Semantics for a Turing-complete Reversible Programming Language with Inductive Types

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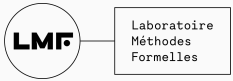

FSCD'24. 12th July 2024

Why study semantics?

Comes from Ancient Greek, *to provide a meaning*.

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- Prove that the language does what it is supposed to do,
- Compile the language and perform optimisations safely,
- Inform on which features can be added to the language.

Reversible Programming

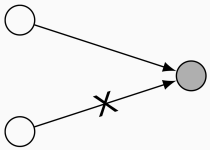
Originally

- Landauer and Bennett, 1961: Reversible Computation and Energy Dissipation.
- Programs are reversible: for a program $\mathtt{t}$, there is $\mathtt{t}^{-1}$ such that $\mathtt{t}; \mathtt{t}^{-1} = \mathtt{skip}$.
- Applications to quantum computing (we can chat about this later).

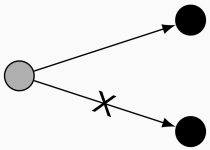
What we do

- Reversibility, but not totality.
- Improve the language through the model.
- Soundness and adequacy.

Backward determinism



Forward determinism



[Kaarsgaard&Rennela21]

The categorical model: inverse categories

Category theory around reversibility is well studied [Axelsen&Kaarsgaard16].

Domain and partiality

Partial inverse

Union of morphisms

Order

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$f \circ \bar{f} = f, \quad \bar{f} \circ \bar{g} = \bar{g} \circ \bar{f}, \quad \overline{f \circ \bar{g}} = \bar{f} \circ \bar{g}, \quad \bar{h} \circ f = f \circ \overline{h \circ f}.$

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Partial inverse — Inverse category: $f \mapsto f^\circ$. $f^\circ \circ f = \bar{f}$ and $f \circ f^\circ = \overline{f^\circ}$.

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Order

$f \leq g$ if $g\overline{f} = f$

Gives a dcpo structure to homsets.

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  - Linear type system.

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Hopefully, there is a way to cheat.

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Given $\quad \Delta \vdash t\colon A \qquad \Delta \vdash t'\colon B$

We form a function $t \mapsto t'\colon A \leftrightarrow B$, Whose semantics is

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Together with pattern-matching

With a sum type $\oplus$:

$$\frac{\Delta \vdash t\colon A}{\Delta \vdash \mathtt{inj}_l\ t\colon A \oplus B} \qquad \frac{\Delta \vdash t\colon B}{\Delta \vdash \mathtt{inj}_r\ t\colon A \oplus B}$$

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Our functions are then:

$$\left\{ \begin{array}{ccc} t_1 & \mapsto & t_1' \\ t_2 & \mapsto & t_2' \\ & \vdots & \\ t_m & \mapsto & t_m' \end{array} \right\} : A \leftrightarrow B$$

whenever $\Delta_i \vdash t_i\colon A$ and $t_j \perp t_k$, $\quad \Delta_i \vdash t_i'\colon B$ and $t_j' \perp t_k'$.

Example and operational semantics

$$\left\{ \begin{array}{lcl} \mathtt{inj}_l\, x & \mapsto & \mathtt{inj}_r\, x \\ \mathtt{inj}_r\, y & \mapsto & \mathtt{inj}_l\, y \end{array} \right\} : A \oplus B \leftrightarrow B \oplus A$$

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Can be seen as:

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- With compatible domains.
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How much can one express with this language?

To Infinity and Beyond

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$\longrightarrow$ Our category $\mathbf{C}$ is parameterised **DCPO**-algebraically ω-compact.

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Summary of the language (mandatory slide)

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\text{(Ground types)} & A, B & ::= & \mathtt{I} \mid A \oplus B \mid A \otimes B \mid \\
\text{(Function types)} & T_1, T_2 & ::= & A \leftrightarrow B \mid \\
 & & & \\
\text{(Unit term)} & t, t_1, t_2 & ::= & * \\
\text{(Pairing)} & & & \mid t_1 \otimes t_2 \\
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λ-calculus thanks to **DCPO**-enrichment.

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Roughly

- Reversible Turing Machines [Axelsen&Glück11].
  - Simulate your favourite Turing machines.
- Encode RTMs in our language:
  - Alphabet & states mapped to $\mathtt{I} \oplus \cdots \oplus \mathtt{I}$.
  - Tape as lists.
  - Functions simulating one-step transition of δ.
  - Iterate until final state.

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$$\lambda g^{A_1 \leftrightarrow B_1}\ .\ \mathsf{fix}\ f^{A_2 \leftrightarrow B_2}\ .\ \left\{ \begin{array}{lcl} \mathtt{inj}_l\,x & \mapsto & \mathtt{inj}_l\,(g\,x) \\ \mathtt{inj}_r\,y & \mapsto & \mathtt{inj}_r\,(f\,y) \end{array} \right\} \colon A_1 \oplus A_2 \leftrightarrow B_1 \oplus B_2$$

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Self promotion:

- Kostia's PhD thesis, `https://theses.hal.science/tel-03959403v1`
- My PhD thesis, `https://theses.hal.science/tel-04625771v1`
- Benoît's habilitation, to be defended on 24th September 2024.