

3-adapted-list coloring problem and Alon-Saks-Seymour conjecture

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Abstract

In this paper we propose a study of the 3-adapted list coloring problem [11] by 2-list colorings. First we prove that this problem is equivalent to the problem of separation of the cliques and the stable sets. This problem is deeply linked with a problem of communication complexity introduced by Yannakakis [18]. Hence we show that these problems are also equivalent with a polynomial bound for an oriented Alon-Saks-Seymour conjecture.

1 Definitions

Let k be an integer. We denote by $[k]$ the set of integers between 0 and $k-1$. Let $G = (V, E)$ be a graph. An edge $uv \in E$ links its two *endpoints* u and v . The *neighbourhood* $N(x)$ of x is the set of vertices y such that $xy \in E$. The *non-neighbourhood* $N^C(x)$ of x is the complementary of the neighbourhood $N(x)$. If the graph is oriented, $N^+(x)$ (resp. $N^-(x)$) denote the incoming (resp. outgoing) neighborhood of x , that is the set of vertices y such that $yx \in E$ (resp. $xy \in E$). The subgraph *induced by* $X \subseteq V$ denoted by $G[X]$ is the graph (X, E') where $E' = E \cap (X \times X)$. A *clique* of G is a complete induced subgraph of G . A *stable set* is an induced subgraph without edges. Observe that the intersection of a clique and a stable set is at most one vertex. A *vertex-coloring* (resp. *edge-coloring*) of G with k colors is a function f_V (resp. f_E) from V (resp. E) to $[k]$. In an edge-colored graph, an $\alpha_1 \dots \alpha_k$ -*clique* is an induced subgraph where the edges are only colored by $\alpha_1 \dots \alpha_k$. A vertex-coloring f_V of G is f_E -*proper* iff for each edge uv of E , $f_V(u) \neq f_V(v)$ or $f_V(u) \neq f_E(uv)$ (at least one extremity is colored differently from the edge). The k -ADAPTED-LIST-COLORING PROBLEM was defined by Hell and Zhu in [11], and then studied in [3] and [13] consists in a proper coloring of the vertices.

k -ADAPTED-LIST-COLORING PROBLEM (k -ALCP)

Input: An edge coloring f_E of the complete graph on n vertices with k colors.

Question: Is there an f_E -proper coloring?

This problem has applications to matrix partitions of graphs, full constraint satisfaction problems and trigraph homomorphisms [5], [6], [4], [10]. It is known to be NP-complete if $k \geq 4$ and it is known to be polynomial if $k \leq 3$ [11]. The question of the case $k = 3$ was open for years and was solved recently by Cygan and al. [2]. We focus our attention on the case $k = 3$. The *majority color* of $x \in V$ is the color which appears the most often for the edges touching x (in case of ties, we arbitrarily cut them). The following definitions are illustrated on Figure 1: a vertex has an l -*constraint* if it can be colored by at most l colors. An l -*list coloring* is a coloring where each vertex has an l -constraint. An l -list coloring is *compatible* with a solution f_V if for each vertex x , $f_V(x)$ is in the l -constraint of x . A set of l -list colorings is *compatible* with a solution if one of the l -list coloring is compatible with the solution. A partition problem is *l -list colored for a function f* if there exists $f(n)$ l -list colorings compatible with all the solutions. A partition problem is said *polynomially l -list colored* if there exists a polynomial Q such that the partition problem is l -list colored for Q . In other words, there exists a polynomial family \mathcal{F} of l -list colorings such that for any solution s , we can find in \mathcal{F} a l -list coloring which is compatible with s .

Observation 1. *Given a polynomial number of 2-list colorings, it is possible to decide in polynomial time if they are compatible with a solution.*

Indeed, each 2-list coloring can be translated into an instance of 2-SAT.

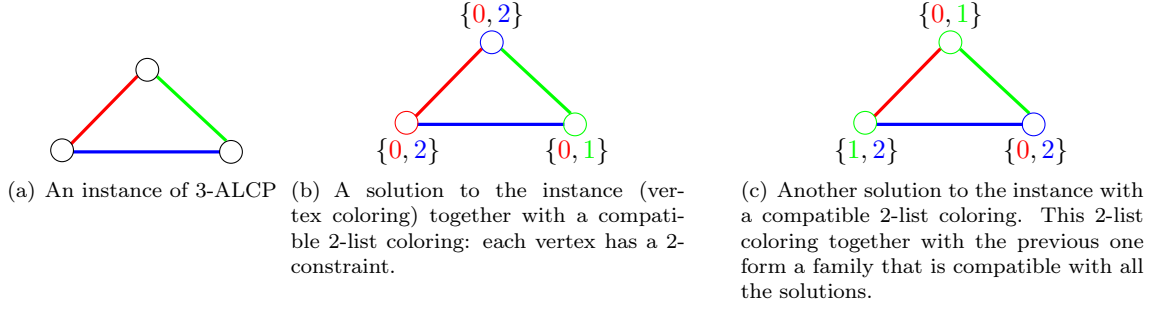


Figure 1: Illustration of definitions. Color correspondence: 0=red ; 1=green ; 2=blue.

Theorem 1. [7] *There exists an algorithm in time $O(n^{\log n})$ which solves the 3-ALCP. This algorithm gives $O(n^{\log n})$ 2-list colorings which are compatible with all the solutions.*

Proof. We give the proof for self completeness. This algorithm consists in testing $n^{\log(n)}$ 2-list colorings which contains all the solutions. Consider the vertex x , without loss of generality we can assume that the majority color of x is 0. If there is a solution in which x is colored by 0 then all its 0-neighbours cannot have the color 0, hence all its 0-neighbours has 2-list constraints. And the number of remaining vertices with 3 possible colors is at most $2n/3$ (recall that the graph is complete). It is possible to make the same for the remaining vertices. Hence consider the following algorithm, for each vertex x try to find a solution for which x is colored by its majority color. If there is no such solution for all x , then all the vertices have 2-list constraints. This algorithm is representable by a research-tree of degree $n + 1$ (one child for the case where no vertex can be colored by its majority color, and one child for each vertex that can be first chosen to be colored by its majority color) and depth $\log(n)$ (we remove $1/3$ of the vertices at each node) in which each of the $n^{\log(n)}$ leaves is a 2-list coloring of the instance. \square

2 Link with the Cliques-Stables separation conjecture

A *cut* is a pair (A, B) such that $A \cup B = V$ and $A \cap B = \emptyset$. A cut *separates* a clique C and a stable set S if $C \subseteq A$ and $S \subseteq B$. A clique and a stable set can be separated iff they do not intersect. Let \mathcal{C}_G be the set of cliques and \mathcal{S}_G be the set of stable sets. We say that a family \mathcal{F} of cuts *separates all the cliques and the stable sets* if for all $(C, S) \in \mathcal{C}_G \times \mathcal{S}_G$ which do not intersect, there exists a cut in \mathcal{F} that separates C and S .

Conjecture 1. (*Clique-Stable Set separation Conjecture*) *There is a polynomial Q , such that for every graph G on n vertices, there exists a family of $Q(n)$ cuts that separates all the cliques and the stable sets.*

Proposition 1. *To prove the Clique-Stable Set separation Conjecture, one only need a polynomial family that separates all the maximal (in sense of inclusion) cliques from the maximal stable sets that do not intersect.*

Consequently, in the following, we will only focus on separating the maximal cliques from the maximal stable sets.

Proof. Assume \mathcal{F} is a polynomial family that separates all the maximal cliques from the maximal stable sets that do not intersect. Add to \mathcal{F} the following cuts, for all $x \in V$:

- Put x and $N(x)$ on the left hand-side, and the remaining vertices on the right hand-side (call it $Cut_{1,x}$).
- Put $N(x)$ on the left and side, and x and the remaning vertices on the right hand side (call it $Cut_{2,x}$).

Note that these additional cuts are in linear number. Now, let (C, S) be a pair of clique and stable set. Grow up C and S by adding vertices to get a maximal clique C' and a maximal stable set S' . Either C' and S' do not intersect, and in this case there is a cut in \mathcal{F} that separate C' from S' , thus C from S ;

or C' and S' intersect in x (recall that the intersection of a clique and a stable set can be at most one vertex): if $x \in C$, then $Cut_{1,x}$ separates C from S , otherwise $Cut_{2,x}$ does. \square

To motivate this polynomial separation let us prove the conjecture for some particular classes of graphs. For a class with a polynomial number of cliques or stable sets, the conjecture is verified. Hence a class of graphs with bounded cliques or stable sets verifies the conjecture. Claims 1, 2 give more interesting examples of classes of graphs which verify the conjecture.

2.1 Example: random graphs

Claim 1. *For random graphs, the clique-stable set conjecture is verified.*

The claim is detailed by the following propositions:

Proposition 2. *Let $a(n)$ (denoted by a in the following) be a function of n such that $2 \leq a(n) \ll \log n$. The random graph $G(n, p)$ with $p = 1 - n^{-2/a} = 1 - 2^{-2 \log n/a}$ has:*

- independent number $\alpha = a - \frac{a \log a}{\log n} + 1 + o(1)$
- clique number $\omega = 2 \ln 2 \cdot 2^{2 \log n/a} \cdot \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n)$.

Moreover, for all $\varepsilon > 0$, there exists a family of $\mathcal{O}(n^{6+\varepsilon})$ cuts (more precisely $\mathcal{O}(n^{5+2/a+\varepsilon})$ if a is a constant, and $\mathcal{O}(n^{5+\varepsilon})$ if $a(n) \rightarrow_{n \rightarrow \infty} \infty$) that separates all the clique and the stable sets.

Observation 2. *If a is a constant, then the previous formulas give:*

- $\alpha = a + 1 + o(1)$
- $\omega = 2 \ln 2 \cdot n^{2/a} \left(\frac{a-2}{a} \log n - \log \log n \right) + \mathcal{O}(\log n)$.

Proof. Let $p' = 1 - p$, $b = 1/p$ and $b' = 1/p'$. The independence number and clique number of $G(n, p)$ are given by the following formulas, depending on p (see [1]):

$$\begin{aligned} \omega &= 2 \log_b(n) - 2 \log_b \log_b n + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \end{aligned}$$

Now $\frac{1}{\log b'} = \frac{a}{2 \log n}$, thus:

$$\begin{aligned} \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= a - \frac{a \log a}{\log n} + 1 + o(1) \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{1}{\log b} &= \frac{-\ln 2}{-2^{-2 \log n/a} + \mathcal{O}(2^{-4 \log n/a})} \\ &= \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) \end{aligned}$$

Thus,

$$- \log \log b = \log \ln 2 + \frac{2 \log n}{a} + \mathcal{O}(2^{-2 \log n/a})$$

Consequently:

$$\begin{aligned} -2 \log_b \log_b n &= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) \cdot (\log \log n - \log \log b) \\ &= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log \log n + \log \ln 2 + \frac{2 \log n}{a}) + \mathcal{O}(\log n) \end{aligned}$$

Hence:

$$\begin{aligned} w &= 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n - \log \ln 2 + \log(e/2)) + \mathcal{O}(\log n) \\ &= 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n) \end{aligned}$$

Now, let $x = 1 - p = 2^{-2 \log n/a}$. Draw a random partition (V_1, V_2) where each vertex is put in V_1 independently from the others with probability x . Let (C, S) be a pair of clique and stable set of the graph. There are at most 4^n such pairs. The probability that $S \subseteq V_1$ and $C \subseteq V_2$ is $x^\alpha(1-x)^\omega$. Let us show that $x^\alpha \geq n^{-(2+2/a+\varepsilon')}$ and $(1-x)^\omega \geq n^{-(2+\varepsilon')}$ for all $\varepsilon' > 0$.

On one hand,

$$\begin{aligned} x^\alpha \geq n^{-(2+2/a+\varepsilon')} &\Leftrightarrow \alpha \log x \geq -(2+2/a+\varepsilon') \log n \\ &\Leftrightarrow 2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2+2/a+\varepsilon') \log n \end{aligned}$$

which is true if n is large enough.

Note that for $a \rightarrow_{n \infty} \infty$, we even have $2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2+\varepsilon') \log n$ for n large enough, which implies $x^\alpha \geq n^{-(2+\varepsilon')}$.

On the other hand, $\log(1-x) = \log b$, thus, using the previous approximations:

$$\begin{aligned} (1-x)^\omega \geq n^{-(2+\varepsilon')} &\Leftrightarrow \omega \log(1-x) \geq -(2+\varepsilon') \log n \\ &\Leftrightarrow \left(2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n) \right) \\ &\quad \cdot \left(\frac{-2^{-2 \log n/a}}{\ln 2} + \mathcal{O}(2^{-4 \log n/a}) \right) \geq -(2+\varepsilon') \log n \\ &\Leftrightarrow 2(\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(1) + \mathcal{O}(2^{-2 \log n/a} \log n) \leq (2+\varepsilon') \log n \end{aligned}$$

which is true if n is large enough.

This leads us to $x^\alpha(1-x)^\omega \geq n^{-(5+2\varepsilon')} = n^{-(5+\varepsilon)}$ for $\varepsilon' = \varepsilon/2$. Thus (C, S) is separated by at least $n^{-(5+\varepsilon)}$ of all the partition. By double counting, there exists a partition that separates at least $n^{-(5+\varepsilon)}$ of all the pairs. We delete these separated pairs, and it remains at most $(1 - 1/n^{5+\varepsilon}) \cdot 4^n$ pairs. The same probability for a pair (C, S) to be cut by a random partition still holds, hence we can iterate the process k times until $(1 - 1/n^{5+\varepsilon})^k \cdot 4^n \leq 1$. This is satisfied by $k = 2n^{6+\varepsilon}$ which is a polynomial in n . If $a \rightarrow_{n \infty} \infty$, we can do the same and get $k = 2n^{5+\varepsilon}$. As a conclusion, we have a polynomial family of random cuts that separates all the clique and the stable sets. \square

However, this does not cover the classic configuration where $p = 1/2$ and $\alpha \approx 2 \log n$. Thus we need the following:

Proposition 3. *Let $a(n) = 2c \log n$ (denoted by a in the following) where $0 \leq c \leq 1$ is a constant. The random graph $G(n, p)$ with $p = 1 - 2^{-c}$ has:*

- independent number $\alpha = 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)$
- clique number $\omega = 2c' \log n - 2c' \log \log n + o(\log \log n)$ where $1/c' = \log(1 - 2^{-1/c})$.

Moreover, for all $\varepsilon > 0$, there exists a family of $\mathcal{O}(n^5)$ cuts that separates all the clique and the stable sets.

Proof. Let $p' = 1 - p$, $b = 1/p$ and $b' = 1/p'$. We proceed the same way as in the previous proposition. The independence number and clique number of $G(n, p)$ are given by the following formulas, depending on p (see [1]):

$$\begin{aligned} \omega &= 2 \log_b(n) - 2 \log_b \log_b n + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \end{aligned}$$

Now $\frac{1}{\log b'} = c$, thus:

$$\begin{aligned}\alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)\end{aligned}$$

Moreover, $\log b = \log(1 - 2^{-1/c}) = 1/c'$, thus, similarly to α , we get:

$$w = 2c' \log(n) - 2c' \log \log n + 2c'(\log(e/2) - \log c') + 1 + o(1)$$

The same reasoning as previously with the same $x = 1 - p'$ gives a probability $x^\alpha(1-x)^\omega \geq n^{-4}$ for a given pair (C, S) to be separated by a random non-uniform cut (V_1, V_2) , so there is a family of $\mathcal{O}(n^5)$ cuts that separate all the clique and the stable sets (see the proof of Proposition 2 for details). \square

2.2 Example: H -free graph with H split graph

A geometric generalization of the clique-stable set conjecture was studied by Lovasz in [15]. The question is to know if the stable set polytope of a graph is the projection of a polytope in higher dimension, with a polynomial number of facets. He claims that if the answer to this question is positive, then there is a polynomial number of cuts that separates all the cliques and the stable sets. Moreover, he claims that this is true for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and t -perfect graphs which are a generalization of series-parallel graphs. Let us generalize the result for comparability graphs, with a much larger class.

A graph $G = (V, E)$ is *split* if $V = V_1 \cup V_2$ and the subgraph induced by V_1 is a clique and the subgraph induced by V_2 is a stable set. Let H be a split graph. We denote by \mathcal{C}_H the class of graph with no induced H . For example, if H the graph with four vertices 1, 2, 3, 4 with the edges 12, 13 and 14, then \mathcal{C}_H is the class of claw-free graphs. Let $\phi(H) = \max(|V_1|, |V_2|)$.

Claim 2. *Let H be a fixed split graph. Then the clique-stable set conjecture is verified on \mathcal{C}_H .*

Before going to the proof, we need a few additional definitions regarding transversality and Vapnik-Chervonenkis dimension. Let $G = (V, E)$ be a hypergraph.

Definition 1. *The transversality τ is the minimum cardinality of a set of vertices intersecting every hyperedges.*

Definition 2. *The extended transversality τ^* is an extended formulation of the transversality : it is a solution to the following linear programming: $\min \sum_{x \in V} w(x)$ subject to*

- for all $x \in V$, $0 \leq w(x) \leq 1$.
- for all $e \in E$, $\sum_{x \in e} w(x) \geq 1$.

Definition 3. *A set T of hyperedges forms a complete Venn diagram if for all $T' \subseteq T$, there exists a vertex v such that for all $e \in T'$, $v \in e$ and for all $e \notin T'$, $v \notin e$. In other words, for all subset T' of hyperedges, there exists a vertex in the intersection of all hyperedges in T' , and in no other hyperedge.*

Definition 4. *The Vapnik-Chervonenkis dimension (VC-dimension for short) of G is the maximum size of a complete Venn diagram in G .*

Proof of claim 2. Let t be a constant depending only on H , to be defined later. Let $G = (V, E) \in \mathcal{C}_H$ be a H -free graph, and let \mathcal{F} be the following family of cuts: for all subset $\{x_1, \dots, x_r\}$ of at most t vertices which is a clique (resp. a stable set), take $U = \cap_{1 \leq i \leq r} N(x_i)$ (resp. $U = \cap_{1 \leq i \leq r} N^C(x_i)$). Add $(U, V \setminus U)$ to \mathcal{F} . Then \mathcal{F} has size $\mathcal{O}(n^t)$, which is polynomial in n . Let (C, S) be a pair of maximal clique and stable set. Let us show that (C, S) is separated by \mathcal{F} .

Let us build an oriented graph B with vertices $C \cup S$ as shown on Figure 2(b). For all $x \in C, y \in S$, put an oriented edge xy if $xy \in E$, and put an oriented edge yx otherwise. From the hyperplane separation theorem applied to the cone of positive vectors on one side, and the convex hull of the columns of the adjacency matrix of B on the other side, we can derive the following lemma:

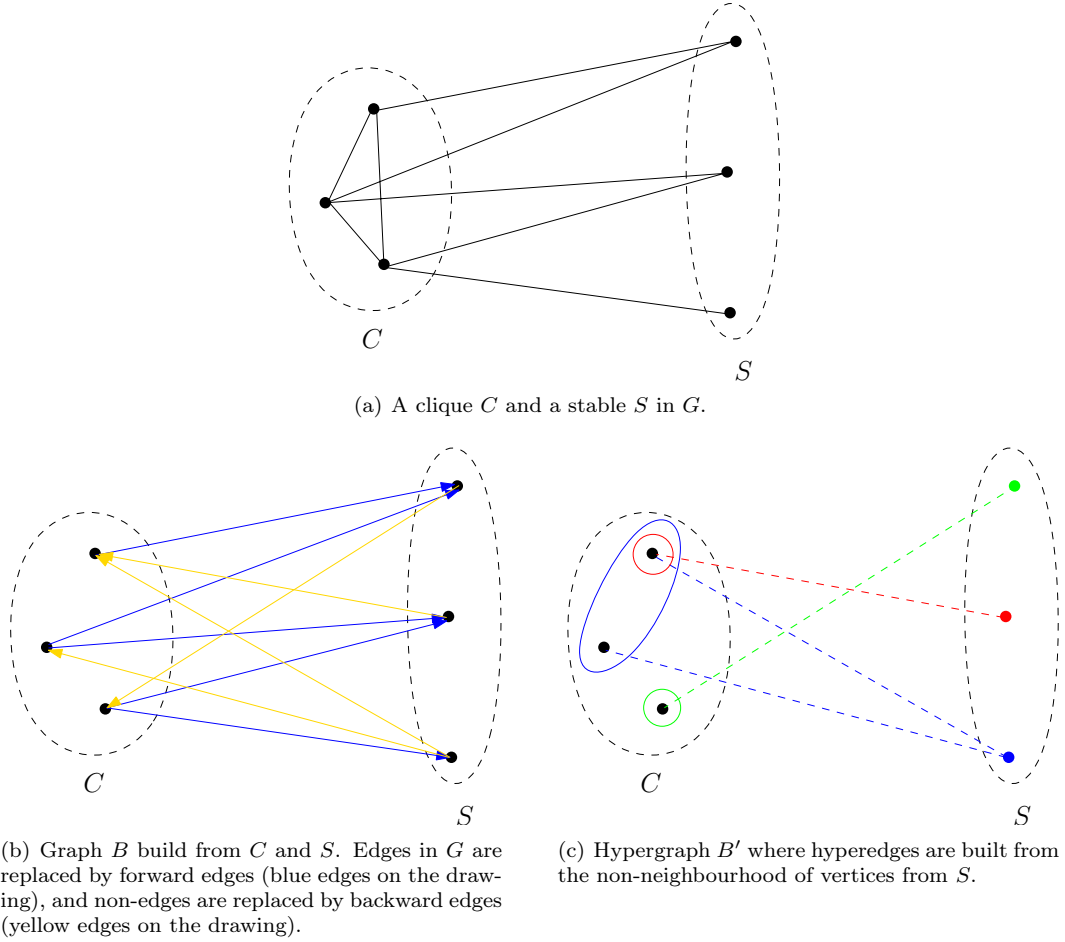


Figure 2: Illustration of proof of Claim 2.

Lemma 1. *For all oriented graph (V, E) , there exists a weight function $w : V \rightarrow [0, 1]$ such that $w(V) = 1$ and for all vertex x , $w(N^+(x)) \geq w(N^-(x))$.*

Corollary 1. *In B , there exists either:*

- (i) *a weight function $w : C \rightarrow \mathbb{R}^+$ such that $w(C) = 2$ and for all vertex $x \in S$, $w(N^+(x)) \geq 1$.*
- or (ii) *a weight function $w : S \rightarrow \mathbb{R}^+$ such that $w(S) = 2$ and for all vertex $x \in C$, $w(N^+(x)) \geq 1$.*

Proof. Take $w : V \rightarrow [0, 1]$ as in Lemma 1. Then $w(V) = 1$ so either $w(C) > 0$ or $w(S) > 0$. Assume $w(C) > 0$ (the other case is handled symmetrically). Take a new weight function defined by $w'(x) = 2w(x)/w(C)$ if $x \in C$, and 0 otherwise. Then for all $x \in S$, on one hand $w'(N^+(x)) \geq w'(N^-(x))$ by extension of the property of w , and on the other hand, $N^+(x) \cup N^-(x) = C$ by construction of B . Thus $w'(N^+(x)) \geq w'(C)/2 = 1$ since $w'(C) = 2$. \square

In the following, let assume we are in case (i). Case (ii) is handled symmetrically by switching C (resp. neighbourhood) and S (resp. non-neighborhood).

Let us now build B' an hypergraph with vertices $C \cup S$ and for all $x \in S$, build the hyperedge $C \setminus N_G(x)$, that is the complementary in C of the neighbors of x in the original graph G . This construction is illustrated on Figure 2(c).

Lemma 2. *The hypergraph B' has extended transversality $\tau^* \leq 2$.*

Proof. Let w be the weight function given by Corollary 1. Then for all hyperedge h built from the non-neighborhood of $x \in S$, we have:

$$\sum_{y \in h} w(y) = w(N^+(x)) \geq 1.$$

Moreover, changing $w(y)$ in 1 in case $w(y) > 1$ will not change the inequality above, and will give us a function with values in $[0, 1]$. Thus w satisfies the constraints of the extended transversality, and $w(V) = w(C) \leq 2$. \square

Let $\varphi = \phi(H)$. Recall that $\phi(H) = \max(|V_1|, |V_2|)$ where V_1 is the clique and V_2 is the stable set composing H .

Lemma 3. *B' has VC-dimension bounded by $2\varphi - 1$.*

Proof. Suppose we have a complete Venn diagram \mathcal{D} of size 2φ in B' . Let $s_1, \dots, s_\varphi, t_1, \dots, t_\varphi$ be the hyperedges composing \mathcal{D} . In the following, we will abuse notation by calling s_i (resp. t_i) both the hyperedge and the vertex of S whose non-neighborhood is precisely the hyperedge s_i (resp. t_i). We denote by $y_1, \dots, y_{\varphi'}$ the vertices of V_2 (with $\varphi' \leq \varphi$). Let $x \in V_1$ and let $\{y_{i_1}, \dots, y_{i_k}\}$ be the set of its neighbors in V_2 . Since \mathcal{D} is a complete Venn diagram in B' , there exists $x' \in C$ such that x' is in the intersection of the hyperedges s_{i_1}, \dots, s_{i_k} , and in no other hyperedge. This means that, in $G[C \cup \{s_1, \dots, s_\varphi\}]$, x' has exactly the same neighborhood in S as x in V_2 . If there is no two vertices $x_1, x_2 \in V_1$ with the same neighborhood in V_2 , the above statement iterated on the other vertices of V_1 proves that there is an induced H in $G[C \cup S]$, thus in G , which is absurd.

Now assume $x_1, \dots, x_r \in V_1$ have the same neighborhood $\{y_{i_1}, \dots, y_{i_k}\}$ in V_2 . Let $\delta_j = \{s_{i_1}, \dots, s_{i_k}\} \cup \{t_j\}$ for all $1 \leq j \leq r$. Then there exists $x'_j \in C$ such that x'_j is exactly in the intersection of all hyperedges in δ_j , and in no other hyperedge. Thus, in $G[C \cup \{s_1, \dots, s_\varphi\}]$, x'_1, \dots, x'_r have the same neighborhood $\{s_{i_1}, \dots, s_{i_k}\}$, just as x_1, \dots, x_r in V_1 . In the same way as before, we find an induced H in $G[C \cup S]$, which is absurd. \square

Observation 3. *In fact, the VC-dimension of B' is bounded by $\varphi + \log \varphi$. Indeed, we need only $t_1, \dots, t_{\log \varphi}$ in addition to s_1, \dots, s_φ : if $x_1, \dots, x_r \in V_1$ have the same neighborhood in V_2 with $r \leq \varphi$, we need r different subsets of $\{t_1, \dots, t_\varphi\}$, which is already possible with $t_1, \dots, t_{\log \varphi}$.*

The following bound [9] links the transversality with VC-dimension and extended transversality.

Lemma 4. *Every hypergraph H with VC-dimension d satisfies*

$$\tau(H) \leq 16d\tau^*(H) \log(d\tau^*(H)).$$

Applying this lemma to B' , we obtain

$$\tau(B') \leq 16d\tau^*(B') \log(d\tau^*(B')) \leq 32\varphi(\log(\varphi) + 1).$$

Let us now define $t = 32\varphi(\log(\varphi) + 1)$ (recall that t was part of the construction of the family \mathcal{F} of cuts in the beginning of the proof). As a consequence, τ is bounded by t which depends only on H . There must be $x_1, \dots, x_t \in C$ such that each hyperedge of B' contains at least one x_i . Let us consider a particular cut of the family \mathcal{F} that is the one built from the subset x_1, \dots, x_t . Then $C \subseteq U = \bigcap_{1 \leq i \leq t} N_G(x_i)$ as C is a clique. Moreover, since C is a maximal clique, each $y \in S$ leads to a non-empty hyperedge in B' , which has to include one x_i . Thus $y \notin U$, so this cut separates C from S . \square

!!! DEF $G[X]$, τ, τ^* , VC-dim, diagramme de Venn complet!!! Dire qu'on ne regarde que les cliques max /stable max car on peut g  rer les autres avec un nombre lin  aire ($2n$) de coupes en mettant pour chaque point tout son voisinage d'un cot  , tout son non-voisinage de l'autre, et en choisissant pour lui de quel cot   on le mets!!!!

2.3 Examples (ancienne version)

To motivate this polynomial separation let us prove the conjecture for some particular classes of graphs. For a class with a polynomial number of cliques or stable sets, the conjecture is verified. Hence a class of graphs with bounded cliques or stable sets verifies the conjecture. Claims 3, 4 and 5 give more interesting examples of classes of graphs which verify the conjecture. The *claw* is the graph with four vertices 1, 2, 3, 4 with the edges 12, 13 and 14.

Claim 3. *For random graphs, the clique-stable set separation conjecture is verified.*

Proof: In random graphs, maximum cliques and maximum stable sets are of size $2\log(n)$. Let (C, S) be a pair of a clique and a stable set which do not intersect. Observe that there is at most 4^n such pairs. Consider a random partition of the vertices into V_1 and V_2 of equal size, the probability that all the vertices of C are in V_1 is $1/2^{2\log(n)}$ and the same holds for V_2 and S . Hence C and S are separated with probability $1/n^4$. Then, a pair (C, S) is separated by $1/n^4$ of all the partitions, hence by double counting there is a partition which separates $1/n^4$ of all the pairs. When the separated pairs are deleted it remains $(1 - 1/n^4) \cdot 4^n$ pairs and the remaining pairs are still separate which probability $1/n^2$. Hence it is possible to separate the pairs after k steps if $(1 - 1/n^4)^k \cdot 4^n \leq 1$, which means that $2n^5$ partitions separates all the pairs. \square

Claim 4. *For comparability graphs, the clique-stable set separation conjecture is verified.*

Proof: A N is isomorph to the poset on four vertices $\{a, b, c, d\}$ with arcs $\{ab, cb, cd\}$. If a poset O contains a maximal antichain and a maximal chain which do not intersect, then O contains a N in its Hasse diagram (formed by two consecutive of the chain and two vertices of the antichain). Now, let G be comparability graph, and O the resulting poset. We subdivide every arc of the Hasse diagram of O , the resulting poset O' contains no N in its Hasse diagram. Hence, we consider all the cuts formed by the neighbourhood and the non-neighbourhood of each vertex in O' . Now, a clique C and a stable set S of G , can be extended into a maximal chain and a maximal antichain in O' which intersect and then are separated by a chosen cut.

Claim 5. *For claw-free graphs, the clique-stable set separation conjecture is verified.*

Proof: Let x be a vertex of the graph. At most two of its neighbours are in the same stable set. Indeed, otherwise there is a claw. Hence, the separation in neighbourhood and non-neighbourhood of a vertex modulo at most two vertices separates the cliques and the stable sets. More formally, for all x and all set X of at most two neighbours of x , the cuts $N(x) \setminus X$ and $(N(x) \setminus X)^c$ separate \mathcal{C} and \mathcal{S} . Then n^3 cuts separates cliques and stable sets. \square

2.4 Equivalence with the polynomial 2-list coloring of 3-ALCP

The following of this section is devoted to the equivalence between the Clique-Stables separation conjecture and the polynomial 2-list coloring of the 3-ALCP.

Recall that a graph $G = (V, E)$ is *split* if $V = V_1 \cup V_2$ and the subgraph induced by V_1 is a clique and the subgraph induced by V_2 is a stable set. A graph with edges colored by a and b is *split* if there is a partition of the vertices into a a -clique and a b -clique.

Proposition 4. *Let x be a vertex. Let A be the 0-neighbourhood of x . Let U be a 12-clique of A . If U is not split then there is no solution where x is colored by 0.*

Proof. Assume that x is colored by 0. Then all the vertices of U are of color 1 or 2 because they are in the 0-neighbourhood of x . The vertices colored by 1 form a 2-clique and all the vertices colored by 2 form a 1-clique and the union is U . Hence U is split. \square

Observe that the same property holds for the 1 and the 2-neighbourhoods. A vertex x is *really 3-colorable* if for each color a , each bc -clique of the a -neighbourhood of x is a split graph. If a vertex is not really 3-colorable then, in a solution, it can be colored by at most 2 different colors. Hence, given a 2-list coloring of the graph without x , there is a natural 2-list coloring of the whole graph by adding the 2-constraint on x . Hence it is possible to assume that the graph has only really 3-colorable vertices. \square

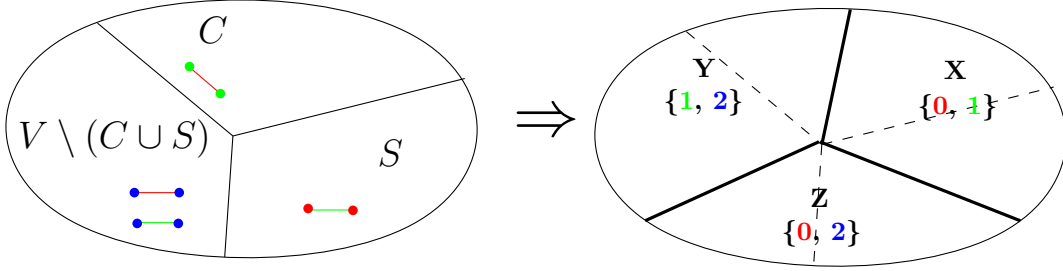


Figure 3: Illustration of proof of lemma 5. On the left hand-side, G is separated in 3 parts: C , S , and the remaining vertices. Each possible configuration of edge- and vertex-colouring are represented. On the right-hand-side, (X, Y, Z) is a 2-list coloring compatible with the solution. X (resp. Y, Z) has constraint $\{0, 1\}$ (resp. $\{1, 2\}, \{0, 2\}$). Color correspondence: 0=red ; 1=green ; 2=blue.

Let us state the main result of this section. For this, we define two functions and we prove that these two functions are polynomially equivalent. We denote by $|G|$ the number of vertices of the graph.

$$f_1(n) = \max_{|G|=n} \min_{\mathcal{F}} (|\mathcal{F}| : \mathcal{F} \text{ set of 2-list colorings compatible with all the solutions})$$

$$f_2(n) = \max_{|G|=n} \min_{\mathcal{F}} (|\mathcal{F}| : \mathcal{F} \text{ family of cuts which separates all the cliques and the stable sets})$$

Theorem 2. *The functions f_1 and f_2 are polynomially equivalent.*

Corollary 2. *The Clique-Stable set separation conjecture is verified iff there exists a polynomial number of 2-list coloring compatible with all the solutions of an instance of 3-ALCP.*

We start the proof of Theorem 2 by the following lemma:

Lemma 5. $f_2(n) \leq n f_1(n)$.

Proof. Let $G = (V, E)$ be a graph with n vertices. Consider the graph F obtained from G by coloring its edges by 0, and replacing all missing edges by 1-colored edges (formally, H is an edge-colored graph with the same vertices as G where xy is colored by 0 if $xy \in E$ and colored by 1 otherwise.) By assumption, we have a set \mathcal{F} of $f_1(n)$ 2-list coloring that is compatible with all solutions of 3-ALCP. The idea is the following: from a pair (C, S) of clique and stable set of G which do not intersect, we find a natural solution of 3-ALCP on H . Hence we have a 2-list coloring in \mathcal{F} which is compatible with this particular solution. We build from this about n cuts that depend only on the previous 2-list coloring, and among which there is for sure a cut that separates C and S . Let C be a clique and S a stable set of G which do not intersect. Observe that the edges of C are colored by 0, and those of S are colored by 1. Then the vertex-coloring 1 for the vertices of C , 0 for those of S and 2 for the other vertices is a solution of the 3-ALCP. Left-hand side of Figure 2.4 illustrates the situation.

There is a 2-list coloring (X, Y, Z) in \mathcal{F} which is compatible with this solution. The set X (resp. Y, Z) is the set of vertices which have the constraint $\{0, 1\}$ (resp. $\{1, 2\}, \{0, 2\}$). The vertices of C cannot be in Z , and those of S cannot be in Y (see right hand-side of Figure 2.4). Then $C \subseteq X \cup Y$ and $S \subseteq X \cup Z$. Since all the edges are of color 0 and 1, and X can be colored properly, the graph restricted to X is split (put all the 0-colored vertices on one side, and the 1-colored vertices on the other side to get a 1-clique and a 0-clique). Given a graph, there is a linear number of decompositions into a split graph [5]. Then the number of possible decompositions of X is linear. Let $(U_k, V_k)_k$ be these decompositions. The pair (C, S) is separated by one of the $(U_k \cup Y, V_k \cup Z)$. Then $f_2(n)$ is bounded by a polynomial function of $f_1(n)$. \square

Before the next lemma, we need the following observation about the union of two cliques:

Observation 4. *If there are p cuts that separate all the clique from the stable sets, then there are p^2 cuts that separates all the $C \cup C'$ (where C and C' are cliques) from the stable sets. Indeed, if (V_1, V_2) separates C from S and (V'_1, V'_2) separates C' from S , then the new cut $(V_1 \cup V'_1, V_2 \cap V'_2)$ satisfies $C \cup C' \subseteq V_1 \cup V'_1$ and $S \subseteq V_2 \cap V'_2$.*

Lemma 6. $f_1 \leq \mathcal{O}(f_2^6(n))$.

Proof. Let G be an instance of 3-ALCP, and x a vertex of G . Let us build a set of 2-list colorings of size polynomial in f_2 that is compatible with all the solutions where x is given color 0. The same can be done for the other colors. As seen before, we can assume that x is really 3-colorable, otherwise there is a natural 2-constraint on it. First of all, give the constraint $\{1, 2\}$ to the 0-neighbourhood of x . Let U be the graph obtained from the 1-neighbourhood of x by removing 0-colored edges. There is a family of $f_2^2(|U|)$ cuts that separates all the union of two cliques and the stable sets in U (see Obs 4). For all cut (V_1, V_2) in this family, give to V_2 the constraint $\{1, 2\}$. We iterate this process to get the constraints on V_1 : let V be the graph obtained from V_1 by keeping only the 2-colored edges. There is a family of $f_2(|V|)$ cuts that separates all the clique and the stable sets in V . For all cut (V_3, V_4) , give to V_3 the constraint $\{0, 1\}$ and to V_4 the constraint $\{0, 2\}$. This gives us $f_2^2(|U|)f_2(|V|)$ 2-list colorings on the 1-neighbourhood of x . Do the same for the 2-neighbourhood and get $\mathcal{O}(f_2^6(n))$ 2-list colorings.

Take a solution of the 3-ALCP on G and assume x is colored by 0. Let us show that one of the 2-list colorings we built is compatible with this solution. Figure ?? illustrates the construction. First of all, the 0-neighbourhood can not be colored by 0, because x already is, so they are colored 1 or 2. Consider the 1-neighbourhood $N_1(x)$ of x : the 1-colored vertices in $N_1(x)$ form a 02-clique, thus it is a split graph (recall that x is really 3-colorable). Then it is the union of a 0-clique A and a 2-clique B . Call C (resp. D) the 12-clique (resp. 01-clique) formed by the 0-colored (resp. 2-colored) vertices of $N_1(x)$. (A, B, C, D) is thus a partition of $N_1(x)$. Remove the 0-colored edges to obtain the graph U . Then A is a stable set, B and C are cliques. Then there exists a cut (V_1, V_2) with $B \cup C \subseteq V_1$ and $A \subseteq V_2$. V_2 is compatible with the constraint $\{1, 2\}$ because it is formed by A and a part D' of D . Let V be the graph obtained from V_1 by keeping only the 2-colored edges. Then B is a clique and $D'' = D \setminus D'$ is a stable set, so there is a cut (V_3, V_4) that separates B from D'' . V_3 is compatible with the constraint $\{0, 1\}$ because it is the union of B and a part of C . V_4 is compatible with the constraint $\{0, 2\}$ because it is the union of D'' and a part of C . We can check the same for the 2-neighbourhood of x . \square

Proof of theorem 2. Lemma 5 and Lemma 6 conclude the proof of the theorem. \square

3 Link with the oriented Alon-Saks-Seymour conjecture

Given a graph G , the *chromatic number* χ of G is the minimum number of colors such that, two vertices connected by an edge do not have the same color. The *bipartite packing* $bp(G)$ of a graph G is the minimum number of edge disjoint complete bipartite graphs needed to cover the edges of G . The Alon-Saks-Seymour conjecture states the following.

Conjecture 2. (*Alon, Saks, Seymour*) If $bp(G) \leq k$, then $\chi \leq k + 1$

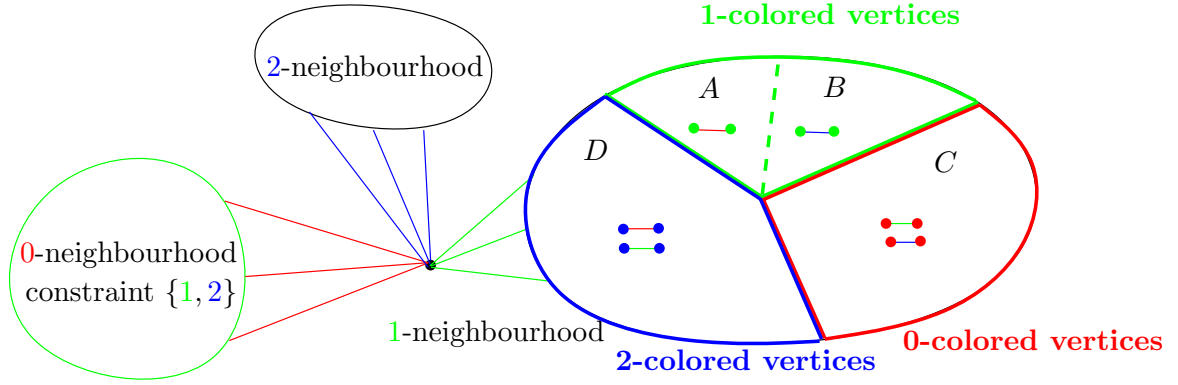
The result is a well-known result for complete graphs, due to Graham and Pollak [8]. Indeed, $n - 1$ edge-disjoint complete bipartite graphs are needed to cover the edges of a complete graph on n vertices. A beautiful proof of this theorem is due to Tverberg [17]. This conjecture was disproved by Huang and Sudakov in [12] who prove that for some graphs $\chi \geq k^{6/5}$ using a construction based on Razborov's graphs [16]. Nevertheless the existence of a polynomial bound is still open. In the following we will use an oriented version of the Alon-Saks-Seymour conjecture.

Conjecture 3. (*Oriented Alon-Saks-Seymour Conjecture*) There exists a polynomial P such that if G is a disjoint union of k complete oriented bipartite graphs, then the chromatic number χ is at most $P(k)$.

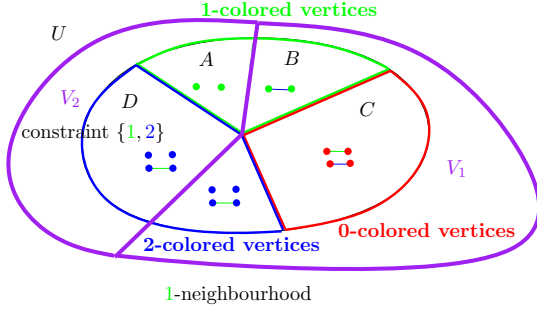
Theorem 3. The oriented Alon-Saks-Seymour conjecture is verified iff the Clique-Stable set separation conjecture is verified.

Lemma 7. If the oriented Alon-Saks-Seymour conjecture is verified, then the Clique-Stable set separation conjecture is verified.

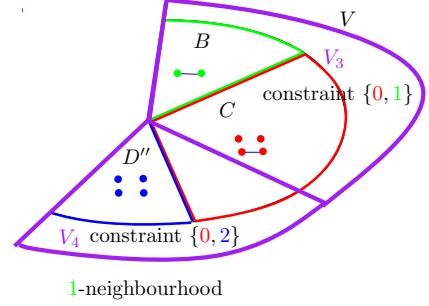
Proof. Let G be a graph on n vertices. We want to separate all pairs of cliques and stable sets which do not intersect. Consider all the pairs (C, S) such that the clique C does not intersect the stable set S . Construct an auxiliary graph H as follows. The vertices of H are the pairs (C, S) and there is an



(a) Vertex x , its 0-neighbourhood subject to the constraint $\{1, 2\}$, and its 1-neighbourhood separated in different parts.



(b) Graph U built from the 1-neighbourhood by removing 0-edges. There is a partition in (V_1, V_2) separating A from $B \cup C$, shown by the purple line. V_2 is subject to the 2-constraint $\{1, 2\}$.



(c) Graph V built from V_1 by removing 1-edges. There is a partition in (V_3, V_4) separating B from D'' , shown by the purple line. V_3 is subject to the 2-constraint $\{0, 1\}$ and V_4 is subject to the 2-constraint $\{0, 2\}$.

Figure 4: Illustration of proof of lemma 6. Color correspondence: 0=red ; 1=green ; 2=blue.

oriented edge between a pair (C, S) and a pair (C', S') iff there is a vertex $x \in C \cap S'$. Observe that the number of vertices of this graph is at most 4^n . The bipartite packing of this graph is at most the number n of vertices of G . Indeed, let H_x be the graph H restricted to the edges for which $x \in C \cap S'$. Observe that the union of the graphs H_x for all x is the graph H because if $(C, S)(C', S')$ is an edge $C \cap S' \neq \emptyset$. In addition, the graph H_x is a complete bipartite graph. Indeed, if there is an edge which starts in (C, S) and if there is an edge which ends in (C', S') then $x \in C$ and $x \in S'$ and finally there is an oriented edge $(C, S)(C', S')$. The graphs H_x and H_y cannot share an oriented edge because otherwise the intersection between a clique and a stable set would be at least 2 which is impossible. Hence the bipartite packing of this graph is at most n .

If the oriented Alon-Saks-Seymour conjecture is verified, $\chi(H) < P(n)$. Consider a color of this polynomial coloring. Let A be the set of vertices of this color. There is no edges between two vertices of A , then the union of all the first components (cliques) of the vertices of A do not intersect the union of all the second components (stable sets) of A . Indeed, if they intersect, there is a clique C which intersects a stable set S , hence there is an edge which is impossible.

The union of the cliques and the union of the stable sets do not intersect, hence it defines a cut which separates all the pairs of A . The same can be done for all the colors. Then we can separate all the pairs (C, S) by $\chi(H)$ cuts, which is a polynomial in n if the Alon-Saks-Seymour conjecture is verified. This achieves the proof of the first implication. \square

Lemma 8. *If the Clique-Stable set separation conjecture is verified, then the oriented Alon-Saks-Seymour conjecture is verified.*

Proof. Let $G = (V, E)$ be an oriented graph with bipartite packing k . Construct an auxiliary graph H as follows. It has k vertices which are the oriented complete bipartite graphs that cover the edges of G . There is an edge between two pairs (A_1, B_1) and (A_2, B_2) iff there is a vertex $x \in A_1 \cap A_2$. Hence the complete bipartite graphs in which x appears at the left form a clique of H (say the clique C_x associated to x) and the complete bipartite graphs for which y appears at the right form a stable set in H (say the stable set S_y associated to x). Indeed, it is quite clear for the clique, and it is also true for the stable set because if $y \in B_1 \cap B_2$ and $x \in A_1 \cap A_2$ there is an edge covered twice which is impossible, then it is a stable set. Note that a clique or a stable set associated to a vertex can be empty, but this does not trigger any problem.

For each vertex x , consider a cut Cut_x that separates C_x and S_x . By assumption there is a polynomial number in k of cuts which separates all the pairs (C, S) , hence which separate this particular pairs of cliques and stable sets. We color each vertex x by the color Cut_x . This coloring is proper: assume there is an edge from x to y , and that x and y are given the same color. Then there exists a bipartite (A, B) that cover the edge xy , hence (A, B) is in the clique associated to x and in the stable set associated to y , which means they intersect: the cut Cut_x can not separate in the same time C_y from S_y , because it would then separate C_x from S_y . This is impossible. Then we have a coloring with at most as many colors as cuts needed to separate all the clique and the stable sets, which is a polynomial in k . \square

Corollary 3. *The oriented Alon-Saks-Seymour conjecture is equivalent to the Clique-Stable set separation conjecture. Moreover, they are also equivalent to having a polynomial number of 2-list coloring which are compatible with all the solutions of an instance of 3-ALCP.*

4 Relations between the t -biclique covering numbers

5 Cliques-Stables separation conjecture and Communication complexity

Yannakakis introduced in [18] the following communication complexity problem, called $CL - IS$: given a publicly known graph Γ on n vertices, Alice and Bob can agree on a protocol, then a clique is given to Alice and a stable set is given to Bob. They don't know which clique or which stable set was given to the other one, and their goal is to decide whether the clique and the stable set intersect or not, by minimizing the number of bits exchanged. This can be formalize by a function $f : X \times Y \rightarrow \{0, 1\}$,

where X is the set of cliques and Y the set of stable sets. Note that the intersection of a clique and a stable set is at most one vertex. In the deterministic version, Alice and Bob send alternatively messages one to each other, and the minimization is on $D(f)$, the number of bits exchanged between them. In the non-deterministic version, for $b \in \{0, 1\}$, an all powerful prover sends a certificate of size $N^b(f)$ in order to convince both Alice and Bob that the result is b . Then, Alice and Bob can exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize $N^b(f)$.

In this particular setting, a certificate proving that $f(x, y) = 1$ is just the name of a vertex in the intersection of the clique and the stable set. Thus $N^1(f) \leq \lceil \log_2(n) \rceil$. Convincing Alice and Bob that the result is 0 is much more complicated. If we have a family of m cuts that separate all the cliques and the stable sets, then a certificate for $f(x, y) = 0$ can be a cut that separates x and y . Thus $N^0(f) \leq \lceil \log_2(m) \rceil$.

It is known in theory of communication complexity that, for general f , $N^b(f) = \lceil \log_2 C^b(f) \rceil$ where $C^b(f)$ is the minimum number of monochromatic combinatorial rectangles needed to cover the b -inputs of the communication matrix M of f , with $M_{x,y} = f(x, y)$. One can observe that a monochromatic combinatorial rectangle that cover some 0-inputs of M is precisely a cut that separates the cliques and the stable sets involved in the rectangle. Thus finding the minimum $N^0(f)$ is equivalent to finding the minimum number of cuts needed to separate all the cliques and the stable sets.

The existing bounds for $CL - IS$ are ([18], [14]):

$$2 \log_2 m \leq D(f) \leq \mathcal{O}(\log_2^2(n)).$$

Moreover, Huang and Sudakov improved in [12] the trivial lower bound $N^0(f) \geq \log_2 n$ thanks to a counterexample to the Alon-Saks-Seyour conjecture: for CL-IS, there exists an infinite collection of graphs such that

$$N^0(f) \geq 6/5 \log(n) - \mathcal{O}(1).$$

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