## Categorical Semantics of Reversible Pattern-Matching

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## Reversible programming

## Reversibility

- Reversible computation: only apply bijections.

[^0]
## Reversibility

- Reversible computation: only apply bijections.
- Application in low-consumption hardware ${ }^{1}$.

[^1]
## Reversibility

- Reversible computation: only apply bijections.
- Application in low-consumption hardware ${ }^{1}$.
- Application in quantum computing, where operations (except measurement) are reversible.

[^2]
## Reversibility

- Reversible computation: only apply bijections.
- Application in low-consumption hardware ${ }^{1}$.
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What we are focusing on today:

[^3]
## Reversibility

- Reversible computation: only apply bijections.
- Application in low-consumption hardware ${ }^{1}$.
- Application in quantum computing, where operations (except measurement) are reversible.

What we are focusing on today:

- Reversible programming language

[^4]
## Reversibility

- Reversible computation: only apply bijections.
- Application in low-consumption hardware ${ }^{1}$.
- Application in quantum computing, where operations (except measurement) are reversible.

What we are focusing on today:

- Reversible programming language
- Detailed denotational semantics

[^5]Terms and types
(Value types) $a, b \quad:=\alpha|a \oplus b| a \otimes b$
(Iso types) $\quad T::=\quad a \leftrightarrow b$
(Values) $\quad v::=c_{\alpha}|x| i n j, v\left|i n j_{r} v\right|\left\langle v_{1}, v_{2}\right\rangle$
(Functions) $\quad \omega::=\quad\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow V_{n}\right\}$
(Terms) $\quad t::=v \mid \omega t$

## Terms and types

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(Functions) $\quad \omega::=\quad\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow V_{n}\right\}$
(Terms) $t::=\quad v \mid \omega t$
$\left\{\left|\operatorname{inj}_{1} x \leftrightarrow \operatorname{inj}_{r} x\right| \operatorname{inj}_{r} x \leftrightarrow \operatorname{inj}_{I} x\right\}$

## Terms and types

(Value types) $a, b::=\alpha|a \oplus b| a \otimes b$
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$$
\left\{\left|\operatorname{inj}_{1} x \leftrightarrow \operatorname{inj}_{r} x\right| \operatorname{inj}_{r} x \leftrightarrow \operatorname{inj}_{1} x\right\}
$$

- Two partial morphisms joined


## Terms and types

$$
\begin{array}{lrl}
\text { (Value types) } & a, b & ::=\alpha|a \oplus b| a \otimes b \\
\text { (Iso types) } & T & ::=a \leftrightarrow b \\
\text { (Values) } & v & ::=c_{\alpha}|x| \text { inj }_{1} v \mid \text { inj }_{r} v \mid\left\langle v_{1}, v_{2}\right\rangle \\
\text { (Functions) } & \omega & ::=\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow V_{n}\right\} \\
\text { (Terms) } & t & ::=v \mid \omega t \\
& \\
& \left\{\mid \text { inj }_{l} x \leftrightarrow \text { inj }_{r} x \mid \text { inj }_{r} x \leftrightarrow \text { inj }_{l} x\right\}
\end{array}
$$

- Two partial morphisms joined
- With compatible domains


## Terms and types

$$
\begin{array}{lrl}
\text { (Value types) } & a, b & ::=\alpha|a \oplus b| a \otimes b \\
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\text { (Terms) } & t & ::=v \mid \omega t \\
& \\
& \left\{\mid \text { inj }_{1} x\right. & \left.\leftrightarrow \text { inj }_{r} x \mid \text { inj }_{r} x \leftrightarrow \text { inj }_{l} x\right\}
\end{array}
$$

- Two partial morphisms joined
- With compatible domains
- Which inverses have compatible domains


## Terms and types

$$
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\text { (Value types) } & a, b & ::=\alpha|a \oplus b| a \otimes b \\
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\text { (Values) } & v & ::=c_{\alpha}|x| \text { inf }_{1} v \mid \text { inf }_{r} v \mid\left\langle v_{1}, v_{2}\right\rangle \\
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\text { (Terms) } & t & ::=v \mid \omega t \\
& \left\{\mid \text { inf }_{1} x \leftrightarrow \text { inf }_{r} x \mid \text { inf }_{r} x \leftrightarrow \text { ind }_{l} x\right\}
\end{array}
$$

- Two partial morphisms joined
- With compatible domains
- Which inverses have compatible domains

Operational semantics:

$$
\left\{\left|\operatorname{inj}_{1} x \leftrightarrow \operatorname{inj}_{r} x\right| \operatorname{inj}_{r} x \leftrightarrow \operatorname{inj}_{1} x\right\}\left(\operatorname{inj}_{r} \star\right) \longrightarrow \operatorname{inj}_{,} \star
$$

## Categorical model

## Restriction and inverse category

Domain and partiality

Partial inverse

Compatibility

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

Partial inverse

Compatibility

Union of partial isos

## Restriction and inverse category

$\begin{array}{ll}\text { Domain and partiality } & \text { Restriction category: } f \mapsto \bar{f} . \\ & f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} .\end{array}$

Partial inverse

Compatibility

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} .
\end{aligned}
$$

Compatibility

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

Partial inverse

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h: \begin{array}{l}
a \\
b \\
c
\end{array} \circlearrowright \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{h}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} \text {. } f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. }
\end{aligned}
$$

Compatibility

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

Partial inverse

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h: \begin{array}{l}
a \\
b \\
c
\end{array} \circlearrowright \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{h}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} \text {. } f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}
\end{aligned}
$$

Compatibility

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

Partial inverse

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h: \begin{array}{l}
a \\
b \\
c
\end{array} \circlearrowright \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{c}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} \text {. } f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}
\end{aligned}
$$

Compatibility
Restriction compatible:

$$
f \smile g: f \bar{g}=g \bar{f}, f \asymp g: f \smile g \text { and } f \smile g^{\circ} .
$$

Union of partial isos

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \quad \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h: \begin{array}{l}
a \\
b \\
c \\
c
\end{array} \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{h}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} . \quad f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{aligned}
& \begin{aligned}
& f \circ \bar{f}=f, \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \\
& h: \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h \quad \begin{array}{c}
a \\
b \\
b
\end{array} \underset{c}{a}\left.\bar{h}: \begin{array}{c}
b \\
c
\end{array}\right] \underset{c}{b}
\end{aligned} \\
& \text { Inverse category: } f \mapsto f^{\circ} . \quad f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

$$
\begin{aligned}
& f \smile g: f \bar{g}=g \bar{f}, f \asymp g: f \smile g \text { and } f \smile g^{\circ} \text {. } \\
& f: \begin{array}{l}
\left.a \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \quad g: \begin{array}{l}
a \\
b \\
b \\
b
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}\right]
\end{array} \\
& \text { Partial order: } f \leq g: g \bar{f}=f .
\end{aligned}
$$

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{aligned}
& f \circ \bar{f}=f, \quad \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \quad \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, \quad \bar{h} \circ f=f \circ \overline{h \circ f} . \\
& h: \begin{array}{l}
a \\
b \\
c
\end{array} \circlearrowright \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{c}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} \text {. } f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\overline{f^{\circ}} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

$$
\begin{aligned}
& f \smile g: f \bar{g}=\bar{g}, f \asymp g: f \smile g \text { and } f^{\circ} \smile g^{\circ} \text {. } \\
& f: \begin{array}{l}
a \longrightarrow a \\
b \\
c
\end{array} \quad g: \begin{array}{l}
a \\
b
\end{array} \quad \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
c
\end{array}
\end{aligned}
$$

Partial order: $f \leq g: g \bar{f}=f$.

$$
k: \begin{aligned}
& a \\
& b \\
& b
\end{aligned} \longrightarrow \begin{aligned}
& a \\
& b \\
& c
\end{aligned}
$$

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{aligned}
& f \circ \bar{f}=f, \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \bar{f} \circ \overline{\bar{g}}=\bar{f} \circ \bar{g}, \bar{h} \circ f=f \circ \overline{h \circ f} \text {. } \\
& h: \begin{array}{l}
a \\
b \\
c
\end{array} \geq \begin{array}{l}
a \\
b \\
c
\end{array} \quad \bar{c}: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \\
& \text { Inverse category: } f \mapsto f^{\circ} . f^{\circ} \circ f=\bar{f} \text { and } f \circ f^{\circ}=\bar{f} \text {. } \\
& h^{\circ}: \begin{array}{l}
a \\
b \\
c
\end{array} \rightarrow \underset{c}{a}
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

$$
\begin{aligned}
& f \smile g: f g=\bar{g}, f \asymp g: f \smile g \text { and } f^{\circ} \smile g^{\circ} . \\
& f: \begin{array}{l}
a \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \quad g: \begin{array}{l}
a \\
c
\end{array} \quad \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow c
\end{array}
\end{aligned}
$$

Partial order: $f \leq g: g \bar{f}=f$.

$$
k: \begin{aligned}
& a \\
& b \\
& c
\end{aligned} \longrightarrow \begin{aligned}
& a \\
& b \\
& c
\end{aligned}
$$

Join: $\bigvee_{s \in S} s$.
if $s \leq t$, then $s \leq \underset{s \in S}{ }{ }_{s} s, \bigvee_{s \in S} s \leq t, \overline{\bigvee_{s \in S} s}=\underset{s \in S}{ }{ }_{s} \bar{s}$,

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{array}{cc}
f \circ \bar{f}=f, & \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \\
a \quad \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, & \bar{h} \circ f=f \circ \overline{h \circ f} . \\
h: \begin{array}{c}
a \\
b \\
c
\end{array} \quad \bar{c} \\
c
\end{array} \quad \bar{b}: \begin{gathered}
b \\
c
\end{gathered}
$$

Partial inverse
Inverse category: $f \mapsto f^{\circ}$. $f^{\circ} \circ f=\bar{f}$ and $f \circ f^{\circ}=\overline{f^{\circ}}$.

$$
h^{\circ}: \begin{aligned}
& a \\
& b \\
& c
\end{aligned} \nearrow \begin{aligned}
& a \\
& b \\
& c
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

$$
\begin{aligned}
& f \smile g: f g=g \bar{f}, f \asymp g: f \smile g \text { and } f \circ \smile g^{\circ} . \\
& f: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \quad g: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
c
\end{array}
\end{aligned}
$$

Partial order: $f \leq g: g \bar{f}=f$.

$$
\begin{aligned}
& \left.k: \begin{array}{c}
a \\
b \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array}\right]
\end{aligned}
$$

Join: $\bigvee_{s \in S} s$.
if $s \leq t$, then $s \leq \bigvee_{s \in S} s, \bigvee_{s \in S} s \leq t, \overline{\bigvee_{s \in S} s}=\bigvee_{s \in S} \bar{s}$,
$f \circ\left(\bigvee_{s \in S} s\right)=\bigvee_{s \in S} f s,\left(\bigvee_{s \in S} s\right) \circ g=\bigvee_{s \in S} s g$.

## Restriction and inverse category

Domain and partiality Restriction category: $f \mapsto \bar{f}$.

$$
\begin{array}{cc}
f \circ \bar{f}=f, & \bar{f} \circ \bar{g}=\bar{g} \circ \bar{f}, \\
a \quad \bar{f} \circ \bar{g}=\bar{f} \circ \bar{g}, & \bar{h} \circ f=f \circ \overline{h \circ f} . \\
h: \begin{array}{c}
a \\
b \\
c
\end{array} \quad \bar{c} \\
c
\end{array} \quad \bar{b}: \begin{gathered}
b \\
c
\end{gathered}
$$

Partial inverse
Inverse category: $f \mapsto f^{\circ}$. $f^{\circ} \circ f=\bar{f}$ and $f \circ f^{\circ}=\overline{f^{\circ}}$.

$$
h^{\circ}: \begin{aligned}
& a \\
& b \\
& c
\end{aligned} \nearrow \begin{aligned}
& a \\
& b \\
& c
\end{aligned}
$$

Compatibility

Union of partial isos
Restriction compatible:

Partial order: $f \leq g: g \bar{f}=f$.

$$
k: \begin{aligned}
& a \\
& b \\
& b
\end{aligned} \longrightarrow \begin{aligned}
& a \\
& b \\
& c
\end{aligned}
$$

Join: $\bigvee_{s \in S} s$.

$$
\text { if } s \leq t \text {, then } s \leq \bigvee_{s \in S} s, \bigvee_{s \in S} s \leq t, \overline{\bigvee_{s \in S} s}=\bigvee_{s \in S} \bar{s}
$$

$$
f \circ\left(\bigvee_{s \in S} s\right)=\bigvee_{s \in S} f s,\left(\bigvee_{s \in S} s\right) \circ g=\bigvee_{s \in S} s g .
$$

$$
\begin{array}{lll}
s \in S & a & \longrightarrow \\
I d: & b & \longrightarrow \\
c & \longrightarrow
\end{array}
$$

$$
\begin{aligned}
& f \smile g: f g=g \bar{f}, f \asymp g: f \smile g \text { and } f^{\circ} \smile g^{\circ} . \\
& f: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
b \\
c
\end{array} \quad g: \begin{array}{l}
a \\
b \\
c
\end{array} \longrightarrow \begin{array}{l}
a \\
c
\end{array}
\end{aligned}
$$

## Example

$S$ a set. Pld ${ }_{S}$ category with one object $*$ and morphisms $Y \subseteq S$.

- Composition: intersection
- Identity: S
- Restriction of $Y: Y$
- Partial inverse of $Y: Y$
- Join: union.

Denotational semantics

## Terms and values

$\llbracket \Delta \rrbracket=\llbracket a_{1} \rrbracket \otimes \cdots \otimes \llbracket a_{n} \rrbracket$ whenever $\Delta \doteq x_{1}: a_{1}, \ldots, x_{n}: a_{n}$.
$\llbracket \Delta \vdash \operatorname{inj}, v: a \oplus b \rrbracket \doteq \iota \circ f$, whenever $f=\llbracket \Delta \vdash v: a \rrbracket$, and similarly for the right-projection.

If $f=\llbracket \Delta_{1} \vdash v_{1}: a_{1} \rrbracket$ and $g=\llbracket \Delta_{2} \vdash v_{2}: a_{2} \rrbracket$,
$\llbracket \Delta_{1}, \Delta_{2} \vdash\left\langle v_{1}, v_{2}\right\rangle: a_{1} \otimes a_{2} \rrbracket \doteq f \otimes g$.

Isos

An iso: $\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow v_{n}^{\prime}\right\}: a \leftrightarrow b$.

## Isos

An iso: $\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow V_{n}^{\prime}\right\}: a \leftrightarrow b$.
The situation:


## Isos

An iso: $\left\{\left|v_{1} \leftrightarrow V_{1}\right| \ldots \mid v_{n} \leftrightarrow V_{n}^{\prime}\right\}: a \leftrightarrow b$.
The situation:


What we'd like to build:


Careful: this diagram does not commute!

## Isos

## Lemma

If $v_{1} \perp v_{2}$ and $v_{1}^{\prime} \perp v_{2}^{\prime}$, then $\llbracket v_{1} \rrbracket \circ \llbracket v_{1} \rrbracket^{\circ} \asymp \llbracket v_{2} \rrbracket \circ \llbracket v_{2} \rrbracket^{\circ}$.

## Isos

## Lemma

If $v_{1} \perp v_{2}$ and $v_{1}^{\prime} \perp v_{2}^{\prime}$, then $\llbracket v_{1} \rrbracket \circ \llbracket v_{1} \rrbracket^{\circ} \asymp \llbracket v_{2} \rrbracket \circ \llbracket v_{2} \rrbracket^{\circ}$.
The definition of isos in the language involves these orthogonalities, thus all $\llbracket v_{i} \rrbracket \circ \llbracket v_{i} \rrbracket^{\circ}$ form a compatible set.

## Isos

## Lemma

$$
\text { If } v_{1} \perp v_{2} \text { and } v_{1}^{\prime} \perp v_{2}^{\prime} \text {, then } \llbracket v_{1} \rrbracket \circ \llbracket v_{1} \rrbracket^{\circ} \asymp \llbracket v_{2} \rrbracket \circ \llbracket v_{2} \rrbracket^{\circ} \text {. }
$$

The definition of isos in the language involves these orthogonalities, thus all $\llbracket v_{i} \rrbracket \circ \llbracket v_{i} \rrbracket^{\circ}$ form a compatible set.

## Definition

$\llbracket \vdash_{\omega}\left\{\left|v_{1} \leftrightarrow V_{1}\right| v_{2} \leftrightarrow V_{2} \ldots\right\}: a \leftrightarrow b \rrbracket=\bigvee_{i} \llbracket v_{i} \rrbracket \circ \llbracket v_{i} \rrbracket^{\circ}: \llbracket a \rrbracket \rightarrow \llbracket b \rrbracket$.

## What we would like to have thus

The language is based is pattern-matching:

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\vdash_{\omega}\left\{\left|v_{1} \leftrightarrow V_{1}\right| v_{2} \leftrightarrow V_{2} \ldots\right\}
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decidable which pattern a term $u$ fits.
It should also be in the denotational semantics:

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\left(\left(\llbracket v_{1} \rrbracket \circ \llbracket v_{1} \rrbracket^{\circ}\right) \vee\left(\llbracket v_{2} \rrbracket \circ \llbracket v_{2} \rrbracket^{\circ}\right)\right) \circ \llbracket u \rrbracket=\llbracket v_{i} \rrbracket \circ \llbracket v_{i} \rrbracket^{\circ} \circ \llbracket u \rrbracket
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for some $i$.
Usually, it relies on disjointness ( $\oplus$ ).

## Pattern-matching and

## consistency

## The origin of consistency

In $(f \vee g) h$, we want $h$ to choose:

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f h \vee g h=f h \text { or } f h \vee g h=g h
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Consistency implies pattern-matching.
What about the other way around?

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## Definition (Strongly non decomposable)

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## Going further

## Recursion

$\mathcal{C}$ a join inverse category.
For any $A, B$ objects of $\mathcal{C}, \operatorname{Hom}_{\mathcal{C}}(A, B)$ is a DCPO.

## Quantum programming language

Patterns are vectors in a basis of a Hilbert space.
Work in progress.

## Thank you


[^0]:    ${ }^{1}$ Rolf Landauer (1961): Irreversibility and Heat Generation in the Computing Process. IBM Journal of Research and Development. 5(3), pp. 183-191, doi:10.1147/rd.53.0183.

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[^5]:    ${ }^{1}$ Rolf Landauer (1961): Irreversibility and Heat Generation in the Computing Process. IBM Journal of Research and Development. 5(3), pp. 183-191, doi:10.1147/rd.53.0183.

