

Alon-Saks-Seymour conjecture, Clique-Stable separation and the stubborn problem

Aurélie LAGOUTTE
Supervized by Stéphan THOMASSÉ

July 18, 2012

1 Geographical context

2 Definitions

- Clique vs Independent Set Problem
- Alon-Saks-Seymour Conjecture
- 3-COMPATIBLE COLORING PROBLEM
- The stubborn problem

3 Results

- Overview
- Equivalence result

4 Prospects

Geographical context



Team MC2, LIP,
ENS Lyon
46, allée d'Italie
69007 Lyon

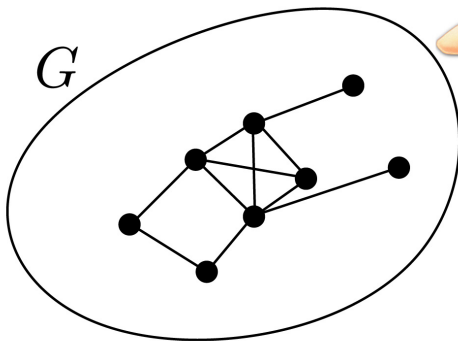
Supervized by Stéphan THOMASSÉ
Joint work with Nicolas BOUSQUET



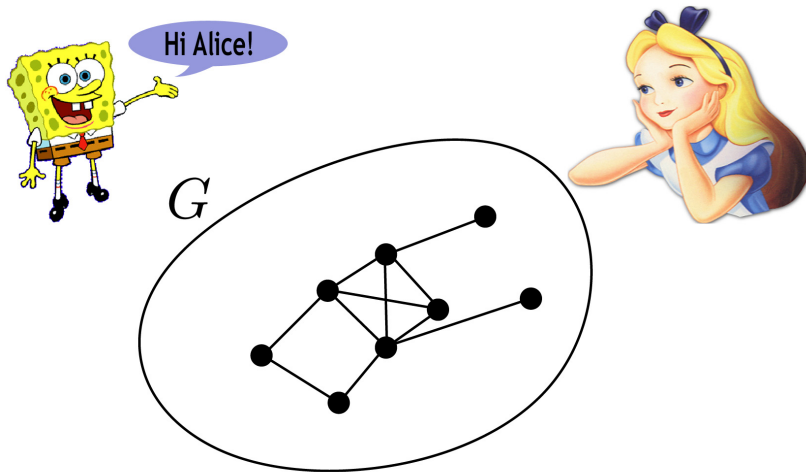
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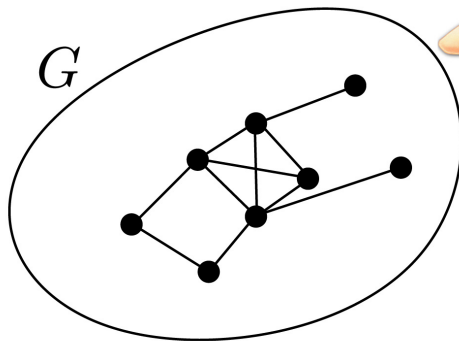
Clique vs Independent Set Problem



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Clique vs Independent Set Problem



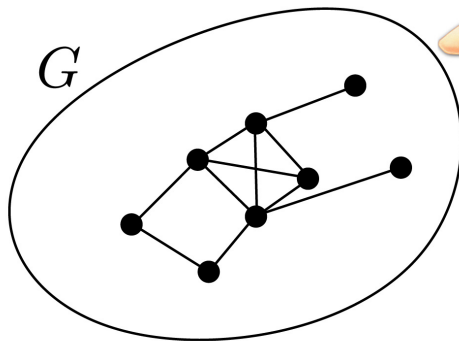
Hi Bob!



Clique vs Independent Set Problem



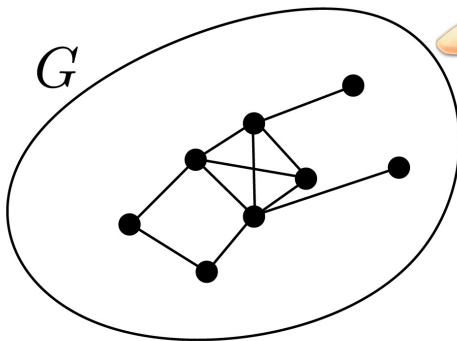
Do you see this graph?



Clique vs Independent Set Problem



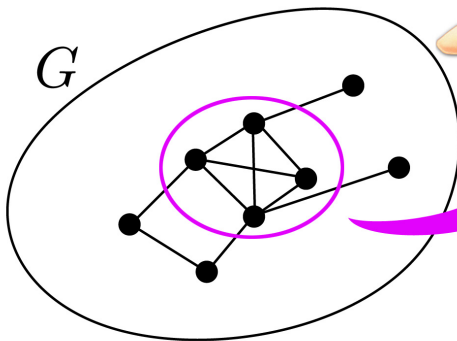
Yes! Let's talk about it.



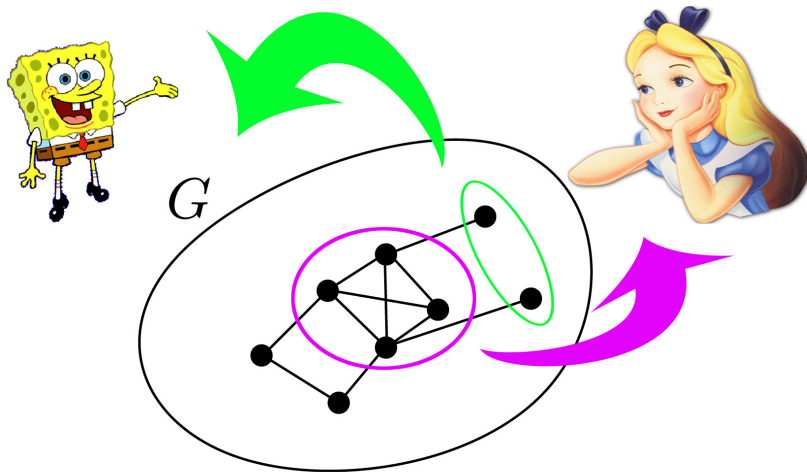
Clique vs Independent Set Problem



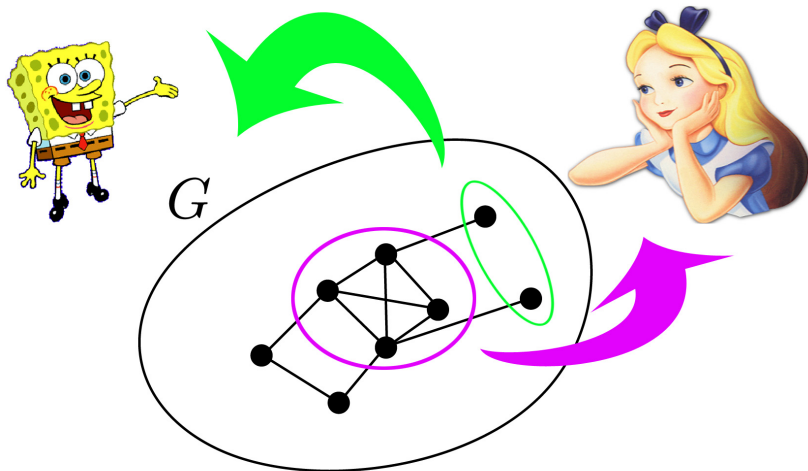
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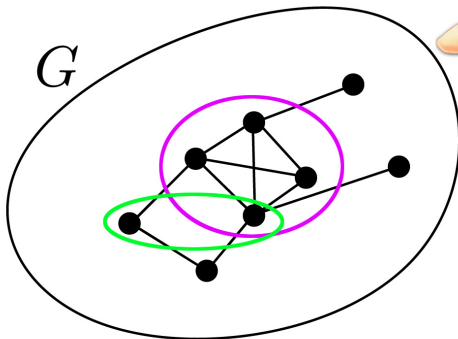


Clique vs Independent Set Problem

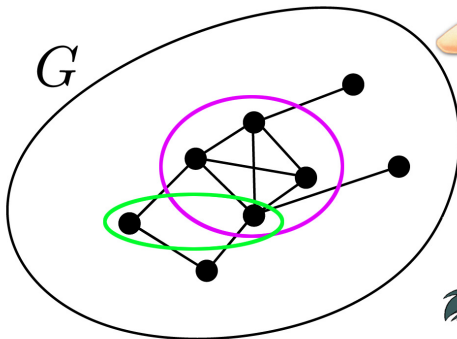


Do the clique and the stable intersect?

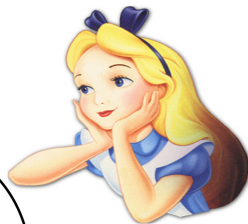
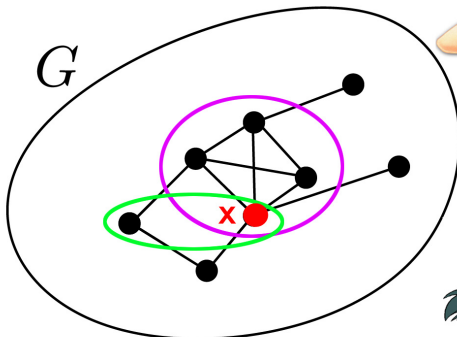
Clique vs Independent Set Problem : Non-det. version



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Yes!
Proof: **x**

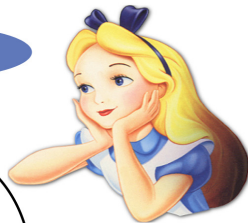
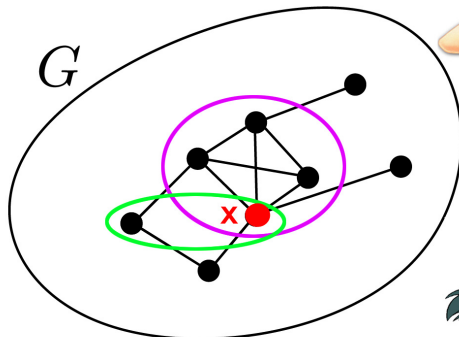


Clique vs Independent Set Problem : Non-det. version



I agree.

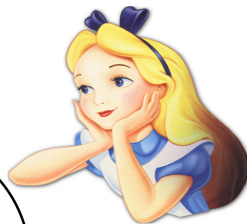
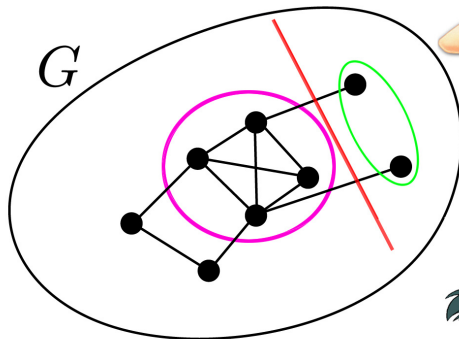
I agree.



Yes!
Proof: **x**



Clique vs Independent Set Problem : Non-det. version



No!
Proof: **cut**

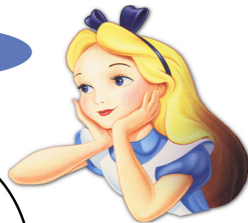
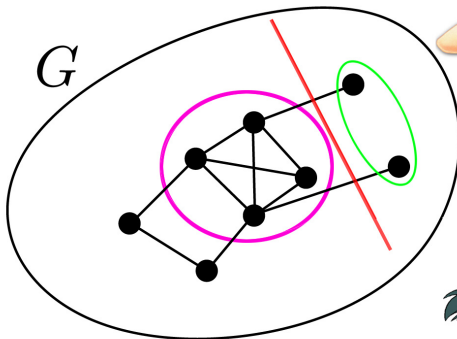


Clique vs Independent Set Problem : Non-det. version



I agree.

I agree.



No!
Proof: **cut**



Clique vs Independent Set Problem

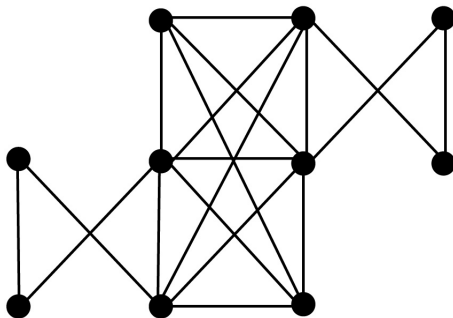
Question: Clique-Stable Set separation

Does there exist a polynomial P , such that for all graph G on n vertices, there exists a family of cuts that separates all the cliques and the stable sets (=a Clique-Stable separator) of size $P(n)$?

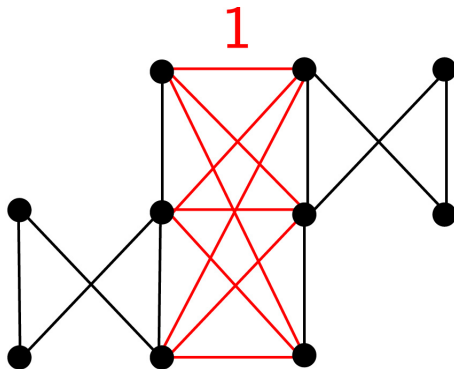
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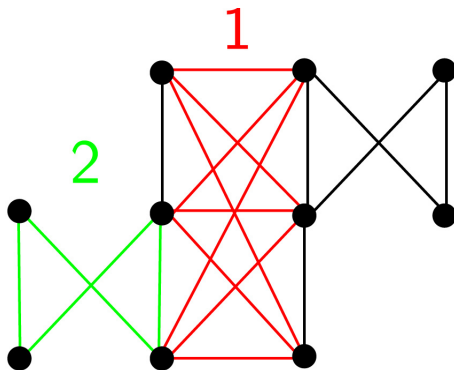
Bipartite packing $bp(G)$



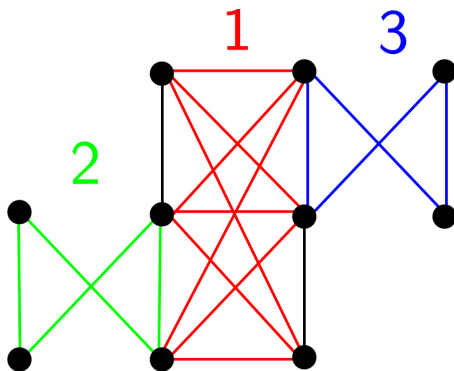
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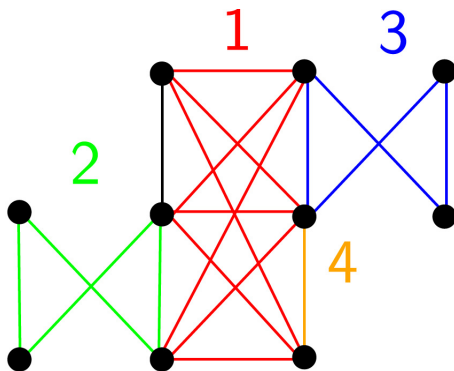
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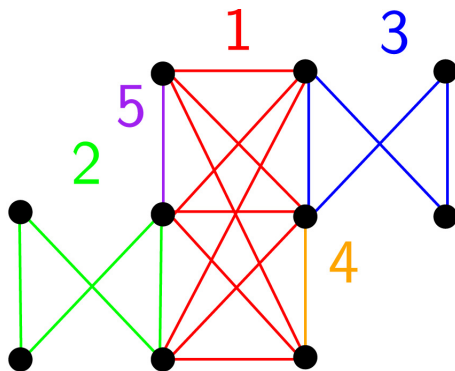
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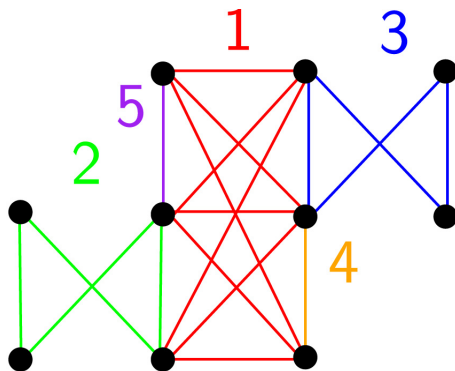
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$$bp(G) = 5$$

Alon-Saks-Seymour conjecture

Alon-Saks-Seymour conjecture (≈ 1974)

If the edges of G can be partitioned into k edge-disjoint complete bipartite graphs, then $\chi \leq k + 1$.

In other words: $\chi \leq \text{bp} + 1$.

Remark

True on complete graphs (Graham-Pollak theorem, 1971).

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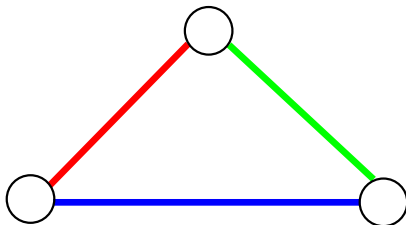
New question: Polynomial Alon-Saks-Seymour conjecture

Does there exist P such that for all G , $\chi \leq P(\text{bp})$.

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3-COMPATIBLE COLORING PROBLEM

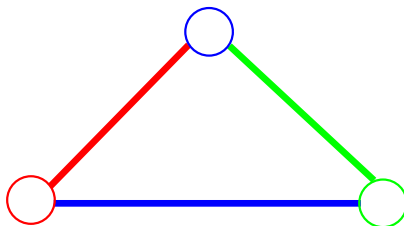


3-COMPATIBLE COLORING PROBLEM (3-CCP)

Input: An edge coloring f_E of the complete graph on n vertices with 3 colors $\{A, B, C\}$.

Question: Is there a coloring of the vertices with the same three colors, such that no edge has the same color as both its extremities?

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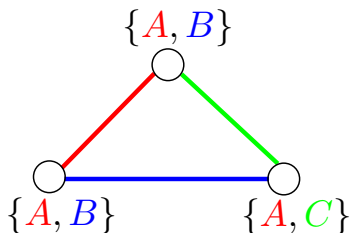


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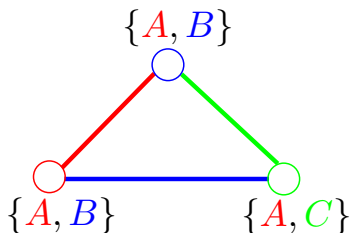


2-list covering for 3-COMPATIBLE COLORING PROBLEM (3-CCP)

Assign to each vertex a list of size 2 compatible with a solution.
 Build a family \mathcal{F} of such objects that covers all the solutions, i.e.
 for all solution, there is 2-list assignment in \mathcal{F} which is compatible
 with the solution.

Goal: have a *polynomial* family.

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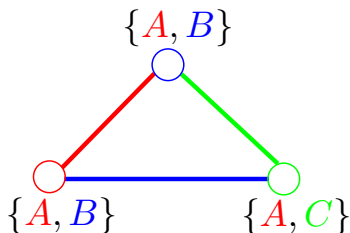


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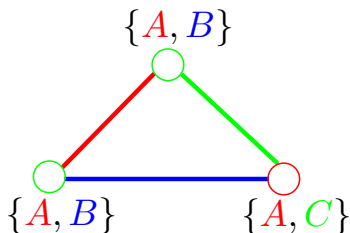
Compatible!

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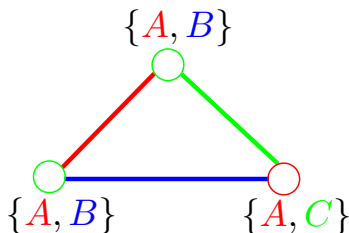


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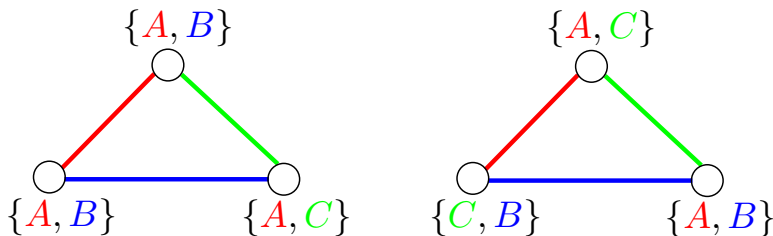
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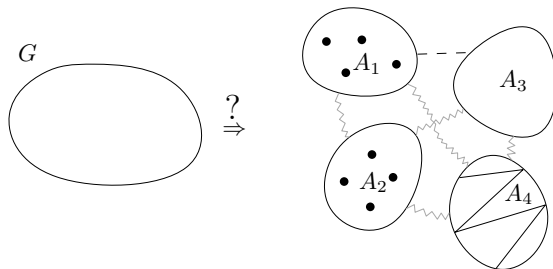
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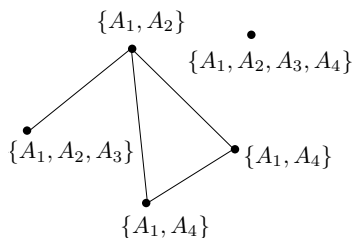


The stubborn problem

Input: A graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.

Question: Can V be partitioned into four sets A_1, \dots, A_4 such that A_4 is a clique, both A_1 and A_2 are stable sets, there is no edge between A_1 and A_3 , and each vertex v belongs to A_i only if $A_i \in \mathcal{L}(v)$?

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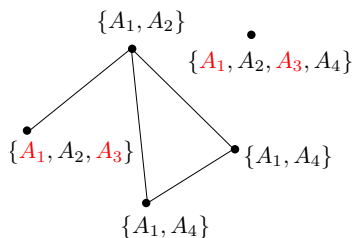


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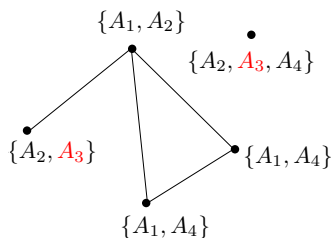


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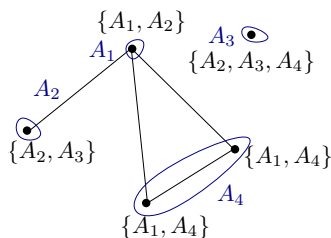


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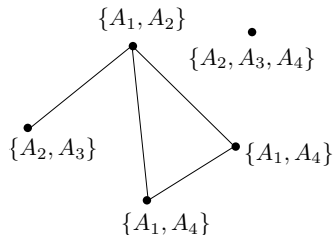


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2-list covering for the stubborn problem

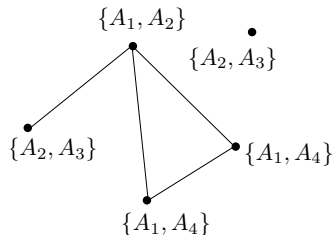


2-list covering for the stubborn problem

Assign to each vertex a list of size 2 compatible with a maximal solution.

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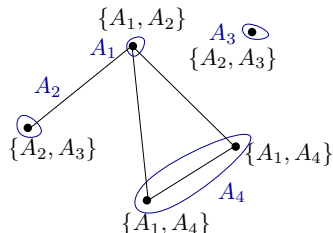


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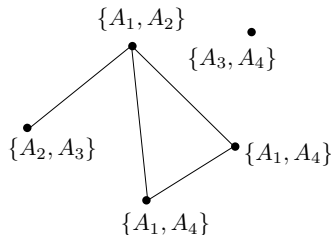
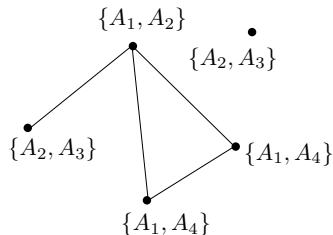


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Overview

Examples

There is a polynomial CS-separator on random graphs and on H -free graphs with H split.

Equivalence theorem I

The following are equivalent:

- For all graph G , there is a polynomial CS-separator.
- There is a polynomial P such that for all graph G , $\chi \leq P(\text{bp})$.

Outline

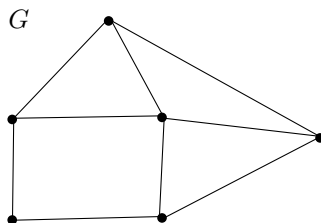
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Second theorem

Equivalence theorem II

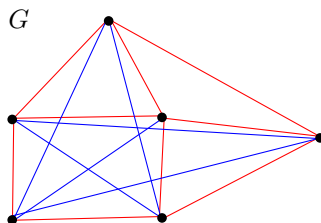
The following are equivalent:

- ① For all graph G and all list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on (G, \mathcal{L}) .
- ② For all n and all edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f) .
- ③ For all graph G , there is a polynomial CS-separator.



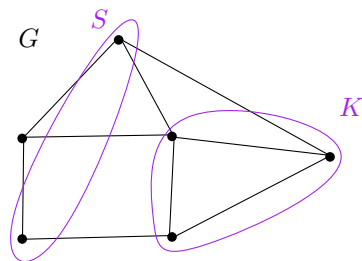
Lemma

$(2 \Rightarrow 3)$: Polynomial 2-list covering for 3-CCP for all $G \Rightarrow$
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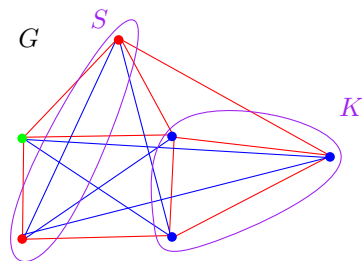
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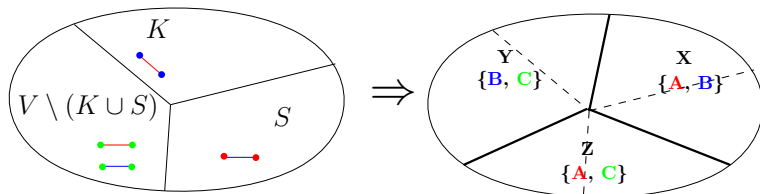
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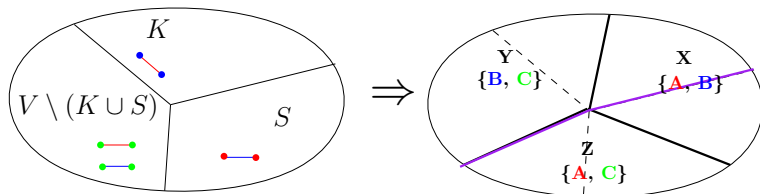
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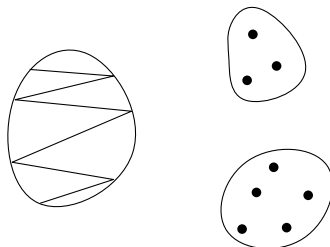


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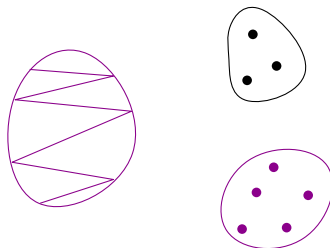
Lemma

$(3 \Rightarrow 1)$: Polynomial CS-separator for all $G \Rightarrow$ Polynomial 2-list covering for the stubborn problem for all G .



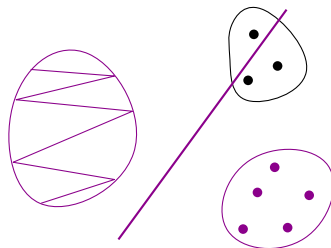
Observation

If there are p cuts that separate all the cliques and the stable sets, then there are p^2 cuts that separate all the cliques and the union of two stables.



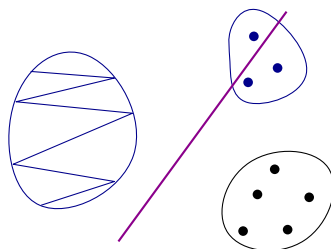
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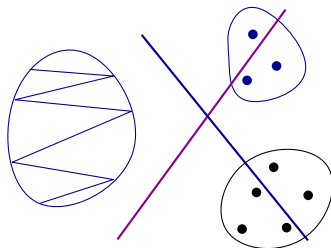
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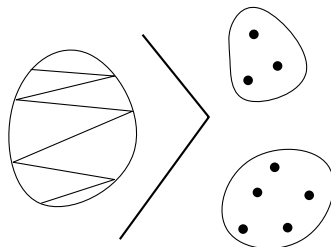
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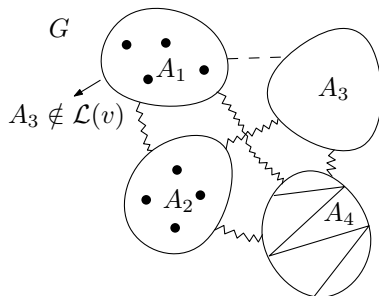
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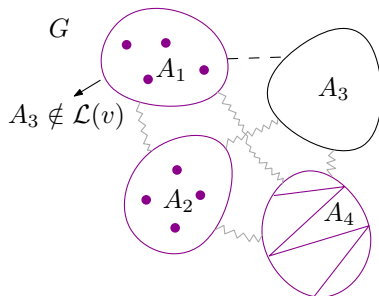
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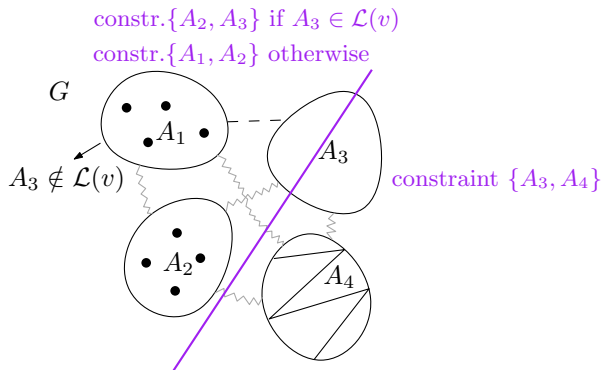
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- Study which graph operations preserves the property of polynomial Clique-Stable separation.

Merci! Avez-vous des questions?