Ph.D. Qualifying Exam – Spring 2002

First problem

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any function. Let E be the set of points $x \in \mathbb{R}$ such that

$$f'_+(x) = \lim_{\substack{h \to 0 \\ h>0}} \frac{f(x+h) - f(x)}{h} \qquad \text{and} \qquad \lim_{\substack{h \to 0 \\ h<0}} \frac{f(x+h) - f(x)}{h}$$

both exist and are different. Prove that E is countable.

Solution

Let $$\mathrm{F} = \left\{ x \in \mathbb{R} \;\middle|\; f'_+(x), f'_-(x) \text{ exist and } f'_+(x) > f'_-(x) \right\}$$

and $$\mathrm{G} = \left\{ x \in \mathbb{R} \;\middle|\; f'_+(x), f'_-(x) \text{ exist and } f'_+(x) < f'_-(x) \right\}$$

so that $$\mathrm{E} = \mathrm{F} \cup \mathrm{G}$$

We will show first that F is countable. If $x \in \mathrm{F}$, since the inequality $f'_+(x) > f'_-(x)$ is strict, the interval $(f'_-(x), f'_+(x))$ is nonempty and thus contains a rational q. Because it is open, there is a positive ω such that $(q - \omega, q + \omega) \subset (f'_-(x), f'_+(x))$. Thus

$$\mathrm{F} \subset \bigcup_{\substack{q \in \mathbb{Q} \\ \omega > 0}} \mathrm{F}_{q,\omega}$$

where $\mathrm{F}_{q,\omega} = \left\{ x \in \mathbb{R} \;\middle|\; f'_+(x,), f'_-(x) \text{ exist and } f'_+(x) > q - \omega > q + \omega > f'_-(x) \right\}$

Fix a rational number q and let x_0 be in $\mathrm{F}_{q,\omega}$ for some positive ω. That is, $x \in \bigcup_{\omega>0} \mathrm{F}_{q,\omega}$. Consider the auxiliary function

$$\forall x \in \mathbb{R} \qquad g(x) = f(x) - qx$$

Then $g'_+(x_0)$ and $g'_-(x_0)$ both exist since $x \longmapsto qx$ is differentiable at x_0, and we have

$$g'_+(x_0) = f'_+(x_0) - q \qquad g'_-(x_0) = f'_-(x_0) - q$$

so that $$g'_+(x_0) > \omega > -\omega > g'_-(x_0) \tag{1}$$

This implies that g has a strict local minimum at x_0. Indeed, the existence of $g'_+(x_0)$ and $g'_-(x_0)$ tell us that there exists a positive η such that

$$\forall h \in (0, \eta) \qquad \frac{g(x_0 + h) - g(x_0)}{h} - g'_+(x_0) > -\frac{\omega}{2} \tag{2}$$

and $\qquad \forall h \in (-\eta, 0) \qquad \dfrac{g(x_0 - h) - g(x_0)}{h} - g'_-(x_0) < \dfrac{\omega}{2}$ **(3)**

From **(1)** and **(2)** $\quad \forall h \in (0, \eta) \qquad g(x_0 + h) - g(x_0) > h\Big(g'_+(x_0) - \dfrac{\omega}{2}\Big) > \dfrac{h\omega}{2} > 0$

while from **(1)** and **(3)** $\quad \forall h \in (-\eta, 0) \qquad g(x_0 + h) - g(x_0) > h\Big(g'_-(x_0) + \dfrac{\omega}{2}\Big)$

$$> -\frac{h\omega}{2} > 0$$

Notice how there was a change in the direction of inequality **(3)** when multiplying both sides by h, due to the fact that $h < 0$. Anyhow, we obtained

$$\forall h \in (-\eta, \eta) \quad h \neq 0 \qquad g(x_0 + h) - g(x_0) > 0$$

which confirms that x_0 is a strict minimum for g in $(x_0 - \eta, x_0 + \eta)$. Of course, there exists an integer n big enough so that $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \subset (-\eta, \eta)$. Therefore, we showed that

$$\Big(\bigcup_{\omega>0} \mathrm{F}_{q,\omega}\Big) \subset \bigcup_{n\in\mathbb{N}^\star} \mathrm{M}_n \tag{4}$$

where $\qquad \mathrm{M}_n = \Big\{x \in \mathbb{R} \;\Big|\; g \text{ has a strict minimum at } x \text{ in } \Big(x - \dfrac{1}{n}, x + \dfrac{1}{n}\Big)\Big\}$

Finally, we show that each M_n is countable. If x and y are in M_n, such that $|x - y| < \frac{1}{n}$, we know that, unless $x = y$, we have

$$g(x) < g(y) \qquad \text{and} \qquad g(y) < g(x)$$

which is impossible. Thus if x and y are distinct in M_n, they are at least at distance $\frac{1}{n}$ apart. Which proves that M_n is countable. By **(4)**, $\bigcup_{\omega>0} \mathrm{F}_{q,\omega}$ is countable, and therefore F is countable. Applying this result to $-f$, we get that G is countable as well. Therefore,

E is countable.

Second problem

Consider functions $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^2(0,1)$ such that $f_n \xrightarrow[n\to\infty]{} f$ and $g_n \xrightarrow[n\to\infty]{} g$ weakly in $\mathrm{L}^2(0,1)$.

1. Show that the L^2 norms of the $(f_n)_{n\in\mathbb{N}}$'s are uniformly bounded.
2. Show by example that $(f_n g_n)_{n\in\mathbb{N}}$ need not converge to fg in the weak star topology of $\mathrm{L}^1(0,1)$.
3. Suppose $(h_n)_{n\in\mathbb{N}}$ and h are in $\mathrm{L}^1(0,1)$ and that the L^1 norms of the $(h_n)_{n\in\mathbb{N}}$ are uniformly bounded. Show that $h_n \xrightarrow[n\to\infty]{} h$ in the weak star topology of L^1 if and only if each Fourier coefficient of h_n converges to the corresponding Fourier coefficient of h.
4. Suppose that the Fourier coefficients of the $(f_n)_{n\in\mathbb{N}}$'s and $(g_n)_{n\in\mathbb{N}}$'s are 0 at negative integers. Show that $f_n g_n \xrightarrow[n\to\infty]{} fg$ in the weak star topology of $\mathrm{L}^1(0,1)$.

Solution

1 This is a consequence of the Banach-Steinhaus theorem, as well as the fact that $\mathrm{L}^2(0,1)$ is its own dual, through Riesz's representation theorem for Hilbert spaces.

Indeed, the $(f_n)_{n\in\mathbb{N}}$ define a collection of bounded linear functionals on $\mathrm{L}^2(0,1)$ as follows:

$$\varphi_n : g \longmapsto \langle f_n | g \rangle$$

and

$$\|\varphi_n\| = \operatorname*{Sup}_{\substack{g\in\mathrm{L}^2 \\ \|g\|=1}} (\varphi, g) = \operatorname*{Sup}_{\substack{g\in\mathrm{L}^2 \\ \|g\|=1}} \langle f_n | g \rangle = \|f_n\|$$

Furthermore, the $(f_n)_{n\in\mathbb{N}}$ converge weakly, which means in particular that the sequence $\big(\langle f_n | g \rangle\big)_{n\in\mathbb{N}}$ is bounded for every g. Since $\mathrm{L}^2(0,1)$ is complete, the principle of uniform boundedness tells us that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is bounded. It follows that $(f_n)_{n\in\mathbb{N}}$ is bounded.

2 Let $\quad \forall n \in \mathbb{N} \quad f_n : x \longmapsto \mathrm{e}^{-2\pi i n x} \quad$ and $\quad g_n : x \longmapsto \mathrm{e}^{2\pi i n x}$

The Riemann-Lebesgue lemma tells us that

$$\forall h \in \mathrm{L}^1(0,1) \qquad \lim_{n\to\infty} \int_{[0,1]} f_n h = \lim_{n\to\infty} \int_{[0,1]} g_n h = 0$$

In particular $\quad \forall h \in \mathscr{C}([0,1]) \qquad \lim_{n\to\infty} \int_{[0,1]} f_n h = \lim_{n\to\infty} \int_{[0,1]} g_n h = 0$

so $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ both converge to 0 in the weak-$\star$ topology of $\mathrm{L}^1(0,1)$. But

$$\forall n \in \mathbb{N} \qquad f_n g_n = 1$$

so $(f_n g_n)_{n\in\mathbb{N}}$ converges to 1 in the weak-$\star$ topology of $\mathrm{L}^1(0,1)$.

3 Suppose $(h_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{L}^1(0,1)$, let h be another L^1 function.

First, if $(h_n)_{n\in\mathbb{N}}$ converges to h in the weak-$\star$ topology of $\mathrm{L}^1(0,1)$, then in particular,

$$\forall p \in \mathbb{Z} \qquad \widehat{h_n}(p) = \int_0^1 h_n(t)\mathrm{e}^{-2\pi\mathrm{i}pt}\,\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^1 h(t)\mathrm{e}^{-2\pi\mathrm{i}pt}\,\mathrm{d}t = \widehat{h}(p) \tag{1}$$

since the functions $t \longmapsto \mathrm{e}^{-2\pi\mathrm{i}pt}$ are continuous on $[0,1]$.

Conversely, suppose that (**1**) holds. Then because integrating and taking limits are linear operations, we get

$$\forall g \in \mathscr{P} \qquad \int_{[0,1]} h_n g \xrightarrow[n\to\infty]{} \int_{[0,1]} hg$$

Here, $\mathscr{P}$ is the set of 1-periodic trigonometric polynomials. Let f be any continuous function such that $f(0) = f(1)$. We know that $\mathscr{P}$ is dense in the set of such functions (the convolutions $\mathrm{K}_n \star f$ converge to f uniformly, where K_n is the n-th Fejér kernel), so if ϵ is a positive number, there exists a trigonometric polynomial g such that $\|f-g\|_\infty < \epsilon$. Now, we have for every integer n:

$$\begin{aligned}\left|\int_{[0,1]}(h_n-h)f\right| &\leqslant \left|\int_{[0,1]}(h_n-h)(f-g)\right| + \left|\int_{[0,1]}(h_n-h)g\right| \\ &\leqslant \|h_n-h\|_1\|f-g\|_\infty + \left|\int_{[0,1]}(h_n-h)g\right|\end{aligned} \tag{2}$$

Because the sequence $(h_n)_{n\in\mathbb{N}}$ is bounded, we can find a positive real number M such that

$$\|h\|_1 \leqslant \mathrm{M} \qquad \text{and} \qquad \forall n \in \mathbb{N} \qquad \|h_n\|_1 \leqslant \mathrm{M}$$

And because g is a trigonometric polynomial,

$$\lim_{n\to\infty}\int_{[0,1]}(h_n-h)g = 0$$

Therefore, taking the limsup in (**2**) yields

$$\limsup_{n\to\infty}\left|\int_{[0,1]}(h_n-h)g\right| \leqslant 2\mathrm{M}\epsilon$$

This holds for every positive ϵ. Thus

$$\lim_{n\to\infty}\int_{[0,1]} h_n f = \int_{[0,1]} hf$$

Finally, if we want to show something similar for functions f that are continuous, but don't satisfy the endpoint condition $f(0) = f(1)$, we can proceed as in the 2003 exam,

problem 5: first let f be the indicator function of an interval and approximate it by functions satisfying the endpoint condition. Then use the fact that linear combinations of indicators of intervals are dense in $\mathscr{C}([0,1])$.

$\boxed{(h_n)_{n\in\mathbb{N}} \text{ converges to } h \text{ in the weak-}\star \text{ topology of } \mathrm{L}^1.}$

4 We go back to the initial situation: $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences in $\mathrm{L}^2(0,1)$, converging weakly respectively to f and g. We suppose that the $(f_n)_{n\in\mathbb{N}}$'s and $(g_n)_{n\in\mathbb{N}}$'s have their Fourier coefficients equal to 0 at negative integers. Then f (and g) have the same property since

$$\forall p \in \mathbb{Z} \qquad \widehat{f}(p) = \int_0^1 f(t)\mathrm{e}^{-2\pi \mathrm{i} p t}\,\mathrm{d}t = \lim_{n\to\infty} \int_0^1 f_n(t)\mathrm{e}^{-2\pi \mathrm{i} p t}\,\mathrm{d}t = \lim_{n\to\infty} \widehat{f_n}(p) = 0 \qquad (\mathbf{3})$$

Define $\qquad \forall n \in \mathbb{N} \qquad h_n = f_n g_n \qquad \text{and} \qquad h = fg$

As we saw in the first question, there exists a positive number M such that

$$\forall n \in \mathbb{N} \qquad \|f_n\|_2, \|g_n\|_2 \leqslant \mathrm{M}$$

Therefore $\qquad \forall n \in \mathbb{N} \qquad \|h_n\|_1 \leqslant \|f_n\|_2 \, \|g_n\|_2 \leqslant \mathrm{M}^2$

Since $h_n = f_n g_n$, its Fourier coefficients are obtained as the convolution of the Fourier coefficients of f_n and g_n:

$$\forall p \in \mathbb{N} \qquad \widehat{h_n}(p) = \sum_{k\in\mathbb{Z}} \widehat{f_n}(k)\, \widehat{g_n}(p-k) = \sum_{k=0}^{p} \widehat{f_n}(k)\, \widehat{g_n}(p-k) \qquad (\mathbf{4})$$

and if $p < 0$, $\qquad \widehat{h_n}(p) = \sum_{k\in\mathbb{Z}} \widehat{f_n}(k)\, \widehat{g_n}(p-k) = \sum_{k\in\mathbb{N}} \widehat{f_n}(k)\, \underbrace{\widehat{g_n}(p-k)}_{=0} = 0$

Similarly $\qquad \forall p \in \mathbb{Z} \qquad \widehat{h}(p) = \begin{cases} 0 & \text{if} \quad p < 0 \\ \displaystyle\sum_{k=0}^{p} \widehat{f}(k)\, \widehat{g}(p-k) & \text{if} \quad p \geqslant 0 \end{cases} \qquad (\mathbf{5})$

Because of (**3**), and because (**4**) and (**5**) only involve finite sums, it follows that

$$\forall p \in \mathbb{Z} \qquad \widehat{h_n}(p) \xrightarrow[n\to\infty]{} \widehat{h}(p)$$

Using the result from the third question,

$\boxed{(h_n)_{n\in\mathbb{N}} \text{ converges to } h \text{ in the weak-}\star \text{ topology of } \mathrm{L}^1(0,1).}$

Third problem

Consider a $\mathscr{C}^\infty$ function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}$, there exists an integer k such that $f^{(k)}(x) = 0$. Let

$$\mathrm{U} = \big\{ x \in \mathbb{R} \;\big|\; f \text{ is a polynomial in a neighbourhood of } x \big\}$$

1. Prove that U is a dense open subset of $\mathbb{R}$.

2. Prove that the complement of U contains no isolated points.

Solution

1 We first show that U is open. Let $x_0 \in \mathrm{U}$. Then, by definition of U, there exists a positive ϵ and a polynomial function P such that $f = \mathrm{P}$ on $\mathrm{I} = (x_0 - \epsilon, x_0 + \epsilon)$.

If x is any point in that interval, letting $r = |x - x_0|$, the interval $\big(x-(\epsilon-r), x+(\epsilon-r)\big)$ is included in I, on which $f = \mathrm{P}$. Thus f is a polynomial in a neighbourhood of x, which shows that x is in U as well, and therefore $\mathrm{I} \subset \mathrm{U}$:

U is open.

Next we show that U is dense in $\mathbb{R}$. Let O be a nonempty open set in $\mathbb{R}$. Because every point x in $\mathbb{R}$ is such that $f^{(k)}(x) = 0$ for some k, we have

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} \mathrm{F}_k$$

where $\qquad \forall k \in \mathbb{N} \qquad \mathrm{F}_k = \big\{ x \in \mathbb{R} \mid f^{(k)}(x) = 0 \big\}$

Therefore $\qquad \overline{\mathrm{O}} = \bigcup_{k \in \mathbb{N}} \big(\overline{\mathrm{O}} \cap \mathrm{F}_k \big)$

By Baire's lemma, there exists a k such that $\overline{\mathrm{O}} \cap \mathrm{F}_k$ has nonempty interior. This interior is actually $\mathrm{O} \cap \overset{\circ}{\mathrm{F}}_k$.

> Remember indeed that $\mathrm{Int}(\mathrm{A} \cap \mathrm{B}) = \overset{\circ}{\mathrm{A}} \cap \overset{\circ}{\mathrm{B}}$. The similar property with unions, however, is false. Take for example
>
> $$\mathrm{A} = \mathbb{Q} \qquad \mathrm{B} = \mathbb{R} \setminus \mathbb{Q} \qquad \overset{\circ}{\mathrm{A}} = \emptyset \qquad \overset{\circ}{\mathrm{B}} = \emptyset$$
>
> Then $\qquad \overset{\circ}{\mathrm{A}} \cup \overset{\circ}{\mathrm{B}} = \emptyset \qquad$ while $\qquad \mathrm{Int}(\mathrm{A} \cup \mathrm{B}) = \overset{\circ}{\mathbb{R}} = \mathbb{R}$
>
> Similarly, for all sets A and B, we have $\overline{\mathrm{A}} \cup \overline{\mathrm{B}} = \overline{\mathrm{A} \cup \mathrm{B}}$, but the property with the intersections is false in general.

So there is an open ball $\mathscr{B}(x_0, \epsilon)$ in $\mathrm{O} \cap \overset{\circ}{\mathrm{F}}_k$. Then in particular

$$\forall x \in (x_0 - \epsilon, x_0 + \epsilon) \qquad f^{(k)}(x) = 0$$

Since f is obtained by multiple integrations of $f^{(k)}$, it follows that f is a polynomial of degree at most $k-1$ in $(x_0 - \epsilon, x_0 + \epsilon)$. Which proves that $x_0 \in \mathrm{U}$, so that $\mathrm{O} \cap \mathrm{U}$ is not empty. Every open subset of $\mathbb{R}$ intersets U nontrivially, or in other words,

$$\boxed{\text{U is dense in } \mathbb{R}.}$$

$\boxed{2}$ We first show that if (a, b) is an open interval contained in U, then f is a polynomial in (a, b). For this, let $x_0 \in (a, b) \subset \mathrm{U}$. Then we know that for some positive ϵ, there exists a polynomial function P such that $f = \mathrm{P}$ on $(x_0 - \epsilon, x_0 + \epsilon)$. Let

$$\mathrm{M} = \mathrm{Sup}\left\{x \in (x_0, b) \;\middle|\; f = \mathrm{P} \text{ in } (x_0, x)\right\}$$

Then $$\epsilon \leqslant \mathrm{M} \leqslant b$$

and we will show that M is actually equal to b. Suppose it is not the case. Then we know that $f = \mathrm{P}$ on (x_0, M) and $\mathrm{M} \in (a, b) \subset \mathrm{U}$. Therefore, there exists a positive η and a polynomial Q such that $f = \mathrm{Q}$ on $(\mathrm{M} - \eta, \mathrm{M} + \eta)$. But then

$$\forall x \in (\mathrm{M} - \eta, \mathrm{M}) \qquad \mathrm{P}(x) - \mathrm{Q}(x) = f(x) - f(x) = 0$$

$\mathrm{P} - \mathrm{Q}$ is a polynomial with infinitely many zeroes, so it has to be the 0 polynomial: $\mathrm{P} = \mathrm{Q}$. It follows that $f = \mathrm{P}$ on $(x_0, \mathrm{M} + \eta)$, which contradicts the definition of M. Thus $\mathrm{M} = b$.

Similarly, we would show that

$$\mathrm{Inf}\left\{x \in (a, x_0) \;\middle|\; f = \mathrm{P} \text{ in } (x, x_0)\right\} = a$$

It follows that $f = \mathrm{P}$ on all of (a, b).

Now, going back to the question asked. Suppose that $x \in \mathbb{R} \setminus \mathrm{U}$ is isolated. Then there exists a positive ϵ such that $(x_0 - \epsilon, x_0)$ and $(x_0, x_0 + \epsilon)$ sit entirely inside U. From what we just showed, there are polynomials P and Q such that

$$\forall x \in (x_0 - \epsilon, x_0) \qquad f(x) = \mathrm{P}(x)$$

and $$\forall x \in (x_0, x_0 + \epsilon) \qquad f(x) = \mathrm{Q}(x)$$

Let k be the biggest number of $\deg(\mathrm{P})$ and $\deg(\mathrm{Q})$. Then $\mathrm{P}^{(k+1)} = 0$ and $\mathrm{Q}^{(k+1)} = 0$. As a consequence,

$$\forall x \in (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} \qquad f^{(k+1)}(x) = \mathrm{P}^{(k+1)}(x) \text{ or } \mathrm{Q}^{(k+1)}(x) = 0$$

Because $f^{(k+1)}$ is continuous at x_0, it follows that $f^{(k+1)}(x_0) = 0$ as well. So $f^{(k+1)} = 0$ on the whole interval $(x_0 - \epsilon, x_0 + \epsilon)$. By integrating $k+1$ times, it follows that f is a polynomial function in that interval. Thus x_0 is in U, which gives us a contradication.

$$\boxed{\text{The complement of U has no isolated points.}}$$

Fourth problem

Prove the inequality

$$\forall a \in \ell^2(\mathbb{Z}) \qquad \sum_{m\in\mathbb{Z}} \left| \sum_{\substack{n\in\mathbb{Z}\\ n\neq m}} \frac{a_n}{n-m} \right|^2 \leqslant \pi^2 \|a\|^2$$

Show also that π^2 cannot be replaced by a smaller constant.

Solution

Let f be the 1-periodic function coinciding with $x \longmapsto x-\frac{1}{2}$ on $(0,1)$ and let's compute its Fourier coefficients. Obviously, since f is symmetric around the point $\frac{1}{2}$, we'll have $\widehat{f}(0) = 0$. So let $n \in \mathbb{Z}$ nonzero and integrate by parts:

$$\begin{aligned}
\widehat{f}(n) &= \int_0^1 \left(x - \frac{1}{2}\right) \mathrm{e}^{-2\pi \mathrm{i} n x}\, \mathrm{d}x = \int_0^1 x \mathrm{e}^{-2\pi \mathrm{i} n x}\, \mathrm{d}x \\
&= \frac{1}{-2\pi \mathrm{i} n} \left[x \mathrm{e}^{-2\pi \mathrm{i} n x}\right]_0^1 + \frac{1}{2\pi \mathrm{i} n} \int_0^1 \mathrm{e}^{-2\pi \mathrm{i} n x}\, \mathrm{d}x \\
\widehat{f}(n) &= -\frac{1}{2\pi \mathrm{i} n}
\end{aligned}$$

Let a be any sequence in $\ell^2(\mathbb{Z})$ and let g be the $\mathrm{L}^2(0,1)$ function whose Fourier coefficients are the terms of the sequence a:

$$g = \sum_{n\in\mathbb{Z}} a_n \mathrm{e}_n \qquad \text{where} \qquad \mathrm{e}_n : x \longmapsto \mathrm{e}^{2\pi \mathrm{i} n x}$$

We know that fg is $\mathrm{L}^2(0,1)$ (since f is bounded), whose Fourier coefficients are obtained by convolution of f and g's coefficients:

$$\forall m \in \mathbb{N} \qquad \widehat{fg}(m) = \sum_{n\in\mathbb{Z}} a_n\, \widehat{f}(m-n) = -\frac{1}{2\pi \mathrm{i}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq m}} \frac{a_n}{m-n}$$

And Parseval's formula tells us that

$$\|fg\|_2^2 = \sum_{m\in\mathbb{Z}} \left|\widehat{fg}\right|^2 = \frac{1}{4\pi^2} \sum_{m\in\mathbb{Z}} \left| \sum_{\substack{n\in\mathbb{Z}\\ m\neq n}} \frac{a_n}{m-n} \right|^2$$

Thus

$$\boxed{\sum_{m\in\mathbb{Z}} \left| \sum_{\substack{n\in\mathbb{Z}\\ m\neq n}} \frac{a_n}{m-n} \right|^2 = 4\pi^2 \|fg\|_2^2 \leqslant 4\pi^2 \|f\|_\infty^2 \, \|g\|_2^2 = \pi^2 \sum_{m\in\mathbb{Z}} |a_n|^2}$$

To show that the inequality is sharp, it is enough to show that

$$\forall g \in \mathrm{L}^2 \qquad \|fg\|_2^2 \leqslant \frac{\|g\|_2^2}{4}$$

is optimal. All we have to do is find a sequence of $\mathrm{L}^2(0,1)$ functions $(g_n)_{n\in\mathbb{N}}$ that will pick up the value $\frac{1}{4}$, which is the supremum of f^2 on $(0,1)$. So let

$$g_n = \sqrt{\frac{n}{2}}\big(\mathbb{1}_{(0,\frac{1}{n})} + \mathbb{1}_{(1-\frac{1}{n},1)}\big)$$

Then

$$|g_n|^2 = \frac{n}{2}\big(\mathbb{1}_{(0,\frac{1}{n})} + \mathbb{1}_{(1-\frac{1}{n},1)}\big)$$

so that

$$\|g_n\|_2^2 = 1$$

Now, we also have, using the fact that f^2 and g_n^2 are symmetric around the line $x = \frac{1}{2}$:

$$\|fg_n\|_2^2 = 2\int_{[0,\frac{1}{2}]} f^2 g_n^2 = n\int_0^{\frac{1}{n}} \left(x - \frac{1}{2}\right)^2 \mathrm{d}x = \frac{n}{3}\left(\left(\frac{1}{n} - \frac{1}{2}\right) - \frac{1}{8}\right) = \frac{1}{3n^2} - \frac{1}{2n} + \frac{1}{4}$$

This tends to $\frac{1}{4}$ as n goes to ∞.

Fifth problem

Let K be an L^1 function on the unit square $[0,1]^2$. Suppose that for every continuous function f on $[0,1]$, we have

$$\int_{[0,1]} \mathrm{K}(x,y) f(y)\,\mathrm{d}y \qquad \text{for almost every } x$$

Prove that $\mathrm{K} = 0$ almost everywhere.

Solution

Denote the unit square $[0,1]^2$ by Q and consider the subspace $\mathscr{F}$ of $\mathscr{C}(\mathrm{Q})$ spanned by functions of the type

$$(x,y) \longmapsto g(x)f(y) \qquad f,g \in \mathscr{C}([0,1])$$

It is easy to check that a product of finite linear combinations of such functions is still a finite linear combination of such functions. Thus $\mathscr{F}$ is a subalgebra of $\mathscr{C}([0,1])$. It separates points, since if (x_1,y_1) and (x_2,y_2) are distinct points in Q, they are separated by the function $(x,y) \longmapsto (x-x_1)^2 + (y-y_1)^2$. Also, $\mathscr{F}$ contains the constant function 1. Therefore, $\mathscr{F}$ is dense in $\mathscr{C}(\mathrm{Q})$.

Now, let f and g be any two continuous functions on $[0,1]$. Since K is $\mathrm{L}^1(\mathrm{Q})$,

$$\int_0^1 \int_0^1 \big|\mathrm{K}(x,y)g(x)f(y)\big|\,\mathrm{d}y\mathrm{d}x \leqslant \|f\|_\infty \|g\|_\infty \|\mathrm{K}\|_1 < \infty$$

and therefore $\displaystyle\int_{\mathrm{Q}} \mathrm{K}(x,y)g(x)f(y)\,\mathrm{d}\lambda(x,y) = \int_0^1 g(x)\left(\int_0^1 \mathrm{K}(x,y)f(y)\,\mathrm{d}y\right)\mathrm{d}x = 0$

by Fubini and by hypothesis on K. By linearity,

$$\forall f \in \mathscr{F} \qquad \int_{\mathrm{Q}} \mathrm{K}f = 0$$

But $$\|\mathrm{K}\|_1 = \operatorname*{Sup}_{\substack{f\in\mathscr{C}(\mathrm{Q})\\ \|f\|_\infty=1}} \int_{\mathrm{Q}} \mathrm{K}f = \operatorname*{Sup}_{\substack{f\in\mathscr{F}\\ \|f\|_\infty=1}} \int_{\mathrm{Q}} \mathrm{K}f$$

since $\mathscr{F}$ is dense in $\mathscr{C}(\mathrm{Q})$. Thus $\|\mathrm{K}\|_1 = 0$, and

K is 0 almost everywhere.

Sixth problem

Show that there do not exist measurable sets A and B with positive measure such that $\mathrm{A} \cap (\mathrm{B} - r) = \emptyset$ for every rational number r.

Solution

Suppose that such sets exist. We might as well assume that A and B have positive *finite* measure, by intersecting them with a big enough compact set. Then $\mathbb{1}_{\mathrm{A}}$ and $\mathbb{1}_{\mathrm{B}}$ are both L^2 functions, and $\mathbb{1}_{\mathrm{A}} \star \mathbb{1}_{-\mathrm{B}}$ is continuous.

> This is a remarkable property of the convolution of L^2 functions. Indeed, if f and g are L^2, then
>
> $$\begin{aligned} \big|f \star g(x+h) - f \star g(h)\big| &= \left| \int_{\mathbb{R}} f(y)\big(g(x+h-y) - g(x-y)\big)\,\mathrm{d}y \right| \\ &\leqslant \|f\|_2 \bigg(\int_{\mathbb{R}} \big|g(y-h) - g(y)\big|^2\,\mathrm{d}y \bigg)^{\frac{1}{2}} \end{aligned}$$
>
> and we know that translations are continuous on $\mathrm{L}^2(\mathbb{R})$ so the righthandside tends to 0.
>
> Another way of seing it is using the Fourier transform:
>
> $$\widehat{f \star g} = \widehat{f}\,\widehat{g}$$
>
> But the Fourier transform is an isometry of L^2 onto itself, therefore $\widehat{f \star g}$ is L^1. So $f \star g$, as the inverse Fourier transform of an L^1 function, is continuous.
>
> In fact, more generally, the convolution of an L^p and an $\mathrm{L}^{p'}$ function yields a continuous function.

We have $\qquad \forall x \in \mathbb{R} \qquad \mathbb{1}_{\mathrm{A}} \star \mathbb{1}_{-\mathrm{B}}(x) = \displaystyle\int_{\mathbb{R}} \mathbb{1}_{\mathrm{A}}(y)\mathbb{1}_{-\mathrm{B}}(x-y)\,\mathrm{d}y$

But $\mathbb{1}_{\mathrm{A}}(y)\mathbb{1}_{-\mathrm{B}}(x-y)$ is 1 if and only if

$$\begin{cases} y \in \mathrm{A} \\ x - y \in -\mathrm{B} \end{cases} \qquad \text{that is} \qquad y \in \mathrm{A} \cap (x + \mathrm{B})$$

and is 0 otherwise. Therefore, $\mathbb{1}_{\mathrm{A}} \star \mathbb{1}_{-\mathrm{B}}$ is 0 at every rational number. But this function is continuous, and rationals are dense in $\mathbb{R}$, therefore

$$\forall x \in \mathbb{R} \qquad \mathbb{1}_{\mathrm{A}} \star \mathbb{1}_{-\mathrm{B}}(x) = 0$$

However $\qquad \big\|\mathbb{1}_{\mathrm{A}} \star \mathbb{1}_{-\mathrm{B}}\big\|_1 = \displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\mathrm{A}}(y)\mathbb{1}_{-\mathrm{B}}(x-y)\,\mathrm{d}y\mathrm{d}x = \mu(\mathrm{A})\mu(\mathrm{B}) \neq 0$

We have our contradiction.

Seventh problem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions on $[0,1]$. Let

$$\mathrm{E} = \Big\{x \in [0,1] \;\Big|\; \sum_{n\in\mathbb{N}} f_n(x) \text{ converges}\Big\}$$

Show that for every positive ϵ, there is a set F and an integer k such that

- F is in the ring of sets generated by sets of the form $f_i^{-1}(\mathrm{A})$ for $i \leqslant k$ and A Borel;
- $m(\mathrm{E}\Delta\mathrm{F}) < \epsilon$.

Solution

Using the fact the a sequence of real or complex numbers converges if and only if it is Cauchy, we see that

$$\mathrm{E} = \bigcap_{p\in\mathbb{N}^\star} \bigcup_{\mathrm{N}\in\mathbb{N}} \bigcap_{n\geqslant \mathrm{N}} \bigcap_{m\in\mathbb{N}} \Big\{ \Big| \sum_{k=n}^{n+m} f_k \Big| < \frac{1}{p} \Big\}$$

For every integers N, n and m, we let

$$\mathrm{A}_{p,n,m} = \Big\{ \Big| \sum_{k=n}^{n+m} f_k \Big| < \frac{1}{p} \Big\}$$

Notice that $\mathrm{A}_{p,n,m}$ is in the σ-algebra generated by sets of the form $f_i^{-1}(\mathrm{B})$ where $i \leqslant n+m$ and B Borel.

> In case you are not sure anymore why this is true, here is a way to do it. Let f and g be two measurable functions, let a be any real number. Then one checks easily that
>
> $$\{f+g<a\} = \{f<a-g\} = \bigcup_{r\in\mathbb{Q}} \big(\{f<r\} \cap \{g<a-r\}\big)$$
>
> which shows that $\{f+g<a\}$ is in the σ-algebra generated by sets of the form $f^{-1}(\mathrm{B}), g^{-1}(\mathrm{B})$ with B Borel. Inductively, this generalizes to sums of an arbitrary number of functions.

For every pair of integers K and M, we let

$$\mathrm{E}_{\mathrm{K,M}} = \bigcap_{p\in\mathbb{N}^\star} \bigcup_{\mathrm{N}\in\mathbb{N}} \bigcap_{n=\mathrm{N}}^{\mathrm{K}} \bigcap_{m=0}^{\mathrm{M}} \Big\{ \Big| \sum_{k=n}^{n+m} f_k \Big| < \frac{1}{p} \Big\}$$

This set is in the σ-algebra generated by the $f_i^{-1}(\mathrm{B})$ for B Borel and $i \leqslant \mathrm{K}+\mathrm{M}$.

It can probably be checked that $\mathbb{1}_{E_{K,M}}$ converges pointwise to $\mathbb{1}_E$ as K and M tend to ∞. Moreover, all the $E_{K,M}$'s are dominated by the constant function equal to 1, which is integrable on $[0,1]$. By the dominated convergence theorem,

$$\lim_{K,M\to\infty} \mathbb{1}_{E_{K,M}} = \mathbb{1}_E \quad \text{in } L^1(0,1)$$

or in other words, $\quad 0 = \lim_{K,M\to\infty} \left\| \mathbb{1}_{E_{K,M}} - \mathbb{1}_E \right\|_1 = \lim_{K,M\to\infty} \mu\big(E \Delta E_{K,M}\big)$

So if ϵ is a positive real number, just take K and M big enough, let $F = E_{K,M}$, so that

$$\boxed{\mu(E \Delta F) \leqslant \epsilon}$$

Eighth problem

Let $\mathrm{C} \in (0,1)$. Show that there are numbers δ_{N}, depending on C, with the following properties:

- If $\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{N}}$ are measurable sets in $[0,1]$ each with measure C, then

$$m(\mathrm{A}_i \cap \mathrm{A}_j) \geqslant (1 - \delta_{\mathrm{N}})\mathrm{C}^2$$

  for some $i \neq j$;

- $\delta_{\mathrm{N}} \xrightarrow[\mathrm{N}\to\infty]{} 0$.

Solution

Let N be a positive integer and $\mathrm{A}_1, \ldots, \mathrm{A}_{\mathrm{N}}$ be measurable sets in $[0,1]$, each with measure C. Consider the function $\mathrm{F} = \sum_{n=1}^{\mathrm{N}} \mathbb{1}_{\mathrm{A}_n}$. Then

$$\mathrm{F}^2 = \sum_{m,n=1}^{\mathrm{N}} \mathbb{1}_{\mathrm{A}_n}\mathbb{1}_{\mathrm{A}_m} = \sum_{n=1}^{\mathrm{N}} \mathbb{1}_{\mathrm{A}_n} + \sum_{n=1}^{\mathrm{N}} \sum_{\substack{m=1 \\ m\neq n}}^{\mathrm{N}} \mathbb{1}_{\mathrm{A}_n \cap \mathrm{A}_m}$$

and

$$\int_{[0,1]} \mathrm{F}^2 = \mathrm{NC} + \sum_{n=1}^{\mathrm{N}} \sum_{\substack{m=1 \\ m\neq n}}^{\mathrm{N}} m\big(\mathrm{A}_n \cap \mathrm{A}_m\big)$$

Now let i and j be distinct and such that

$$m(\mathrm{A}_i \cap \mathrm{A}_j) = \mathrm{Max}\left\{ m(\mathrm{A}_n \cap \mathrm{A}_m) \;\middle|\; m \neq n \right\}$$

Then

$$\mathrm{NC} + \mathrm{N}(\mathrm{N}-1)m(\mathrm{A}_i \cap \mathrm{A}_j) \geqslant \int_{[0,1]} \mathrm{F}^2 \geqslant \left(\int_{[0,1]} \mathrm{F} \right)^2 = \mathrm{N}^2\mathrm{C}^2$$

This last step is just Hölder's inequality applied to $\mathrm{F} = \mathrm{F} \times 1$. Anyhow, it follows that

and

$$m(\mathrm{A}_i \cap \mathrm{A}_j) \geqslant \frac{\mathrm{N}^2\mathrm{C}^2 - \mathrm{NC}}{\mathrm{N}(\mathrm{N}-1)} = \frac{\mathrm{N} - \frac{1}{\mathrm{C}}}{\mathrm{N}-1}\,\mathrm{C}^2$$

so take

$$\boxed{\delta_{\mathrm{N}} = \frac{1-\mathrm{C}}{\mathrm{NC}-\mathrm{C}}}$$

Ninth problem

Suppose the Banach space X is uniformly convex. That is, for every positive ϵ, there exists a positive δ such that

$$\forall x, y \in \mathrm{X} \qquad \left(\|x\| = \|y\| = 1 \quad \text{and} \quad \left\| \frac{x+y}{2} \right\| > 1 - \delta \right) \implies \|x - y\| \leqslant \epsilon$$

Let f be a bounded linear functional on X with norm 1. Show that there is a unique point $x \in \mathrm{X}$ with $\|x\| = 1$ such that $f(x) = 1$.

Solution

Since f has norm 1, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit sphere of X such that

$$\lim_{n\to\infty} (f, x_n) = 1 \tag{1}$$

Let ϵ be a positive real number; uniform convexity of X provides us with a δ such that

$$\forall x, y \in \mathrm{X} \qquad \left(\|x\| = \|y\| = 1 \quad \text{and} \quad \left\| \frac{x+y}{2} \right\| > 1 - \delta \right) \implies \|x - y\| \leqslant \epsilon$$

And (**1**) provides us with a positive integer N such that

$$\forall n \geqslant \mathrm{N} \qquad 1 - \delta < (f, x_n)$$

Thus
$$\forall n \geqslant \mathrm{N} \quad \forall p \in \mathbb{N} \qquad 1 - \delta < \left(f, \frac{x_n + x_{n+p}}{2} \right)$$

which implies in turn that
$$\left\| \frac{x_n + x_{n+p}}{2} \right\| \geqslant 1 - \delta$$

Therefore
$$\forall n \geqslant \mathrm{N} \quad \forall p \in \mathbb{N} \qquad \|x_{n+p} - x_n\| \leqslant \epsilon$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and therefore converges to some x. Of course,

$$\|x\| = 1 \qquad \text{and} \qquad (f, x) = \lim_{n\to\infty} (f, x_n) = 1$$

Now let $y \in \mathrm{X}$ satisfy this same property:

$$\|y\| = 1 \qquad \text{and} \qquad (f, y) = 1$$

Then
$$\left\| \frac{x+y}{2} \right\| \geqslant \left(f, \frac{x+y}{2} \right) = 1$$

which implies, by uniform convexity, that $\|x - y\| \leqslant \epsilon$ for every positive ϵ.

There exists a unique $x \in \mathrm{X}$ such that $(f, x) = 1$.

Tenth problem

1. Let μ be a finite measure on $\mathbb{R}$ and let ν be the measure with density e^{-x^2} with respect to μ. Show that the Fourier transform of ν is the restriction to $\mathbb{R}$ of an entire function F.

2. Express the n-th derivative $\mathrm{F}^{(n)}(0)$ in terms of μ.

3. Show that the set

$$\mathrm{S} = \left\{ p(x)\,\mathrm{e}^{-x^2} \;\middle|\; p \text{ polynomial} \right\}$$

is dense in $\mathscr{C}_0(\mathbb{R})$, the space of continuous functions tending to 0 at $\pm\infty$.

Solution

1 For every complex number z, define

$$\mathrm{F}(z) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zx}\,\mathrm{d}\nu(x) = \int_{\mathbb{R}} \mathrm{e}^{-x^2-\mathrm{i}zx}\,\mathrm{d}\mu(x)$$

Since the function $x \longmapsto \mathrm{e}^{-x^2+x\mathrm{Im}z}$ is continuous and tends to 0 at $\pm\infty$, it is bounded by some $\mathrm{M} > 0$ and we have

$$\int_{\mathbb{R}} \left|\mathrm{e}^{-x^2-\mathrm{i}zx}\right| \mathrm{d}\mu(x) = \int_{\mathbb{R}} \mathrm{e}^{-x^2+x\mathrm{Im}z}\,\mathrm{d}\mu(x) \leqslant \mathrm{M}\|\mu\| < \infty$$

Therefore, F is well defined at every $z \in \mathbb{C}$.

Now, let γ be any closed curve in $\mathbb{C}$. Since γ is compact, there exists a positive number C such that

$$\forall z \in \gamma \qquad \mathrm{Im}z \leqslant \mathrm{C}$$

Therefore $\displaystyle\int_\gamma \int_{\mathbb{R}} \left|\mathrm{e}^{-x^2-\mathrm{i}zx}\right| \mathrm{d}\mu(x)\mathrm{d}|z| \leqslant \int_\gamma \int_{\mathbb{R}} \mathrm{e}^{-x^2+\mathrm{C}x}\,\mathrm{d}\mu(x)\mathrm{d}|z| \leqslant \mathrm{F(C)length}(\gamma) < \infty$

which allows us to Fubinise the double integral

$$\int_\gamma \mathrm{F}(z)\,\mathrm{d}z = \int_{\mathbb{R}} \mathrm{e}^{-x^2} \int_\gamma \mathrm{e}^{-\mathrm{i}xz}\,\mathrm{d}z\mathrm{d}\mu(x) = 0$$

since $z \longmapsto \mathrm{e}^{-\mathrm{i}xz}$ is holomorphic in a neighbourhood of γ. Since this holds for every closed curve γ, F is analytic in $\mathbb{C}$. And of course, the restriction to $\mathbb{R}$ of F is the Fourier transform of ν.

$\widehat{\nu}$ is the restriction to $\mathbb{R}$ of an entire function.

> In the next question, we will see another strategy that could have been used to solve that question, and gives us at the same time the derivatives of F at 0. I simply wanted to present both methods because each of them is interesting and worth seeing at least once.

2 The function $x \longmapsto \mathrm{e}^{|zx|-x^2}$ is continuous and tends to 0 at $\pm\infty$, thus is bounded. We let $\mathrm{M} > 0$ be such that

$$\forall x \in \mathbb{R} \qquad \mathrm{e}^{|zx|-x^2} \leqslant \mathrm{M}$$

Thus
$$\int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \frac{|zx|^n}{n!} \mathrm{e}^{-x^2}\, \mathrm{d}\mu(x) = \int_{\mathbb{R}} \mathrm{e}^{|zx|-x^2}\, \mathrm{d}\mu(x) \leqslant \mathrm{M}\|\mu\| < \infty$$

which allows us, thanks to Fubini, to compute as we wish the double integral:

$$\mathrm{F}(z) = \int_{\mathbb{R}} \mathrm{e}^{-x^2-\mathrm{i}zx}\, \mathrm{d}\mu(x) = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} \frac{(-\mathrm{i}zx)^n}{n!}\, \mathrm{e}^{-x^2}\, \mathrm{d}\mu(x) = \sum_{n\in\mathbb{N}} \frac{z^n}{n!} \int_{\mathbb{R}} (-\mathrm{i}x)^n \mathrm{e}^{-x^2}\, \mathrm{d}\mu(x)$$

This gives us a power series expansion of F around zero. In particular, we get for free the derivatives of F at 0:

$$\boxed{\forall n \in \mathbb{N} \qquad \mathrm{F}^{(n)}(0) = \int_{\mathbb{R}} (-\mathrm{i}x)^n \mathrm{e}^{-x^2}\, \mathrm{d}\mu(x)}$$

> Of course, this is the method you would use during the exam, since it overkills questions 1 and 2.

3 Let
$$\mathrm{S} = \left\{ p(x)\mathrm{e}^{-x^2} \;\middle|\; p \text{ polynomial} \right\}$$

and let μ be a finite measure on $\mathbb{R}$ that vanishes on S:

$$\forall p \in \mathbb{C}[\mathrm{X}] \qquad \int_{\mathbb{R}} p(x)\mathrm{e}^{-x^2}\, \mathrm{d}\mu(x) = 0$$

In particular
$$\forall n \in \mathbb{N} \qquad \int_{\mathbb{R}} (-\mathrm{i}x)^n\, \mathrm{e}^{-x^2}\, \mathrm{d}\mu = 0$$

Hence, by questions 1 and 2, the Fourier transform of $\mathrm{e}^{-x^2}\, \mathrm{d}\mu$ is 0. And, by injectivity of the Fourier transform, this measure itself is 0:

$$\forall f \in \mathscr{C}_c(\mathbb{R}) \qquad \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-x^2}\, \mathrm{d}\mu(x) = 0$$

Now, if f is continuous with compact support, $x \longmapsto f(x)\mathrm{e}^{x^2}$ is also continuous with compact support and as a consequence,

$$\int_{\mathbb{R}} f \mathrm{d}\mu = \int_{\mathbb{R}} \left(f(x)\,\mathrm{e}^{x^2}\right) \mathrm{e}^{-x^2}\,\mathrm{d}\mu = 0$$

So μ vanishes on $\mathscr{C}_c(\mathbb{R})$. This set is dense in $\mathscr{C}_0(\mathbb{R})$, so μ also vanishes there and it follows that $\mu = 0$. Therefore,

$$\boxed{\text{S is dense in } \mathscr{C}_0(\mathbb{R}).}$$