

Alon-Saks-Seymour conjecture, Clique-Stable separation and the stubborn problem

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Supervized by Stéphan THOMASSÉ

July 23, 2012

1 Geographical context

2 Definitions

- Clique vs Independent Set Problem
- Alon-Saks-Seymour Conjecture
- The stubborn problem
- 3-COMPATIBLE COLORING PROBLEM

3 Results

- Overview
- Equivalence result
- On particular classes

4 Prospects

Geographical context



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46, allée d'Italie
69007 Lyon

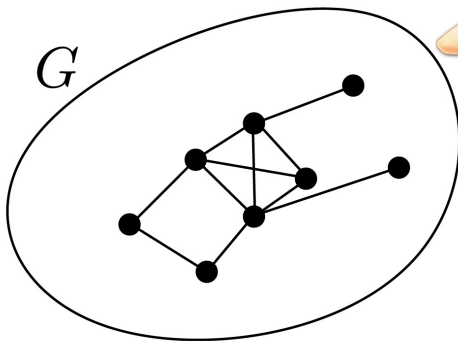
Supervised by Stéphan THOMASSÉ
Joint work with Nicolas BOUSQUET



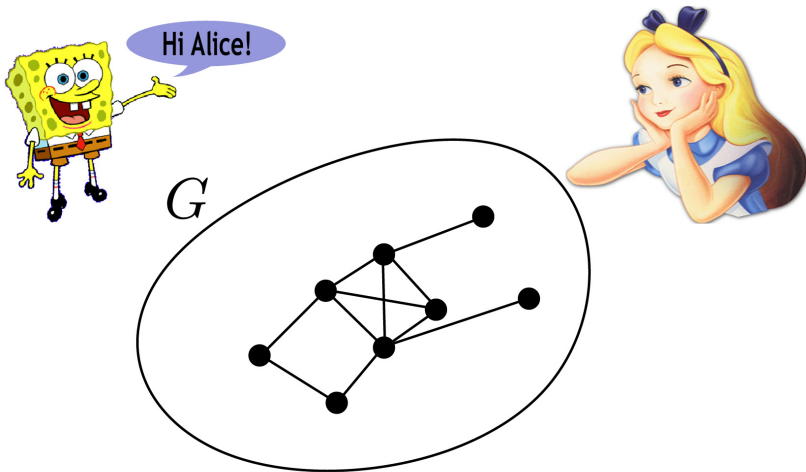
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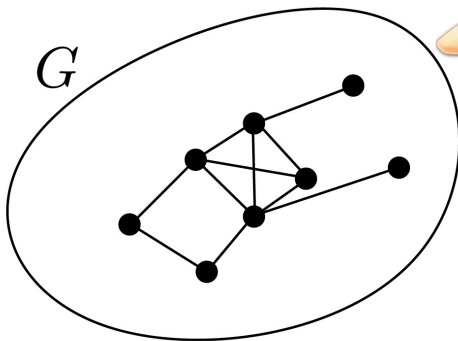
Clique vs Independent Set Problem



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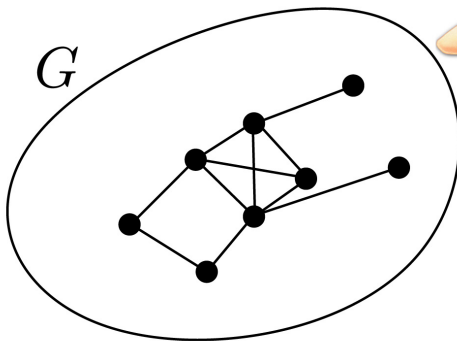
Hi Bob!



Clique vs Independent Set Problem



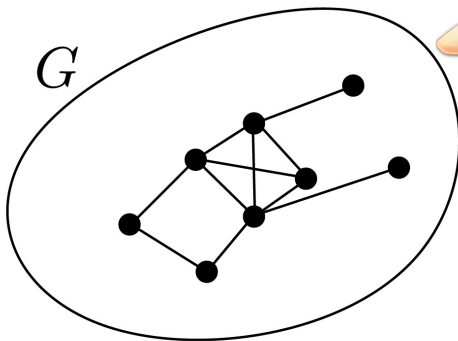
Do you see this graph?



Clique vs Independent Set Problem



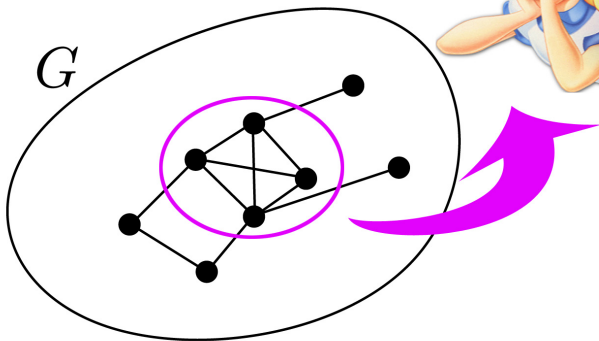
Yes! Let's talk about it.



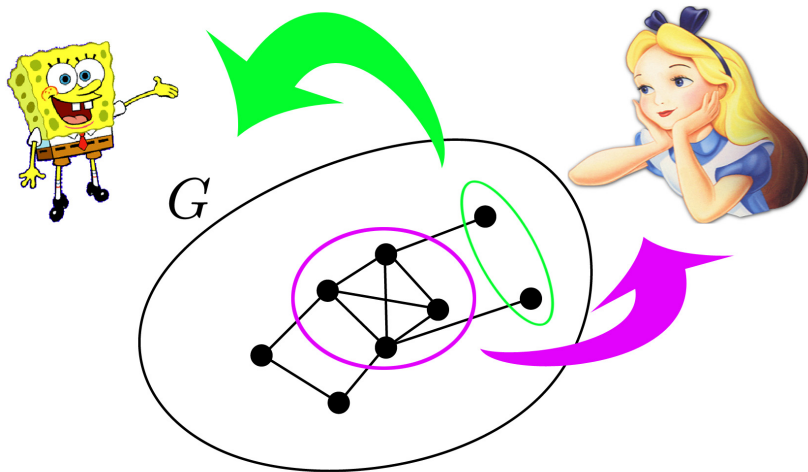
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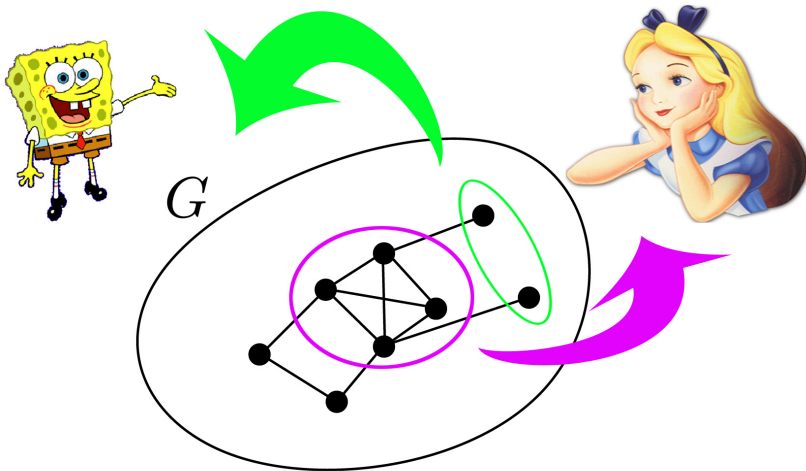
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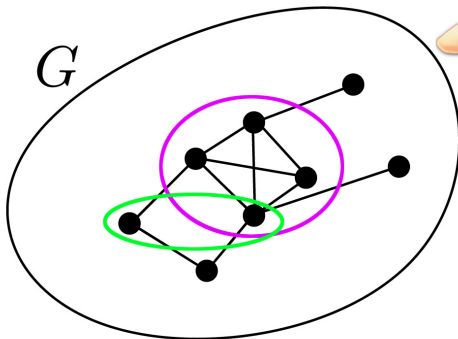


Clique vs Independent Set Problem

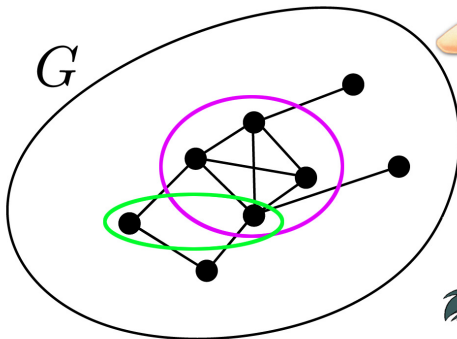


Do the clique and the stable intersect?

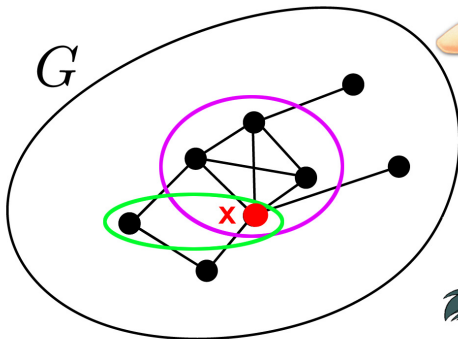
Clique vs Independent Set Problem : Non-det. version



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Yes!
Proof: **x**

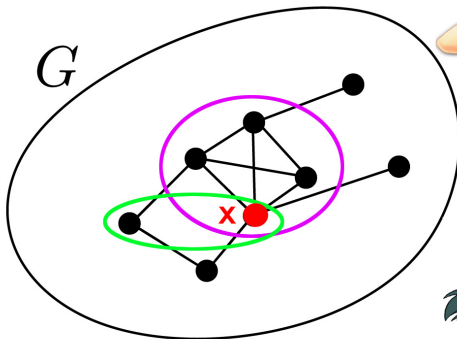
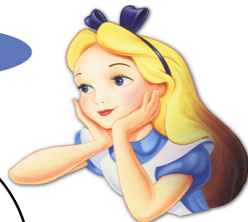


Clique vs Independent Set Problem : Non-det. version



I agree.

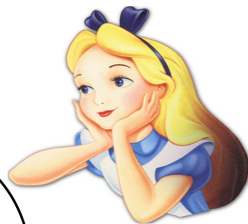
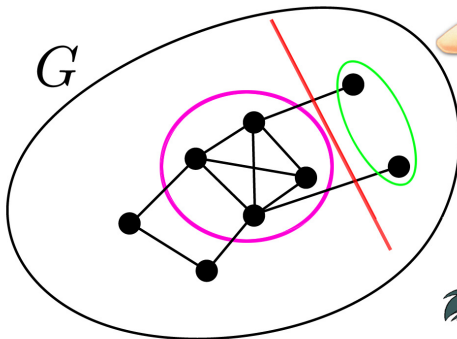
I agree.



Yes!
Proof: **x**



Clique vs Independent Set Problem : Non-det. version



No!
Proof: **cut**

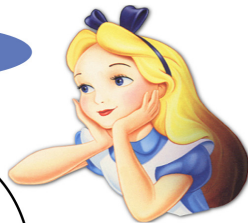
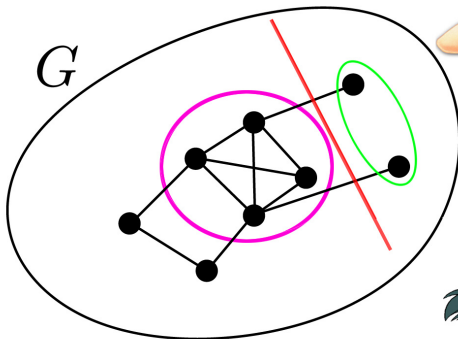


Clique vs Independent Set Problem : Non-det. version



I agree.

I agree.



No!
Proof: **cut**



Clique vs Independent Set Problem

Motivation: Yannakakis, 1991

Slack matrix for $STAB(G)$ in perfect graphs.

Existing bound

There is a Clique-Stable separator of size $n^{\log n}$.

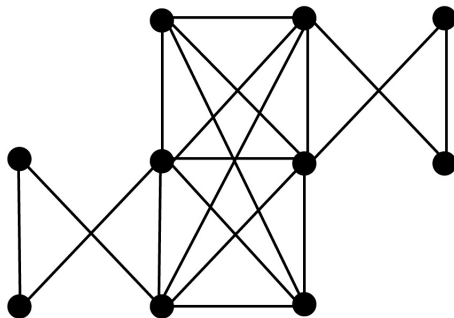
Question

Does there exists a polynomial P , such that for all graph G on n vertices, there exists a Clique-Stable separator of size $P(n)$?

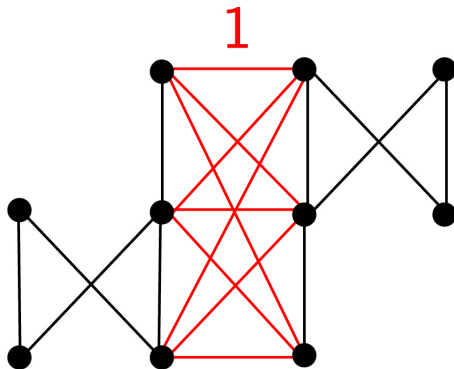
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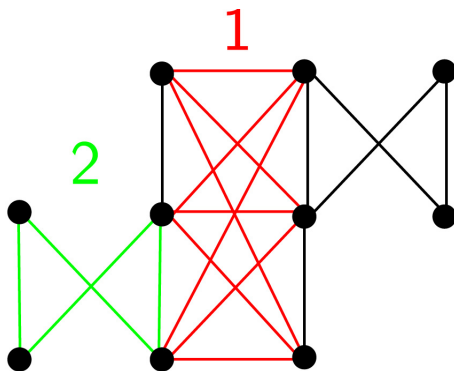
Bipartite packing $bp(G)$



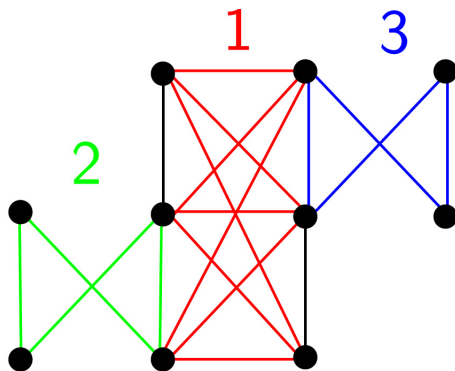
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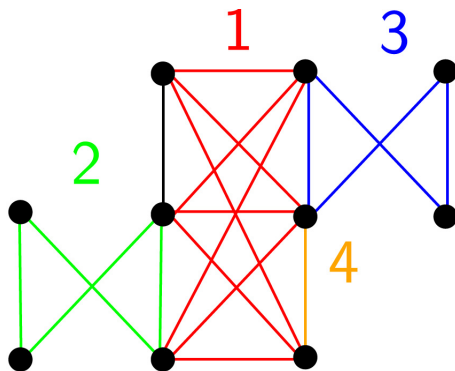
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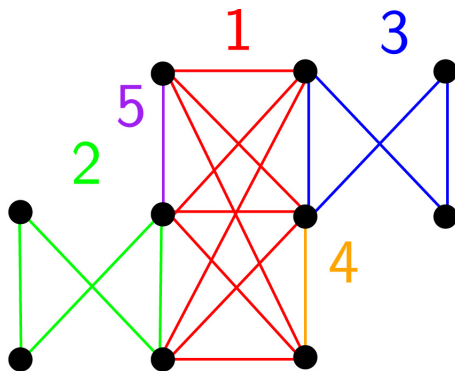
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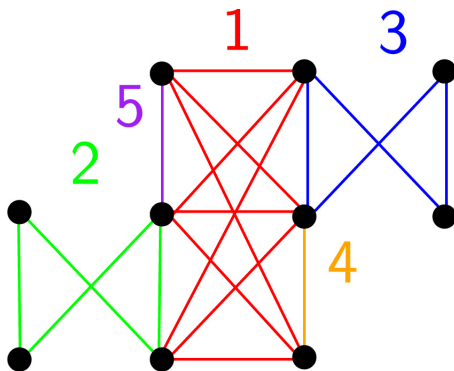


Bipartite packing $bp(G)$



Bipartite packing $\text{bp}(G)$





$$bp(G) = 5$$

Alon-Saks-Seymour conjecture

Motivation

Graham-Pollak theorem, 1971: $\text{bp}(K_n) = n - 1$.

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Alon-Saks-Seymour conjecture (≈ 1974)

If the edges of G can be partitioned into k edge-disjoint complete bipartite graphs, then $\chi \leq k + 1$.

In other words: $\chi \leq \text{bp} + 1$.

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Counter-example

Huang & Sudakov, 2010: There exists G such that $\chi \geq \text{bp}^{6/5}$.

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Question: Polynomial Alon-Saks-Seymour conjecture

Does there exist P such that for all G , $\chi \leq P(\text{bp})$.

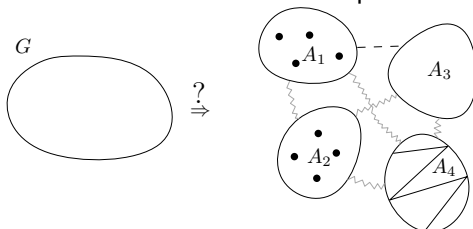
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List-M partition problems

$$\begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}$$

Matrix M for the stubborn problem.



Feder & Hell: Dichotomy theorem?

Classification P or NP-complete for small matrices, $k \leq 3$.

Classification for $k = 4$ except for the stubborn problem.

Existing bound

The stubborn problem can be solved in time $\mathcal{O}(n^{\log n})$ via decomposition into $\mathcal{O}(n^{\log n})$ instances of 2-SAT.

Question

Decomposing the problem into a polynomial number of 2-SAT instances?

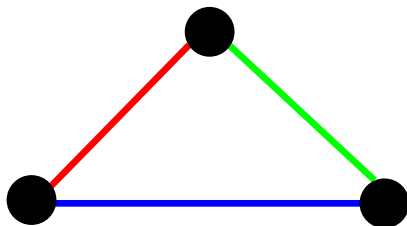
Complexity result

Cygan et al, 2010: The stubborn problem is polynomially solvable.

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3-COMPATIBLE COLORING PROBLEM

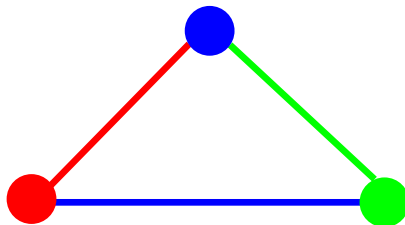


3-COMPATIBLE COLORING PROBLEM (3-CCP)

Input: An edge coloring f_E of the complete graph on n vertices with 3 colors $\{A, B, C\}$.

Question: Is there a coloring of the vertices with the same three colors, such that no edge has the same color as both its extremities?

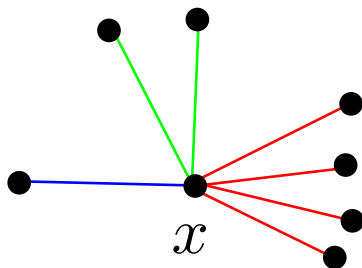
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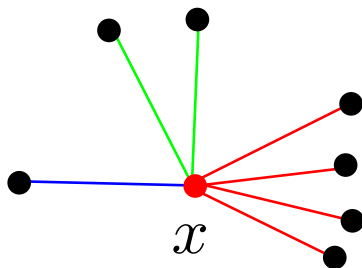


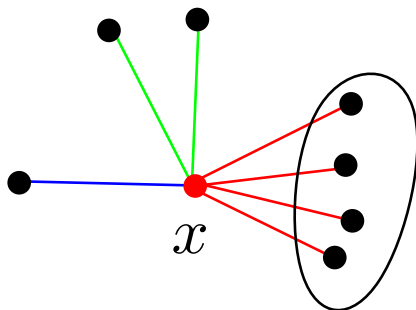
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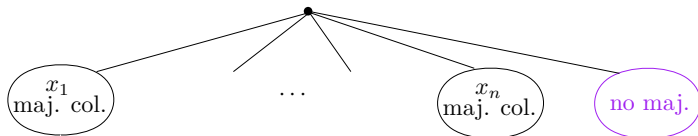
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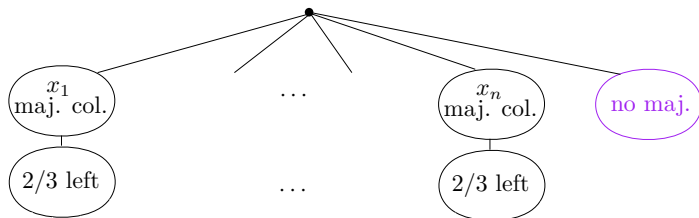


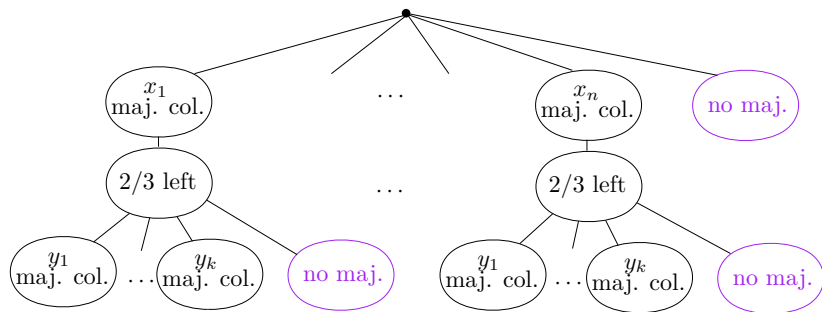


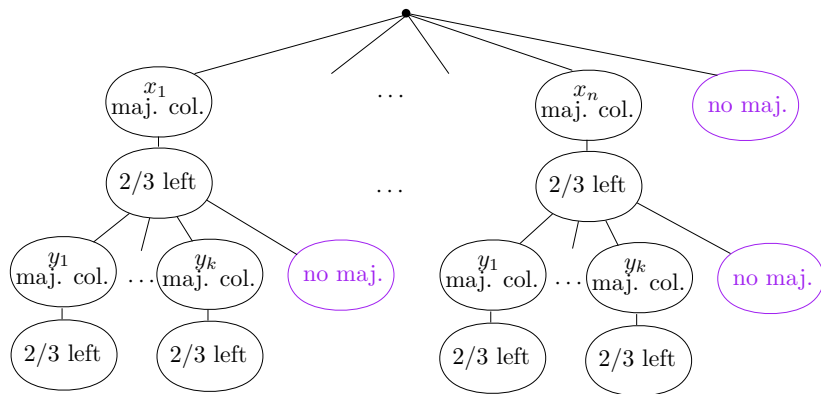


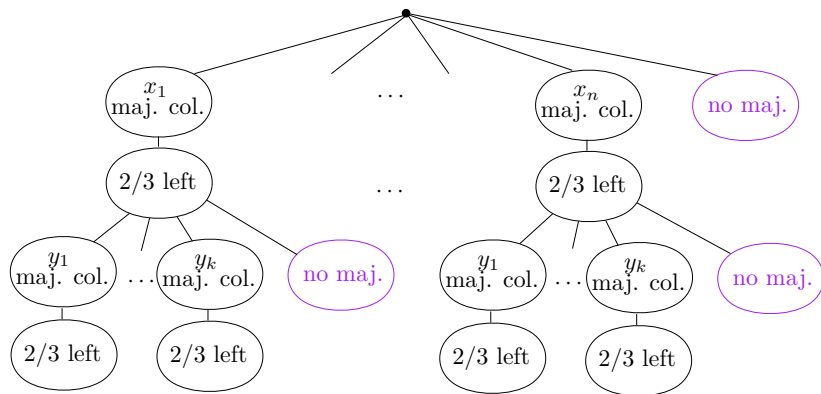
constraint $\{B, C\}$











Tree: degree $n + 1$, height $C \log n \rightarrow \mathcal{O}(n^{\log n})$ leaves

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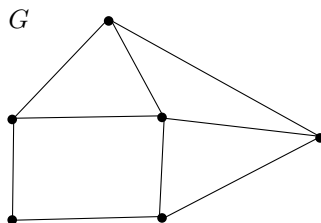
Equivalence theorem

The following are equivalent:

- ① There is a polynomial P such that for all graph G , $\chi \leq P(\text{bp})$.
- ② For all graph G , there is a polynomial CS-separator.
- ③ For all graph G and all list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on (G, \mathcal{L}) .
- ④ For all n and all edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f) .

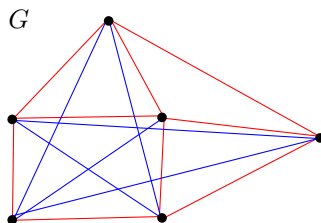
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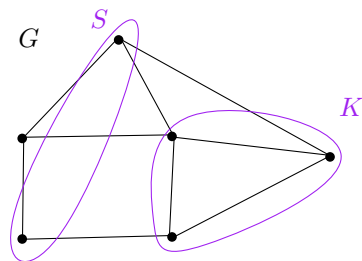
Lemma

$(2 \Rightarrow 3)$: Polynomial 2-list covering for 3-CCP for all $G \Rightarrow$
Polynomial CS-separator for all G .



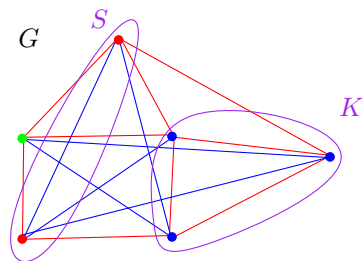
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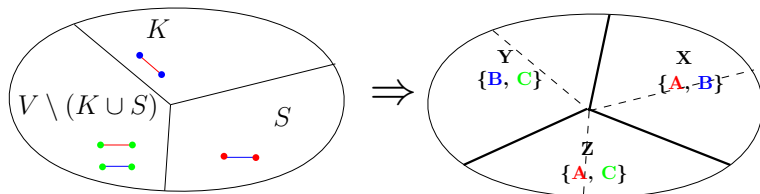
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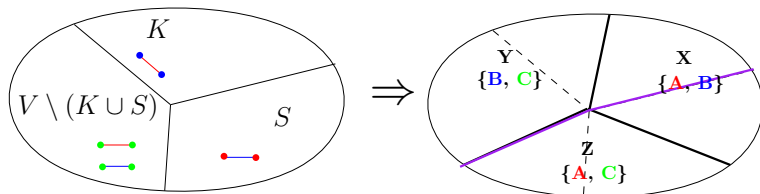
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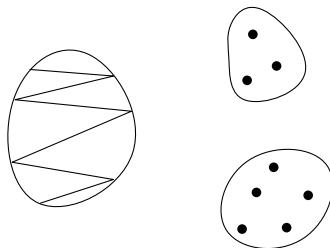


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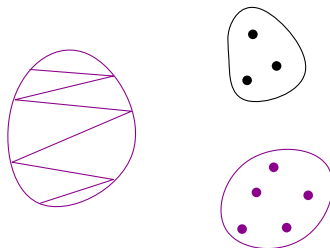
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$(3 \Rightarrow 1)$: Polynomial CS-separator for all $G \Rightarrow$ Polynomial 2-list covering for the stubborn problem for all G .



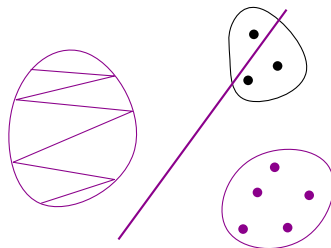
Observation

If there are p cuts that separate all the cliques and the stable sets, then there are p^2 cuts that separate all the cliques and the union of two stables.



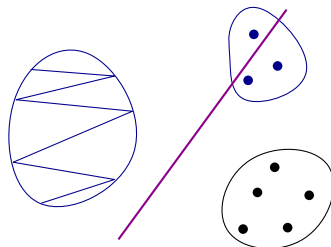
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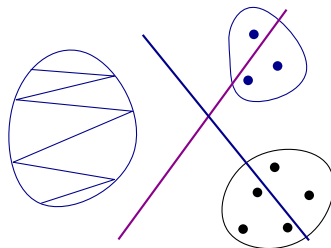
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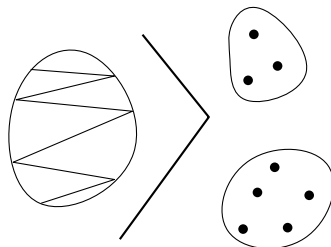
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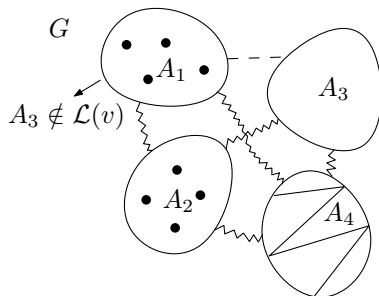
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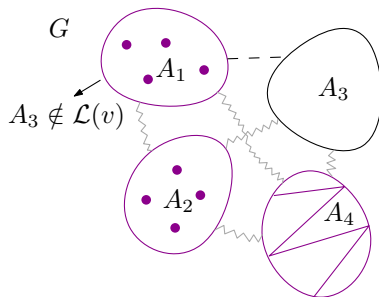
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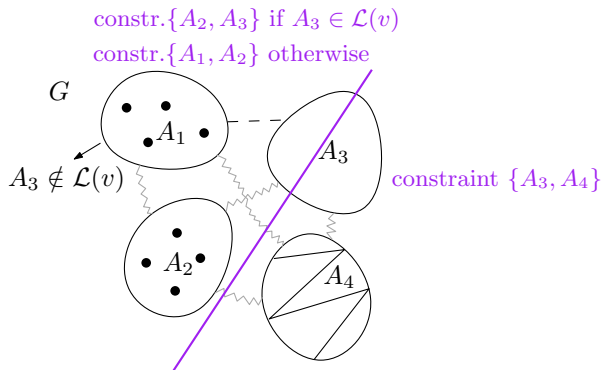
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On particular classes

Random graphs

There is a polynomial Clique-Stable separator on random graphs.

Split-free

Let H be a union of a clique and a stable set (=split graph). Then there is a polynomial Clique-Stable separator for H -free graphs.

Remark: These classes of graphs contain claw-free graphs and comparability graphs.

Prospects

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- Solve one problem and deduce the others!

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- Study the Clique-Stable separation on perfect graphs thanks to structure theorem.

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- Solve one problem and deduce the others!
- Study the Clique-Stable separation on perfect graphs thanks to structure theorem.
- Study which graph operations preserves the property of polynomial Clique-Stable separation.

Thank you for listening!
Do you have any questions?