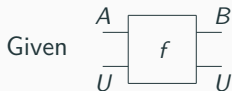
This is a trace for unitaries between **finite dimension** Hilbert space.

However:

Lemma (Towards no go)

The canonical trace on infinite dimension Hilbert spaces is not a trace.

The Quantum Case



The canonical trace formula is:

$$\text{Tr}(f) = f_{AB} + \sum_n f_{UB} f_{UU}^n f_{AU}$$

This is a trace for unitaries between **finite dimension** Hilbert space.

However:

Lemma (Towards no go)

The canonical trace on infinite dimension Hilbert spaces is not a trace.

My take:

Conjecture: No go

There is **no trace** for (sub)unitaries between infinite dimension Hilbert spaces.

Is quantum recursion not possible?

Until we meet again

Is quantum recursion not possible?

If you ask people in the community:

- Classical recursion allows **non termination**.
- *“Non-terminating quantum-controlled program do not make sense.”*
- But no mathematical account of this point yet.

Until we meet again

Is quantum recursion not possible?

If you ask people in the community:

- Classical recursion allows **non termination**.
- *“Non-terminating quantum-controlled program do not make sense.”*
- But no mathematical account of this point yet.

Termination needs to be ensured!

A technique called **guarded** recursion. **To be continued...**

Until we meet again

Is quantum recursion not possible?

If you ask people in the community:

- Classical recursion allows **non termination**.
- *“Non-terminating quantum-controlled program do not make sense.”*
- But no mathematical account of this point yet.

Termination needs to be ensured!

A technique called **guarded** recursion. **To be continued...**

Thank you!