# Categorical Semantics of Reversible Pattern-Matching





September 1st, 2021

CALCO2021 MFPSXXXVII

# **Reversible programming**

• Reversible computation: only apply bijections.

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Reversible programming language

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- Reversible computation: only apply bijections.
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What we are focusing on today:

- Reversible programming language
- Detailed denotational semantics

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(Value types)	a, b ::=	$lpha \mid a \oplus b \mid a \otimes b$
(lso types)	<i>T</i> ::=	$a \leftrightarrow b$
(Values)	v ::=	$c_{lpha} \mid x \mid \operatorname{inj}_{l} v \mid \operatorname{inj}_{r} v \mid \langle v_{1}, v_{2} \rangle$
(Functions)	$\omega$ ::=	$\{ \mid v_1 \leftrightarrow v'_1 \mid \ldots \mid v_n \leftrightarrow v'_n \}$
(Terms)	t ::=	$v \mid \omega t$

 $\{ \mid \operatorname{inj}_{I} x \leftrightarrow \operatorname{inj}_{r} x \mid \operatorname{inj}_{r} x \leftrightarrow \operatorname{inj}_{I} x \}$ 

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- With compatible domains
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Operational semantics:

 $\{ | \operatorname{inj}_{I} x \leftrightarrow \operatorname{inj}_{r} x | \operatorname{inj}_{r} x \leftrightarrow \operatorname{inj}_{I} x \} (\operatorname{inj}_{r} \star) \longrightarrow \operatorname{inj}_{I} \star$ 

# **Categorical model**

Domain and partiality

Partial inverse

Compatibility

Domain and partiality Restriction category:  $f \mapsto \overline{f}$ .

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#### Example

S a set.  $PId_S$  category with one object \* and morphisms  $Y \subseteq S$ .

- Composition: intersection
- Identity: S
- Restriction of Y: Y
- Partial inverse of Y: Y
- Join: union.

# **Denotational semantics**

$$\llbracket \Delta \rrbracket = \llbracket a_1 \rrbracket \otimes \cdots \otimes \llbracket a_n \rrbracket$$
 whenever  $\Delta \doteq x_1 : a_1, \dots, x_n : a_n$ .

 $\llbracket \Delta \vdash \operatorname{inj}_{I} v : a \oplus b \rrbracket \doteq \iota_{I} \circ f$ , whenever  $f = \llbracket \Delta \vdash v : a \rrbracket$ , and similarly for the right-projection.

If 
$$f = \llbracket \Delta_1 \vdash v_1 : a_1 \rrbracket$$
 and  $g = \llbracket \Delta_2 \vdash v_2 : a_2 \rrbracket$ ,  
 $\llbracket \Delta_1, \Delta_2 \vdash \langle v_1, v_2 \rangle : a_1 \otimes a_2 \rrbracket \doteq f \otimes g$ .

## $\mathsf{An iso:} \ \{ \ \mid \ v_1 \leftrightarrow v_1' \mid \ldots \mid \ v_n \leftrightarrow v_n' \ \} : a \leftrightarrow b.$

An iso:  $\{ | v_1 \leftrightarrow v'_1 | \dots | v_n \leftrightarrow v'_n \} : a \leftrightarrow b.$ 

The situation:



An iso:  $\{ | v_1 \leftrightarrow v'_1 | \dots | v_n \leftrightarrow v'_n \} : a \leftrightarrow b.$ 

The situation:



What we'd like to build:



Careful: this diagram does not commute!

#### Lemma

If  $v_1 \perp v_2$  and  $v'_1 \perp v'_2$ , then  $\llbracket v'_1 \rrbracket \circ \llbracket v_1 \rrbracket^{\circ} \asymp \llbracket v'_2 \rrbracket \circ \llbracket v_2 \rrbracket^{\circ}$ .

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The definition of isos in the language involves these orthogonalities, thus all  $\llbracket v_i \rrbracket \circ \llbracket v_i \rrbracket^\circ$  form a compatible set.

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#### Definition

$$\llbracket \vdash_{\omega} \{ | v_1 \leftrightarrow v'_1 | v_2 \leftrightarrow v'_2 \dots \} : a \leftrightarrow b \rrbracket = \bigvee_i \llbracket v'_i \rrbracket \circ \llbracket v_i \rrbracket^\circ : \llbracket a \rrbracket \to \llbracket b \rrbracket.$$

The language is based is pattern-matching:

$$\vdash_{\omega} \{ \mid v_1 \leftrightarrow v'_1 \mid v_2 \leftrightarrow v'_2 \dots \}$$

decidable which pattern a term u fits. It should also be in the denotational semantics:

$$\left(\left(\llbracket v_1'\rrbracket \circ \llbracket v_1\rrbracket^\circ\right) \lor \left(\llbracket v_2'\rrbracket \circ \llbracket v_2\rrbracket^\circ\right)\right) \circ \llbracket u\rrbracket = \llbracket v_i'\rrbracket \circ \llbracket v_i\rrbracket^\circ \circ \llbracket u\rrbracket$$

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Usually, it relies on disjointness  $(\oplus)$ .

# Pattern-matching and consistency

In  $(f \lor g)h$ , we want h to choose:

 $fh \lor gh = fh$  or  $fh \lor gh = gh$ 

A join inverse category with this property will be called *pattern-matching* category.

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For this, we need h to be not decomposable either.

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**Consistency:** for any morphism *k*, if *h* is non decomposable, so is *kh*. A join inverse category with this property will be called *consistent* category.

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Consistency implies pattern-matching.

In  $(f \lor g)h$ , we want h to choose:

 $fh \lor gh = fh$  or  $fh \lor gh = gh$ 

A join inverse category with this property will be called *pattern-matching* category. With disjointness ( $\oplus$ ), rather fh = 0 or gh = 0. **The idea:** translate it as a notion of non decomposability, to have  $fh \lor gh$  not decomposable.

For this, we need h to be not decomposable either.

**Consistency:** for any morphism *k*, if *h* is non decomposable, so is *kh*. A join inverse category with this property will be called *consistent* category.

Consistency implies pattern-matching.

What about the other way around?

#### Definition (Strongly non decomposable)

Thus  $m = u \lor v \Rightarrow u = 0$  or v = 0

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Definition (Linearly non decomposable)

$$0 - \cdots - f - m$$

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Definition (Weakly non decomposable)



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Definition (Weakly non decomposable)



Thus  $m = u \lor v \Rightarrow u = m$  or v = m

strongly 
$$\Rightarrow$$
 linearly  $\Rightarrow$  weakly

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#### Theorem

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#### Recursion

 $\mathcal C$  a join inverse category.

For any A, B objects of C,  $Hom_{\mathcal{C}}(A, B)$  is a DCPO.

#### Quantum programming language

Patterns are vectors in a basis of a Hilbert space. Work in progress.

# Thank you