

Central Submonads and Notions of Computation: Soundness, Completeness and Internal Languages

Titouan Carette, Louis Lemonnier, Vladimir Zamdzhiev



Laboratoire
Méthodes
Formelles



LICS'23. Boston, MA. June, 29th 2023

Motivation

p1:	p2:
do	do
x <- op1	y <- op2
y <- op2	x <- op1
f x y	f x y

Two examples of monadic sequencing in Haskell.

- Monads represent computational effects.
- Monads have an algebraic flavour (category theory).
- Centre (algebraically): elements that commute with all others.
- **Intuition:** If `op1` **or** `op2` is central, `p1` and `p2` should be equivalent.

Question

What is centrality for monads?

Background: Premonoidal categories

[Power and Robinson, 1997]: premonoidal categories as model of effects.

Tensor \otimes is not a bifunctor.



- A premonoidal category \mathcal{P} has a centre $Z(\mathcal{P})$;
- \otimes is a bifunctor in $Z(\mathcal{P})$;
- $Z(\mathcal{P})$ is a monoidal category.

Link with Monads

The Kleisli category of a strong monad is a premonoidal category.

Strong monads

Correspondence:

- Effects \leftrightarrow monads;
- pairing \leftrightarrow monoidal structure \otimes .

Strong monad: combines the two with a strength τ .

$$\tau_{X,Y}: X \otimes TY \rightarrow T(X \otimes Y).$$

Operationally:

Input: $\langle x, M \rangle$

Output: `do $y \leftarrow M$; return $\langle x, y \rangle$`

Examples on Set

Writer monad

- Monoid M with centre $Z(M)$.
- Monad $- (M \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$.
- Centre $- (Z(M) \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$.

Powerset monad

- Commutative.
- Centre $-$ itself.

Link with Lawvere theories

- Lawvere theory \mathbf{T} with centre $Z(\mathbf{T})$.
- Monad $-$ induced by \mathbf{T} .
- Centre $-$ induced by $Z(\mathbf{T})$.

Central cone

Commutative monad \mathcal{T} :

$$\begin{array}{ccccc}
 & TX \otimes TY & \xrightarrow{\tau'} & T(X \otimes TY) & \\
 & \swarrow & & \downarrow T\tau & \\
 TX \otimes TY & & & T^2(X \otimes Y) & \\
 \downarrow \tau & & & \downarrow \mu & \\
 T(TX \otimes Y) & \xrightarrow{T\tau'} & T^2(X \otimes Y) & \xrightarrow{\mu} & T(X \otimes Y)
 \end{array}$$

commutes for every objects X and Y .

Central cone of \mathcal{T} at **fixed** X :

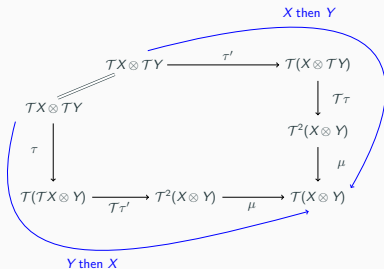
A pair $(Z, \iota: Z \rightarrow TX)$ such that

$$\begin{array}{ccccc}
 & Z \otimes TY & \xrightarrow{\iota \otimes TY} & TX \otimes TY & \xrightarrow{\tau'} & T(X \otimes TY) \\
 & \downarrow \iota \otimes TY & & & \downarrow T\tau & \\
 TX \otimes TY & & & T^2(X \otimes Y) & & \\
 \downarrow \tau & & & \downarrow \mu & & \\
 T(TX \otimes Y) & \xrightarrow{T\tau'} & T^2(X \otimes Y) & \xrightarrow{\mu} & T(X \otimes Y)
 \end{array}$$

commutes for every object Y .

Central cone

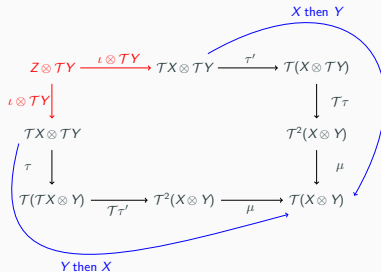
Commutative monad \mathcal{T} :



commutes for every objects X and Y .

Central cone of \mathcal{T} at **fixed** X :

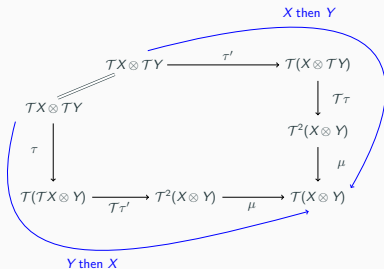
A pair $(Z, \iota: Z \rightarrow TX)$ such that



commutes for every object Y .

Central cone

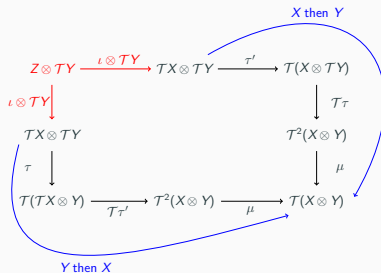
Commutative monad \mathcal{T} :



commutes for every objects X and Y .

Central cone of \mathcal{T} at **fixed** X :

A pair $(Z, \iota: Z \rightarrow TX)$ such that



commutes for every object Y .

If the universal central cone at X exists, write $\mathcal{Z}X \stackrel{\text{def}}{=} Z$.

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$.

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$.

Corollaries

- (1) All strong monads on **Set**, **DCPO**, **Top**, **Vect**, ... are centralisable.

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong \mathcal{Z}(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow \mathcal{Z}(\mathbf{C}_{\mathcal{T}})$.

Corollaries

- (1) All strong monads on **Set**, **DCPO**, **Top**, **Vect**, ... are centralisable.
- (2) A commutative monad is its own centre.

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$.

Corollaries

- (1) All strong monads on **Set**, **DCPO**, **Top**, **Vect**, ... are centralisable.
- (2) A commutative monad is its own centre.
- (3) If \mathbf{C} closed and total, every strong monad on it admits a centre.

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow Z(\mathbf{C}_{\mathcal{T}})$.

Corollaries

- (1) All strong monads on **Set**, **DCPO**, **Top**, **Vect**, ... are centralisable.
- (2) A commutative monad is its own centre.
- (3) If \mathbf{C} closed and total, every strong monad on it admits a centre.

All strong monads centralisable?

Main Result: Centralisability

Theorem

Equivalent conditions for a strong monad to be **centralisable**:

1. Existence of all universal central cones.
2. Existence of a monad \mathcal{Z} s.t. $\mathbf{C}_{\mathcal{Z}} \cong \mathcal{Z}(\mathbf{C}_{\mathcal{T}})$.
3. Left adjoint $\mathbf{C} \rightarrow \mathbf{C}_{\mathcal{T}}$ corestricts to a left adjoint $\mathbf{C} \rightarrow \mathcal{Z}(\mathbf{C}_{\mathcal{T}})$.

Corollaries

- (1) All strong monads on **Set**, **DCPO**, **Top**, **Vect**, ... are centralisable.
- (2) A commutative monad is its own centre.
- (3) If \mathbf{C} closed and total, every strong monad on it admits a centre.

All strong monads centralisable? No, but only artificial counterexamples!

Theorem

Given a submonad \mathcal{S} with $\iota: \mathcal{S} \hookrightarrow \mathcal{T}$, equivalent conditions:

1. ι_X is a central cone for all X ;
2. there exists a canonical embedding $\mathbf{C}_{\mathcal{S}} \hookrightarrow Z(\mathbf{C}_{\mathcal{T}})$.

When the centre \mathcal{Z} exists, also equivalent to:

3. \mathcal{S} is a submonad of \mathcal{Z} .

Definition: \mathcal{S} – *central* submonad if it satisfies 1, 2 or 3.

Remark: the centre is the universal central submonad.

Computational Interpretation

Language	Model
Simply-typed λ -calculus (ST λ C)	Cartesian Closed Category (CCC)
Moggi's metalanguage	CCC with strong monad \mathcal{T}
???	CCC with central submonad $\mathcal{S} \hookrightarrow \mathcal{T}$

Computational Interpretation

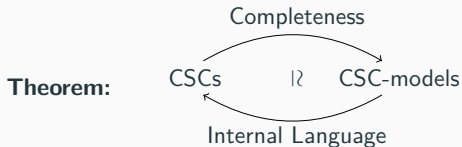
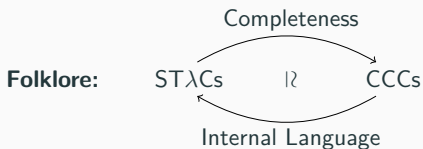
Language	Model
Simply-typed λ -calculus (ST λ C)	Cartesian Closed Category (CCC)
Moggi's metalanguage	CCC with strong monad \mathcal{T}
CSC	CCC with central submonad $\mathcal{S} \hookrightarrow \mathcal{T}$

CSC (Central Submonad Calculus):

- Simply-typed λ -calculus;
- + new types: $\mathcal{S}X$ and $\mathcal{T}X$;
- + terms for monadic computation (à la Moggi);
- + equational rules, such as:

$$\begin{aligned} \Gamma \vdash \text{do}_{\mathcal{T}} x \leftarrow \iota M; \text{do}_{\mathcal{T}} y \leftarrow N; P \\ = \text{do}_{\mathcal{T}} y \leftarrow N; \text{do}_{\mathcal{T}} x \leftarrow \iota M; P : \mathcal{T}C \end{aligned}$$

Completeness and Internal Language



What we have done:

- notion of centre for strong monads;
- equivalent conditions for a strong monad to have a centre;
- equivalent conditions for a submonad to be central;
- computational interpretation: completeness and internal language.

Thank you!

More details: <https://arxiv.org/abs/2207.09190>;
and my PhD thesis, available before September 2024.



Power, J. and Robinson, E. P. (1997).

Premonoidal categories and notions of computation.

Math. Struct. Comput. Sci., 7:453–468.