

# 3-adapted-list coloring problem and Alon-Saks-Seymour conjecture

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## Abstract

In this paper we propose a study of the 3-adapted list coloring problem [7] by 2-list colorings. First we prove that this problem is equivalent to the problem of separation of the cliques and the stable sets. This problem is deeply linked with a problem of communication complexity introduced by Yannakis [13]. Hence we show that these problems are also equivalent with a polynomial bound for an oriented Alon-Saks-Seymour conjecture. Finally we look at the case where a color is a matching, and solve some particular cases.

## 1 Definitions

Let  $k$  be an integer. We denote by  $[k]$  the set of integers between 0 and  $k - 1$ . Let  $G = (V, E)$  be a graph. An edge  $uv \in E$  links its two *endpoints*  $u$  and  $v$ . The *neighbourhood*  $N(x)$  of  $x$  is the set of vertices  $y$  such that  $xy \in E$ . The subgraph *induced by*  $X \subseteq V$  is the graph  $(X, E')$  where  $E' = E \cap (X \times X)$ . A *clique of*  $G$  is a complete induced subgraph of  $G$ . A *stable set* is an induced subgraph without edges. Observe that the intersection of a clique and a stable set is at most one vertex. A *vertex-coloring* (resp. *edge-coloring*) of  $G$  with  $k$  colors is a function  $f_V$  (resp.  $f_E$ ) from  $V$  (resp.  $E$ ) to  $[k]$ . In an edge-colored graph, an  $\alpha_1 \dots \alpha_k$ -*clique* is an induced subgraph where the edges are only colored by  $\alpha_1 \dots \alpha_k$ . A vertex-coloring  $f_V$  of  $G$  is  *$f_E$ -proper* iff for each edge  $uv$  of  $E$ ,  $f_V(u) \neq f_E(uv)$  or  $f_V(v) \neq f_E(uv)$ . The  $k$ -ADAPTED-LIST-COLORING PROBLEM was defined by Hell and Zhu in [7], and then studied in [1] and [9] consists in a proper coloring of the vertices.

$k$ -ADAPTED-LIST-COLORING PROBLEM ( $k$ -ALCP)

**Input:** An edge coloring  $f_E$  of the complete graph on  $n$  vertices with  $k$  colors.

**Question:** Is there an  $f_E$ -proper coloring?

This problem has applications to matrix partitions of graphs, full constraint satisfaction problems and trigraph homomorphisms [3], [4], [2], [6]. It is known to be NP-complete if  $k \geq 4$  and it is known to be polynomial if  $k \leq 2$  [7]. The question remains open for  $k = 3$ . We focus our attention on this last case. The *majority color of*  $x \in V$  is the color which appears the most often for the edges touching  $x$  (in case of ties, we arbitrarily cut them). A vertex has an  *$l$ -constraint* if it can be colored by at most  $l$  colors. An  *$l$ -list coloring* is a coloring where each vertex has an  $l$ -constraint. An  $l$ -list coloring is *compatible* with a solution

$f_V$  if for each vertex  $x$ ,  $f_V(x)$  is in the  $l$ -constraint of  $x$ . A set of  $l$ -list colorings is *compatible* with a solution if one of the  $l$ -list coloring is compatible with the solution. A partition problem is  *$l$ -list colored for a function  $f$*  if there exists  $f(n)$   $l$ -list colorings compatible with all the solutions. A partition problem is said *polynomially  $l$ -list colored* if there exists a polynomial  $Q$  such that the partition problem is  $l$ -list colored for  $Q$ .

**Observation 1** *Given a polynomial number of 2-list colorings, it is possible to decide in polynomial time if they are compatible with a solution.*

**Theorem 1** *REF? There exists an algorithm in time  $O(n^{\log n})$  which solves the 3-ALCP.*

Proof: We give the proof for self completeness. This algorithm consists in testing  $n^{\log(n)}$  2-list colorings which contains all the solutions. Consider the vertex  $x$ , without loss of generality we can assume that the majority color of  $x$  is 0. If there is a solution in which  $x$  is colored by 0 then all its 0-neighbours cannot have the color 0, hence all its 0-neighbours has 2-list constraints. And the number of remaining vertices with 3 possible colors is at most  $2n/3$ . It is possible to make the same for the remaining vertices. Hence consider the following algorithm, for each vertex  $x$  try to find a solution for which  $x$  is colored by its majority color. If there is no such solution for all  $x$ , then all the vertices have 2-list constraints. This algorithm is representable by a research-tree of degree  $n+1$  and depth  $\log(n)$  in which each of the  $n^{\log(n)}$  leaves is 2-list colorings of the instance.  $\square$

## 2 Link with the Cliques-Stables separation conjecture

A *cut* is a pair  $(A, B)$  such that  $A \cup B = V$  and  $A \cap B = \emptyset$ . A cut *separates* a clique  $C$  and a stable set  $S$  if  $C \subseteq A$  and  $S \subseteq B$ . A clique and a stable set can be separated iff they do not intersect. Let  $\mathcal{C}$  be the set of cliques and  $\mathcal{S}$  be the set of stable sets.

**Conjecture 1** (*Clique-Stable Set separation Conjecture*) *There is a polynomial  $Q$ , such that for every graph  $G$  on  $n$  vertices, all pairs  $(C, S) \in \mathcal{C}_G \times \mathcal{S}_G$  which do not intersect are separated by  $Q(n)$  cuts.*

To motivate this polynomial separation let us prove the conjecture for some particular classes of graphs. For a class with a polynomial number of cliques or stable sets, the conjecture is verified. Hence a class of graphs with bounded cliques or stable sets verifies the conjecture. Claims 1, 2 and 3 give more interesting examples of classes of graphs which verify the conjecture. The *claw* is the graph with four vertices 1, 2, 3, 4 with the edges 12, 13 and 14.

**Claim 1** *For random graphs, the clique-stable set separation conjecture is verified.*

Proof: In random graphs, maximum cliques and maximum stable sets are of size  $\log(n)$ . Let  $(C, S)$  be a pair of a clique and a stable set which do not intersect. Observe that there is at most  $4^n$  such pairs. Consider a random partition of

the vertices into  $V_1$  and  $V_2$  of equal size, the probability that all the vertices of  $C$  are in  $V_1$  is  $1/2^{\log(n)}$  and the same holds for  $V_2$  and  $S$ . Hence  $C$  and  $S$  are separated with probability  $1/n^2$ . Then, a pair  $(C, S)$  is separated by  $1/n^2$  of all the partitions, hence by double counting there is a partition which separates  $1/n^2$  of all the pairs. When the separated pairs are deleted it remains  $(1 - 1/n^2) \cdot 4^n$  pairs and the remaining pairs are still separate which probability  $1/n^2$ . Hence it possible to separate the pairs after  $k$  steps if  $(1 - 1/n^2)^k \cdot 4^n \leq 1$ , which means that  $2n^3$  partitions separates all the pairs.  $\square$

**Claim 2** *For comparability graphs, the clique-stable set separation conjecture is verified.*

Proof: A  $N$  is isomorph to the poset on four vertices  $\{a, b, c, d\}$  with arcs  $\{ab, cb, cd\}$ . If a poset  $O$  contains a maximal antichain and a maximal chain which do not intersect, then  $O$  contains a  $N$  in its Hasse diagram (formed by two consecutive of the chain and two vertices of the antichain). Now, let  $G$  be comparability graph, and  $O$  the resulting poset. We subdivide every arc of the Hasse diagram of  $O$ , the resulting poset  $O'$  contains no  $N$  in its Hasse diagram. Hence, we consider all the cuts formed by the neighbourhood and the non-neighbourhood of each vertex in  $O'$ . Now, a clique  $C$  and a stable set  $S$  of  $G$ , can be extended into a maximal chain and a maximal antichain in  $O'$  which intersect and then are separated by a chosen cut.

**Claim 3** *For claw-free graphs, the clique-stable set separation conjecture is verified.*

Proof: Let  $x$  be a vertex of the graph. At most two of its neighbours are in the same stable set. Indeed, otherwise there is a claw. Hence, the separation in neighbourhood and non-neighbourhood of a vertex modulo at most two vertices separates the cliques and the stable sets. More formally, for all  $x$  and all set  $X$  of at most two neighbours of  $x$ , the cuts  $N(x) \setminus X$  and  $(N(x) \setminus X)^c$  separate  $C$  and  $S$ . Then  $n^3$  cuts separates cliques and stable sets.  $\square$

This problem is not so far from a problem of communication complexity (see [10] for more details). A graph  $G$  is given to Alice and Bob. Alice and Bob try to minimize the bits exchanged to determine if a clique and a stable set intersect. Alice and Bob decide a protocol for this graph. Then a clique is given to Alice and a stable set is given to Bob. They apply the protocol. The goal is to minimize the communication between Alice and Bob ie the number of exchanged bits. This problem was introduced by Yannakakis [13] which proposes a deterministic algorithm which uses  $\log^2(n)$  bits. The best known lower bound for non-deterministic algorithm is  $6/5 \cdot \log(n)$  is given by [8] using a counterexample to the Alon-Saks-Seymour conjecture.

The following of this section is devoted to the equivalence between the Clique-Stables separation conjecture and the polynomial 2-list coloring of the 3-ALCP.

**Observation 2** *A solution of the 3-ALCP is a partition of the graph into  $(X, Y, Z)$  such that the graph restricted to  $X$  (resp.  $Y$ , resp  $Z$ ) has only color 01 (resp. 02, resp. 12).*

Proof: Consider a solution of the 3-ALCP. Let  $X$  be (resp.  $Y$ ,  $Z$ ) the set of vertices colored by 0 (resp. 1, 2) in the solution. There is no edge of color 0

(resp. 1, 2) in the graph restricted to  $X$  (resp.  $Y, Z$ ) since an edge cannot have the color of its endpoints then  $X$  is a 12-clique. The reverse is also clearly true.  $\square$

Hence the 3-ALCP is polynomially 2-list colored iff the problem of finding such a division into 3 cliques is polynomially 2-list colored because the solutions are the same.

A graph  $G = (V, E)$  is *split* if  $V = V_1 \cup V_2$  and the subgraph induced by  $V_1$  is a clique and the subgraph induced by  $V_2$  is a stable set. A graph with edges colored by  $a$  and  $b$  is *split* if there is a partition of the vertices into a  $a$ -clique and a  $b$ -clique.

**Proposition 1** *Let  $x$  be a vertex. Let  $A$  be the 0-neighbourhood of  $x$ . Let  $U$  be a 12-clique of  $A$ . If  $U$  is not split then there is no solution where  $x$  is colored by 0.*

Proof: Assume that  $x$  is colored by 0. Then all the vertices of  $U$  are of color 1 or 2 because they are in the 0-neighbourhood of  $x$ . The vertices colored by 1 form a 2-clique and all the vertices colored by 2 form a 1-clique and the union is  $U$ . Hence  $U$  is split.  $\square$

Observe that the same property holds for the 1 and the 2-neighbourhoods. A vertex  $x$  is *really 3-colorable* if for each color  $a$ , each  $bc$ -clique of the  $a$ -neighbourhood of  $x$  is a split graph. If a vertex is not really 3-colorable then, in a solution, it can be colored by at most 2 different colors. A global solution is necessarily a solution of the graph without  $x$  and  $x$  colored by one of the two possible colors. Hence, given a 2-list coloring of the graph without  $x$ , there is a natural 2-list coloring of the graph which is 2-list coloring of the graph without  $x$  and the 2-constraint for  $x$ . Hence it is possible to assume that the graph has only really 3-colorable vertices.

**Theorem 2** *The 3-ALCP is polynomially 2-list colored iff the clique-stable set separation conjecture is verified.*

Proof: Assume that the 3-ALCP has a 2-list coloring and let us prove that it is possible to separate all pairs of cliques and stable sets. Let  $G = (V, E)$  be a graph. Consider a edge-colored graph with the same vertices as  $G$  where  $xy$  is colored by 0 if  $xy \in E$  and colored by 1 otherwise. Let  $C$  be a clique and  $S$  a stable set of  $G$  which do not intersect. Observe that the edges of  $C$  are colored by 0, and those of  $S$  are colored by 1. Then the vertex-coloring 1 for the vertices of  $C$ , 0 for those of  $S$  and 2 for the other vertices is a solution of the 3-ALCP. By assumption, the 3-ALCP is polynomially 2-list colored. Hence there exists a polynomial number of 2-list colorings which are compatible with all the solutions. Consider a partition  $(X, Y, Z)$  compatible with this solution. The set  $X$  (resp.  $Y, Z$ ) is the set of vertices which can be colored by 01 (resp. 12, 02). The vertices of  $Z$  cannot be in  $C$ , and those of  $Y$  cannot be in  $S$ . Then  $C \subseteq X \cup Y$  and  $S \subseteq X \cup Z$ .

Since all the edges are of color 0 and 1, the graph restricted to  $X$  is split, because otherwise  $X$  cannot be colored properly. Given a graph, there is a linear number of decompositions into a split graph. Then the number of possible decompositions of  $X$  is linear. Let  $(U_k, V_k)_k$  be these decompositions. The pair  $(C, S)$  is separated by one of the  $(U_k \cup Y, V_k \cup Z)$ .

Then, the clique-stable set conjecture is verified because of the polynomial number of 2-list colorings and the linear number of decomposition into split graphs.

We will now prove the reverse of Theorem 2. The proof is based on two lemmas. First the 3-ALCP is polynomially 2-list colored if a problem a little bit different is polynomially 2-list colored: a particular case of a 4-partition problem. Next this new problem can be polynomially 2-list colored if the cliques-stable set separation conjecture is verified.

The 3-ALCP is polynomially 2-list colored iff there is a polynomial number of 2-list colorings  $(A, B, C)$  for which each vertex in  $A$  (resp.  $B$ , resp.  $C$ ) can be colored by 0 or 1 (resp. 0 or 2, resp. 1 or 2) which are compatible with all the solutions of the 3-ALCP. Let us introduce the 4-PARTITION-2-CLIQUE PROBLEM (4P2CP) and prove that if this problem has a 2-list coloring then it is also true for the 3-ALCP. Let us define the 4-PARTITION-2-CLIQUE PROBLEM. 4-PARTITION-2-CLIQUE PROBLEM (4P2CP)

**Input:** A complete graph on  $n$  vertices, edges-colored by 3 colors.

**Question:** Is there a partition of the vertices into a 0-clique, a 2-clique, a 12-clique and a 01-clique.

**Lemma 1** *If 4P2CP is polynomially 2-list colored, then the 3-ALCP is polynomially 2-list colored.*

Proof: We treat all the vertices which are not really 3-colorable as we explain before. Let  $x$  be a vertex. Hence we can assume that  $x$  is really 3-colorable. Our goal is to prove that the solutions of the 3-ALCP in which  $x$  is colored by 0 are polynomially 2-list colored. Since the same can be done if  $x$  is colored by 1 and 2 and a general solution is necessarily a solution of one of these forms, the 3-ALCP is polynomially 2-list colored because the union of three polynomial 2-list colorings is still a polynomial 2-list coloring.

Consider a solution in which  $x$  is colored by 0, all its 0-neighbours are not colored by 0. Then for these vertices, there are 2-constraints. A 2-list coloring for the remaining vertices with the 2-constraints for  $x$  and its 0-neighbours give a 2-list coloring of the complete graph.

Consider the graph restricted to the 1-neighbours of  $x$ . Since  $x$  is really 3-colorable, the 02-cliques are split graphs by Proposition 1. Let us prove that if the 4P2CP is polynomially 2-list colored then the 3-ALCP is polynomially 2-list colored. The graph restricted to the 02-cliques has to be split. Hence the 02-clique of the solution is a union of a 0-clique and a 2-clique. Hence, a solution of the 3-ALCP is also a solution for the 4P2CP where the 12-clique and the 01-clique are the same and the 02-clique is the union of the 0-clique and the 2-clique. If a vertex can be in at most two sets for the 4P2CP, it can be in at most two sets for the 3-ALCP. Hence there is a polynomial 2-list coloring for the 3-ALCP restricted to the 1-neighbours of  $x$ . A global solution is necessarily a solution in all subsets, then the union of the 2-constraints for 0, 1 and 2-neighbourhood give all the possible solutions. Hence if the 4P2CP problem is polynomially 2-list colored, the 3-ALCP is polynomially 2-list colored.  $\square$

**Lemma 2** *If the clique-stable set separation conjecture is verified, the 4P2CP is polynomially 2-list colored.*

Proof: Let  $G$  be an edge colored graph. We want to find a polynomial number of 2-list colorings which are compatible to all the solutions of the 4P2CP. Consider a solution  $(U, V, W, X)$  of the 4P2CP where  $U$  is a 0-clique,  $V$  a 2-clique,  $W$  a 12-clique and  $X$  of 01-clique. Construct an auxiliary graph  $H$  as follows.  $H$

has the same vertices as  $G$  and there is an edge  $xy$  in  $H$  iff  $xy$  is colored by 1 or 2 in  $G$ . Let  $C$  be the set  $U$  and  $S$  be the set  $V \cup W$ . Observe that  $(C, S)$  is a pair of a clique and a stable set which do not intersect in  $H$ . Hence by assumption there is a partition  $(A, B)$  which separates  $(C, S)$ . The  $A$  contains  $U$  and a subset of the  $X$  and  $B$  contains the  $V, W$  and a subset of the  $X$ . Hence the vertices of  $A$  have 2-constraints but those of  $B$  have 3-constraints. Hence the vertices of  $A$  can be at most in 2 sets of the partitions.

Consider the graph  $G$  restricted to  $B$ . Consider the following auxiliary graph with vertices  $B$  where there is an edge  $xy$  iff  $xy$  is colored by 2 in  $G$ . The set  $V$  is a clique and the set  $X$  is a stable set. By assumption there is a set  $(A', B')$  which separates  $V$  from  $X$ . Hence the vertices of  $B$  has 2-constraints. Hence all the vertices of the graph have 2-constraints. Since the clique-stable set separation conjecture is satisfied and we apply it a constant number of times, the 4P2CP is polynomially 2-list colored. This achieves the proof of the lemma  $\square$

Lemma 1 and Lemma 2 conclude the proof of the theorem.  $\square$

### 3 Link with the oriented Alon-Saks-Seymour conjecture

Given a graph  $G$ , the *chromatic number*  $\chi$  of  $G$  is the minimum number of colors such that, two vertices connected by an edge do not have the same color. The *bipartite packing*  $bp(G)$  of a graph  $G$  is the minimum number of edge disjoint complete bipartite graphs needed to cover the edges of  $G$ . The Alon-Saks-Seymour conjecture states the following.

**Conjecture 2** (*Alon, Saks, Seymour*) *If  $bp(G) \leq k$ , then  $\chi \leq k + 1$*

The result is a well-known result for complete graphs, due to Graham and Pollak [5]. Indeed,  $n - 1$  edge-disjoint complete bipartite graphs are needed to cover the edges of a complete graph on  $n$  vertices. A beautiful proof of this theorem is due to Tverberg [12]. This conjecture was disproved by Huang and Sudakov in [8] who prove that for some graphs  $\chi \geq k^{6/5}$  using a construction using Razborov's graphs [11]. Nevertheless the existence of a polynomial bound is still open. In the following we will use an oriented version of the Alon-Saks-Seymour conjecture.

**Conjecture 3** (*Oriented Alon-Saks-Seymour Conjecture*) *There exists a polynomial  $P$  such that if  $G$  is a disjoint union of  $k$  complete oriented bipartite graphs, then the chromatic number  $\chi$  is at most  $P(k)$ .*

**Theorem 3** *The oriented Alon-Saks-Seymour conjecture is verified iff the clique-stable set separation conjecture is verified.*

*Proof:* Let us first prove the direct implication. Let  $G$  be a graph on  $n$  vertices. We want to separate all pairs of cliques and stable sets which do not intersect. Consider all the pairs  $(C, S)$  such that the clique  $C$  does not intersect the stable set  $S$ . Construct an auxiliary graph  $H$  as follows. The vertices of  $H$  are the pairs  $(C, S)$  and there is an oriented edge between a pair  $(C, S)$  and a pair  $(C', S')$  iff there is a vertex  $x \in C \cap S'$ . Observe that the number of vertices of this graph is at most  $4^n$ . The bipartite packing of this graph is at most the number  $n$  of

vertices of  $G$ . Indeed, let  $H_x$  be the graph  $H$  restricted to the edges for which  $x \in C \cap S'$ . Observe that the union of the graphs  $H_x$  for all  $x$  is the graph  $H$  because if  $(C, S)(C', S')$  is an edge  $C \cap S' \neq \emptyset$ . In addition, the graph  $H_x$  is a complete bipartite graph. Indeed, if there is an edge which starts in  $(C, S)$  and if there is an edge which ends in  $(C', S')$  then  $x \in C$  and  $x \in S'$  and finally there is an oriented edge  $(C, S)(C', S')$ . The graphs  $H_x$  and  $H_y$  cannot share an oriented edge because otherwise the intersection between a clique and a stable set would be at least 2 which is impossible. Hence the bipartite packing of this graph is at most  $n$ .

Since the oriented Alon-Saks-Seymour conjecture is verified,  $\chi(H) < P(n)$ . Consider a color of this polynomial coloring. Let  $A$  be the set of vertices of this color. There is no edges between two vertices of  $A$ , then the union of all the cliques of  $A$  do not intersect the union of the stable sets of  $A$ . Indeed, if they intersect, there is a clique  $C$  which intersects a stable set  $S$ , hence there is an edge which is impossible.

The union of the cliques and the union of the stable sets do not intersect, hence it defines a cut which separates all the pairs of  $A$ . The same can be done for all the colors. Then we can separate all the pairs  $(C, S)$  by a polynomial number of cuts if the Alon-Saks-Seymour conjecture is verified. This achieves the proof of the first implication.

Let us now prove the reverse. If the clique-stable set separation conjecture is verified then the Alon-Saks-Seymour is verified. Let  $G = (V, E)$  be an oriented graph with bipartite packing  $k$ . Consider its bipartite incidence graph  $I$  where the vertices are  $V \cup V$  with an edge between  $x_1$  and  $y_2$  iff there is an edge  $xy$  or  $yx$  in the oriented graph. The bipartite packing of this graph is less than  $2k$  since the associated edges of  $(A, B)$  are covered by  $(A, B)$  and  $(B, A)$ . Then there is  $2k$  oriented pairs of vertices  $(A, B)$  which cover the edges of the incidence graph.

Construct an auxiliary graph  $H$  as follows. It has  $2k$  vertices which are the complete bipartite graphs. There is an edge between two pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  iff there is a vertex  $x \in A_1 \cap A_2$ . Hence the complete bipartite graphs in which  $x$  appears at the left form a clique of  $H$  and the complete bipartite graphs for which  $y$  appears at the right form a stable set in  $H$ . Indeed, it is quite clear for the clique, and it is also true for the stable set because if  $y \in B_1 \cap B_2$  and  $x \in A_1 \cap A_2$  there is an edge covered twice which is impossible, then it is a stable set.

In the incidence graph  $I$ , there is a non-edge between  $x$  and  $y$  iff they never appear on the opposite sides of a same bipartite, ie if the clique associated to  $x$  and the stable set associated to  $y$  do not intersect. Observe that the incidence bipartite graph  $I$  is a subgraph of the bipartite graph of separation between cliques and stable sets which do not intersect. By assumption there is a polynomial number in  $k$  of cuts which separates all the pairs  $(C, S)$ , hence which separate this particular pairs of cliques and stable sets. If a cut separates a set of cliques and stable sets, there is no edge in  $G$  between the vertices associated to the cliques and the vertices associated to the stable sets of  $H$ . Assume that there is an edge between two vertices  $x$  and  $y$  of the same color. Hence there are the edges  $x_1y_2$  and  $x_2y_1$  in  $I$  and then the clique associated to  $x$  and the stable set associated to  $y$  intersect, hence they cannot be separated by a cut. Hence there is no edge. Hence, give to  $x$  the color of the partition which separates  $x_1x_2$  is a proper coloration of the vertices. Hence it is a proper coloring which

achives the proof.  $\square$

## 4 One color is a matching

In this section, we will assume that a color is a perfect matching.

**PARTITION-3-COLORS-WITH-A-MATCHING (P4CM)**

**Input:** A complete graph on  $V$  whose edge are colored by 4 colors  $\{0, 1, 2\}$  where the edges colored 0 formed a matching.

**Question:** Is there a partition of  $V$  into 3 subsets  $V_0, V_1, V_2$  such that the graph induced on  $V_i$  do not contain edge with color  $i$  for  $i = 0, 1, 2$ .

Observe that it is the easiest case of the 3-ALCP because if a vertex has a 0-degree equal to 0 then there is a global solution iff there is a global solution with this vertex colored by 0. The 0-edges are denoted by  $x_i y_i$  and  $L = \{x_i\}$  and  $R = \{y_i\}$ . Observe that it is possible to give the value 0 to  $x_i$  or  $y_i$  for each pair and give color 0 to one of these two vertices decrease the constraints on the other vertices. The *left (resp. right) graph* is the graph restricted to the vertices of  $L$  (resp  $R$ ). Our goal is to find some polynomial algorithm for particular cases using 2-list coloring.

The first easy case is the case where the color between each pair  $(x, y_i)$  and  $(x_j, y_i)$  is a  $C_4$ . Indeed color  $x_i$  or  $y_i$  with 0 does not change anything. Hence there is a solution iff the left graph is 2-colorable which can be done in polynomial time. To motivate the study of a little bit more complicated cases, let us prove that the problem is *NP*-complete if there is 4 colors.

**Lemma 3** *The P4CM problem is NP-complete*

**Proof.** Inspired from [7], we reduce this problem from 3-coloring of graphs: let  $G$  be a graph, we construct an instance  $I$  of the P4CM problem equivalent to the 3-colorability of  $G$ . In  $I$ , the edges with color 0 will form a matching.

First, we define a 1-gadget attached on two given vertices  $x$  and  $y$ , it is drawn in Figure 1. Using only colors 0, 1 and 2, this gadget avoids  $x$  and  $y$  having both color 1. Similarly, we define a 2-gadget by exchanging colors 1 and 2 in Figure 1. Now, every vertex  $x$  of  $G$  is replaced by a set  $B_x$  of  $d(x)$  vertices in  $I$ :

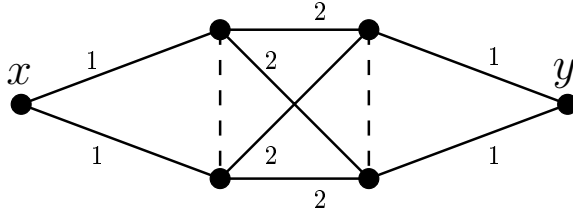


Figure 1: A 1-gadget attached on vertices  $x$  and  $y$ , dotted edges are colored 0.

for every neighbour  $z$  of  $x$ ,  $B_x$  contains a vertex  $x_z$ . For every edge  $xy$  of  $G$ , we add a edge colored 0 between  $x_y$  and  $y_x$  and a 1-gadget and a 2-gadget attached on  $B_x$  and  $B_y$ . Finally, to every set  $B_x$  we add 3 attached sets  $B_x^1$ ,  $B_x^2$  and  $B_x^{12}$  as shown in Figure 3. This gives  $G'$  an enhanced representation of  $G$ . To obtain the final instance  $I$ , we take the disjoint union of  $G'$  and  $G''$ , a copy of



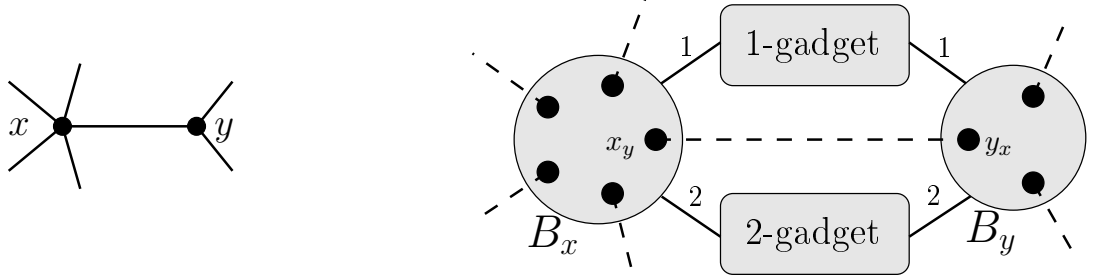


Figure 2: Two vertices  $x$  and  $y$  linked in  $G$  and their corresponding sets in  $I$ . Edges between two sets stand for all possible edges between the two sets.

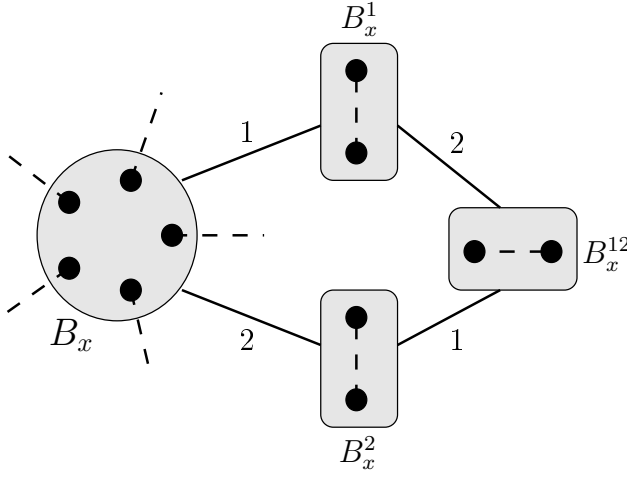


Figure 3: A set  $B_x$  and  $B_x^1$ ,  $B_x^2$  and  $B_x^{12}$ , its three attached sets.

$G'$ , and replace all non-edges by edges with color 3 to achieve a complete graph on  $V(G') \cup V(G'')$ .

Now, we prove that  $G$  is 3-colorable if, and only if,  $I$  is a YES-instance of the P4CM problem. If  $G$  admits a 3-coloring  $c : V(G) \rightarrow \{0, 1, 2\}$ , then, for every set  $B_x$  of  $G'$  or  $G''$ , we give the color  $c(x)$  to every vertex of  $B_x$ . It is easy now to extend the coloring to the other vertices of  $I$  (which are in the different gadgets) such that the coloring induced a partition into three sets  $V_0$ ,  $V_1$  and  $V_2$  that satisfy the condition of the P4CM problem (the set  $V_3$  is empty).

Conversely, assume that  $I$  is a YES-instance of the P4CM problem. As there is a complete bipartite with colored 3 edges between  $V(G')$  and  $V(G'')$ , one of this set do not intersect  $V_3$ , assume that  $G'$  does. So, we give color  $i$  to every vertex of  $G' \cap V_i$ , for  $i = 0, 1, 2$ . For every set  $B_x$  of  $G'$ , the three attached sets  $B_x^1$ ,  $B_x^2$  and  $B_x^{12}$  avoid  $B_x$  to contain a vertex with color 1 and a vertex with color 2. So, if  $B_x$  contains only vertices with color 0, we color  $x$  in  $G$  with color 0. If  $B_x$  contains at least one vertex with color 1 (and then no vertex with color 2), we color  $x$  in  $G$  with color 1. And conversely, if  $B_x$  contains at least one vertex with color 2 (and then no vertex with color 1), we color  $x$  in  $G$  with color 2. Eventually, because of the edges 0, the 1-gadgets and the 2-gadgets in

$I$ , we check that to adjacent vertices in  $G$  have not received the same color, and provide a 3-coloring of  $G$ .  $\square$

**Lemma 4** *If the graph restricted to  $x_i, y_i, x_j, y_j$  consists in three matchings then there is a polynomial algorithm to solve the problem.*

Proof: Consider the left graph in which the edges of color 1 are edges and the edges of color 2 are non-edges. If the left graph has a solution then all the vertices of the left can be divided in two parts: one which is a clique and which is colored by 2 in the solution and the other vertices form a stable set and which can be colored by 1. If there is a solution then it is clearly of this form: it is a split graph.

But there is not necessarily a solution on  $L$ , in a solution there are vertices of  $L$  and  $R$ . But if we choose a vertex  $x_i \in L$  and  $y_j \in R$ , then the color of  $x_i y_j$  is the complement of the color of  $x_i y_i$ . Hence  $x_i$  and  $y_j$  can have the same color 2 (resp. 1) iff there is an edge (resp. non-edge) in the left-graph.

If there is a solution it uses some left and some right vertices. Consider the vertices colored by 1, then the restriction to the left graph on these vertices there is two stables set which are the vertices restricted to  $L$  and to  $R$  and between them there are all the edges. The same holds for the restriction to the vertices colored by 2 by replacing edges by non-edges. Hence, there is a solution iff there is a subdivision of the graph into two cliques which share no edges and two stable sets totally linked by edges.

Hence we just have to prove that the partition of the graph into this form can be decided in polynomial time. Let  $C_1$  and  $C_2$  be the cliques and  $S_3$  and  $S_4$  be the stable sets. Let us prove that the problem is polynomially 2-list colored and the partitions can be found in polynomial time. We can assume that each part  $C_1 \cup S_3$  and  $C_2 \cup S_4$  do not have few vertices. Indeed if they have less than size 20 vertices, hence the graph without these vertices forms a split graph which can be decided in polynomial time. For each set of 20 vertices we just have to test if the graph without these vertices is split which takes a polynomial time. The parts  $C_1 \cup S_3$  and  $C_2 \cup S_4$  have vertices. We can also assume that the parts  $C_1 \cup C_2$  are of big size, because otherwise, the graph is nearly a complete bipartite graph and that  $S_3 \cup S_4$  is of big size, because otherwise the graph is nearly the union of two cliques.

Hence we can assume that there is at least 10 vertices in  $C_1, C_2, S_3, S_4$ . For each choice of 40 vertices, we just have to check if there is a solution with this initial partition in polynomial time. But a vertex  $x$  can be added only in 2 of the 4 parts. Indeed, for example, if a vertex can be added in  $C_1$  then it cannot be added in  $C_2$  because all the vertices of  $C_1$  must be linked to  $x$  because  $x$  can be added in 1. But if  $x$  can be added in 2 then there is no link between the vertices of 1 and 2, the same kind of argument holds for  $S_3$  and  $S_4$ . Then two vertices can only be in two of the four sets of the partition.

Assume that a vertex  $y$  is in 1 – 3. And let  $x$  be a vertex we choose to add in 1. Then the  $xy$  is not in  $E$  then  $y$  cannot be in 1 ie  $y$  is in 3. And this gives other constraints if  $y$  is in 2 – 4 and so on. Then if we treat all the conflicts and there is no contradiction, let us prove that if there is a problem with a constraint latest it is not because of  $x$ . Indeed, assume that it is because of  $x$ . Then we have a problem with  $z$  in 1 or in 3. If it is in 1 then there is no edge  $xz$  and  $z$  was in 1 – 3 or 1 – 4, but it is not possible but we would have add this constraint when

we treat  $x$ . Hence if there is no problem when we add  $x$ , there is no backtrack and then the algorithm to solve the problem is polynomial.  $\square$

We will treat another particular case which is the following. Between a pair  $(x_1, y_1)$  and  $(x_2, y_2)$  there is two edges of color 1 and two of color 2 and the two edges intersect.

**Proposition 2** *If between a pair  $(x_1, y_1)$  and  $(x_2, y_2)$  there is two edges of color 1 and two of color 2 and the two edges intersect then there is a polynomial algorithm to solve the problem.*

Proof: There are two edges of color 1 and 2 between two pairs,  $(x_1, y_1)$  and  $(x_1, y_2)$  are in 1 the two others are in 2. In this case we represent the graph on  $2n$  vertices by a graph on  $n$  vertices (two vertices of the matching is contracted into one vertex) and we put an oriented edge of color 1 between  $x$  and  $y$  if  $(x_1, y_1)$  and  $(x_1, y_2)$  are in 1, we put an oriented edge of color 2 between  $x$  and  $y$  if  $(x_1, y_1)$  and  $(x_1, y_2)$  are in 2. Hence we have a complete oriented tournament of two colors 1 and 2.

If there is a solution just with the vertices of the left it means that we have a 1-clique and a 2-clique which can be linked by anything. But in our solution we can have vertices of the left part and of the right part. If we choose a vertex and decide to put the 0 color on the vertex at the right instead of the vertex at the left, it does not change the color of its in-neighbourhood but it change the colors of its out-neighbourhood. Indeed, if the arrow  $yx$  arrive on  $x$  then it means that both edges  $y_1x_1$  and  $y_1x_2$  have the same color and the same holds for the vertex  $y_2$ . Hence, change  $x_1$  into  $x_2$  has no impact on the color of the edges which arrive on  $x$  but it change the color of the edges which leaves  $x$ . Hence, we search a partition of the vertices into four 4 parts: two 1-cliques (1-3) and two 2-cliques (2-4). The edges from 1 to 2 and from 3 to 4 are of color 1 and the edges from 2 to 1 and 4 to 3 are of color 2.

In a tournament there is a dominating set of size at most  $\log(n)$ . Hence there is a set of size  $\log(n)$  such that there is an edge from any vertex to one of these vertices. Whatever the color of this edge, this edge forbids one of the four sets for each vertex. Then we just have to try all the  $4^{\log(n)}$  initial positions for this dominating set. And all the other vertices can be added in at most 3 of the 4 parts.

Let  $x$  be a vertex with a big out-neighbourhood of color 1 and 2 and which can be in  $12 - 3$ . Then if we add  $x$  to 1 the blue out neighbours of color blue cannot be in 1 or 2. The same holds for the neighbours of color red if we put  $x$  in 2. Hence in each case we loose an high number of vertex then we have a logarithmic depth. If we restrict to the the graph 12-3, if we take a vertex with a big incoming degree of one color and out degree of the other color, then we can delete one new part. Hence we can assume that the graph restricted to the vertices of  $12 - 3$  is nearly monochromatic. (for example, at most  $1/100$  of the edges are of color 2).

The max clique in 12-3 of one of the color is at most 3. In 12-3 there are the two 1-cliques and one 2-clique. If there is a 4-clique of color two, then there is a vertex of out-degree at least 2. This vertex cannot be in the 1-clique on the same side of the 2-clique. Indeed assume that this vertex is this 1-clique. Because all the edges from this 1-clique to its associate 2-clique are of color 1, hence the two neighbours are not in the 2-clique. Hence 2 of the 3 vertices are

on the same 1-clique which is impossible. Hence we just have to guess the 3 vertices of the particular color for  $1 - 2 - 3$ ,  $2 - 3 - 4$  and so on and try to glue the conclusion to conclude.  $\square$

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