

# 3-adapted-list coloring problem and Alon-Saks-Seymour conjecture

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## Abstract

In this paper we propose a study of the 3-adapted list coloring problem [11] by 2-list colorings. First we prove that this problem is equivalent to the problem of separation of the cliques and the stable sets. This problem is deeply linked with a problem of communication complexity introduced by Yannakakis [18]. Hence we show that these problems are also equivalent with a polynomial bound for an oriented Alon-Saks-Seymour conjecture.

## 1 Definitions

Let  $k$  be an integer. We denote by  $[k]$  the set of integers between 0 and  $k - 1$ . Let  $G = (V, E)$  be a graph. An edge  $uv \in E$  links its two *endpoints*  $u$  and  $v$ . The *neighbourhood*  $N(x)$  of  $x$  is the set of vertices  $y$  such that  $xy \in E$ . The *non-neighbourhood*  $N^C(x)$  of  $x$  is the complementary of the neighbourhood  $N(x)$ . For oriented graphs,  $N^+(x)$  (resp.  $N^-(x)$ ) denote the incoming (resp. outgoing) neighborhood of  $x$ , that is the set of vertices  $y$  such that  $yx \in E$  (resp.  $xy \in E$ ). The subgraph *induced by*  $X \subseteq V$  denoted by  $G[X]$  is the graph  $(X, E')$  where  $E' = E \cap (X \times X)$ . A *clique* of  $G$  is a complete induced subgraph of  $G$ . A *stable set* is an induced subgraph without edges. Observe that a clique and a stable set intersect on at most one vertex. A *vertex-coloring* (resp. *edge-coloring*) of  $G$  with  $k$  colors is a function  $f_V$  (resp.  $f_E$ ) from  $V$  (resp.  $E$ ) to  $[k]$ . For edge-colored graphs, an  $(\alpha_1 \dots \alpha_k)$ -*clique* is a clique where the edges are only colored by  $\alpha_1 \dots \alpha_k$ . A vertex-coloring  $f_V$  of  $G$  is  $f_E$ -*proper* if for each edge  $uv$  of  $E$ ,  $f_V(u) \neq f_E(uv)$  or  $f_V(v) \neq f_E(uv)$ . In other words, an endpoint of each edge does not have the color of the edge. The  $k$ -ADAPTED-LIST-COLORING PROBLEM was defined by Hell and Zhu in [11], and then studied in [3] and [13]. It is defined as follows:

$k$ -ADAPTED-LIST-COLORING PROBLEM ( $k$ -ALCP)

**Input:** An edge coloring  $f_E$  of the complete graph on  $n$  vertices with  $k$  colors.

**Question:** Is there an  $f_E$ -proper coloring?

This problem has applications to matrix partitions of graphs, full constraint satisfaction problems and trigraph homomorphisms [6], [7], [5], [10]. Hell and Zhu proved that the problem is NP-complete if  $k \geq 4$  and polynomial if  $k \leq 2$  [11]. The complexity when  $k = 3$  was solved by Cygan and al. [2] [Upgrader la ref.](#) We focus our attention on the case  $k = 3$ . The *majority color* of  $x \in V$  is the color which appears the most often for the edges with endpoint  $x$  (in case of ties, we arbitrarily cut them). We now introduce some definitions relative to list coloring. For list-coloring, each vertex has a set of possible colors.

The following definitions are illustrated on Figure 1 and are relative with list coloring. A set of possible colors, called *constraint*, is associated to each vertex. A vertex has an  $l$ -*constraint* if its set of possible colors has size  $l$ . An  $l$ -*list coloring* is a coloring where each vertex has an  $l$ -constraint. An  $l$ -list coloring is *compatible* with a solution  $f_V$  if for each vertex  $x$ ,  $f_V(x)$  is in the  $l$ -constraint of  $x$ . A set of  $l$ -list colorings is *compatible* with a solution if one of the  $l$ -list coloring is compatible with the solution. A partition problem is  $l$ -*list colored for a function*  $f$  if there exist  $f(n)$   $l$ -list colorings compatible with all the solutions. A partition problem is said *polynomially  $l$ -list colored* if there exists a polynomial  $Q$  such that the partition problem is  $l$ -list colored for  $Q$ . In other words, there exists a polynomial family  $\mathcal{F}$  of  $l$ -list colorings such that for any solution  $s$ , we can find in  $\mathcal{F}$  a  $l$ -list coloring which is compatible with  $s$ .

**Observation 1.** *Given a polynomial number of 2-list colorings, it is possible to decide in polynomial time if they are compatible with a solution.*

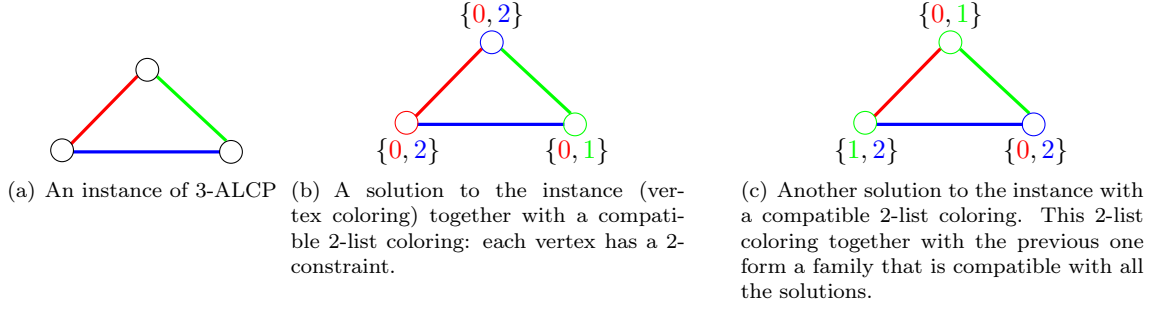


Figure 1: Illustration of definitions. Color correspondence: 0=red ; 1=green ; 2=blue.

Indeed, each 2-list coloring can be translated into an instance of 2-SAT.

**Theorem 1.** [4] *There exists an algorithm in time  $O(n^{\log n})$  which solves the 3-ALCP. This algorithm gives  $O(n^{\log n})$  2-list colorings which are compatible with all the solutions.*

*Proof.* We give the proof for self completeness. This algorithm consists in testing  $n^{\log(n)}$  2-list colorings which contains all the solutions. Consider the vertex  $x$ , without loss of generality we can assume that the majority color of  $x$  is 0. If there is a solution in which  $x$  is colored by 0 then all its 0-neighbours cannot have the color 0, hence all its 0-neighbours has 2-list constraints. And the number of remaining vertices with 3 possible colors is at most  $2n/3$  (recall that the graph is complete). It is possible to make the same for the remaining vertices. Hence consider the following algorithm, for each vertex  $x$  try to find a solution for which  $x$  is colored by its majority color. If there is no such solution for all  $x$ , then all the vertices have 2-list constraints. This algorithm is representable by a research-tree of degree  $n + 1$  (one child for the case where no vertex can be colored by its majority color, and one child for each vertex that can be first chosen to be colored by its majority color) and depth  $\log(n)$  (we remove  $1/3$  of the vertices at each node) in which each of the  $n^{\log(n)}$  leaves is a 2-list coloring of the instance.  $\square$

## 2 Link with the Cliques-Stables separation conjecture

A *cut* is a pair  $(A, B)$  such that  $A \cup B = V$  and  $A \cap B = \emptyset$ . A cut *separates* a clique  $C$  and a stable set  $S$  if  $C \subseteq A$  and  $S \subseteq B$ . Note that a clique and a stable set can be separated if and only if they do not intersect. Let  $\mathcal{C}_G$  be the set of cliques of  $G$  and  $\mathcal{S}_G$  be the set of stable sets of  $G$ . We say that a family  $\mathcal{F}$  of cuts *separates all the cliques and the stable sets* if for all  $(C, S) \in \mathcal{C}_G \times \mathcal{S}_G$  which do not intersect, there exists a cut in  $\mathcal{F}$  that separates  $C$  and  $S$ .

**Conjecture 1.** (*Clique-Stable Set separation Conjecture*) *There is a polynomial  $Q$ , such that for every graph  $G$  on  $n$  vertices, there exists a family of  $Q(n)$  cuts that separates all the cliques and the stable sets.*

**Proposition 1.** *To prove the Clique-Stable Set separation Conjecture, one only need a polynomial family that separates all the maximal (in sense of inclusion) cliques from the maximal stable sets that do not intersect.*

*Proof.* Assume  $\mathcal{F}$  is a polynomial family that separates all the maximal cliques from the maximal stable sets that do not intersect. Add to  $\mathcal{F}$  the following cuts, for all  $x \in V$ :

- Put  $x$  and  $N(x)$  on the left hand-side, and the remaining vertices on the right hand-side (call it  $Cut_{1,x}$ ).
- Put  $N(x)$  on the left and side, and  $x$  and the remaning vertices on the right hand side (call it  $Cut_{2,x}$ ).

Note that these additional cuts are in linear number. Now, let  $(C, S)$  be a pair of clique and stable set. Grow up  $C$  and  $S$  by adding vertices to get a maximal clique  $C'$  and a maximal stable set  $S'$ . Either  $C'$  and  $S'$  do not intersect, and in this case there is a cut in  $\mathcal{F}$  that separate  $C'$  from  $S'$ , thus  $C$  from  $S$ ; or  $C'$  and  $S'$  intersect in  $x$  (recall that a clique and a stable set intersect on at most one vertex): if  $x \in C$ , then  $Cut_{1,x}$  separates  $C$  from  $S$ , otherwise  $Cut_{2,x}$  does.  $\square$

Consequently, in the following, we will only focus on separating the maximal cliques from the maximal stable sets.

To motivate this polynomial separation let us prove the conjecture for some particular classes of graphs. For classes with a polynomial number of cliques or a polynomial number of stable sets, the conjecture is satisfied. Hence a class of graphs with bounded cliques or stable sets verifies the conjecture. Theorem 2 and 3 give more interesting classes of graphs satisfying the conjecture.

## 2.1 Random graphs

Let  $n$  be a positive integer and  $p \in [0, 1]$ . The random graph  $G(n, p)$  is a probability space over the set of graphs on the vertex set  $\{1, \dots, n\}$  determined by  $\Pr[\{i, j\} \in G] = p$ , with these events mutually independent. We say that  $G(n, p)$  has clique number  $\omega$  if  $\omega$  satisfies  $\mathbb{E}(\text{number of cliques of size } \omega) = 1$ . We define similarly the independent number of  $G(n, p)$ .

**Theorem 2.** *For random graphs, the clique-stable set conjecture is satisfied.*

The claim is detailed by the following propositions, in which  $\ln$  denotes the logarithm to the base  $e$ ,  $\log_b$  denotes the logarithm to the base  $b$ , and  $\log$  denotes the logarithm to the base 2.

**Proposition 2.** *Let  $a(n)$  (denoted by  $a$  in the following) be a function of  $n$  such that  $2 \leq a(n) \ll \log n$ . The random graph  $G(n, p)$  with  $p = 1 - n^{-2/a} = 1 - 2^{-2 \log n / a}$  has:*

- independent number  $\alpha = a - \frac{a \log a}{\log n} + 1 + o(1)$
- clique number  $\omega = 2 \ln 2 \cdot 2^{2 \log n / a} \cdot \left( \log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(2^{2 \log n / a}) + \mathcal{O}(\log n)$ .

Moreover, for all  $\varepsilon > 0$ , with high probability there exists a family of  $\mathcal{O}(n^{6+\varepsilon})$  cuts (more precisely  $\mathcal{O}(n^{5+2/a+\varepsilon})$  if  $a$  is a constant, and  $\mathcal{O}(n^{5+\varepsilon})$  if  $a(n) \rightarrow_{n \rightarrow \infty} \infty$ ) that separates all the clique and the stable sets.

**Observation 2.** *If  $a$  is a constant, then the previous formulas give:*

- $\alpha = a + 1 + o(1)$
- $\omega = 2 \ln 2 \cdot n^{2/a} \left( \frac{a-2}{a} \log n - \log \log n \right) + \mathcal{O}(\log n)$ .

*Proof.* Let  $p' = 1 - p$ ,  $b = 1/p$  and  $b' = 1/p'$ . The independence number and clique number of  $G(n, p)$  are given by the following formulas, depending on  $p$  (see [1]):

$$\begin{aligned} \omega &= 2 \log_b(n) - 2 \log_b(\log_b n) + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'}(\log_{b'} n) + 2 \log_{b'}(e/2) + 1 + o(1) \end{aligned}$$

Now  $\frac{1}{\log b'} = \frac{a}{2 \log n}$ , thus:

$$\begin{aligned} \alpha &= 2 \log_{b'}(n) - 2 \log_{b'}(\log_{b'} n) + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= a - \frac{a \log a}{\log n} + 1 + o(1) \end{aligned}$$

Moreover, using  $\log(1-x) = \frac{x + \mathcal{O}(x^2)}{\ln 2}$  then  $\frac{1}{1-x} = 1 + \mathcal{O}(x)$  we have:

$$\begin{aligned} \frac{1}{\log b} &= \frac{-\ln 2}{-2^{-2 \log n / a} + \mathcal{O}(2^{-4 \log n / a})} \\ &= 2^{2 \log n / a} \cdot \ln 2 \cdot (1 + \mathcal{O}(2^{-2 \log n / a})) \end{aligned}$$

Thus,

$$-\log \log b = \log \ln 2 + \frac{2 \log n}{a} + \mathcal{O}(2^{-2 \log n/a})$$

Consequently:

$$\begin{aligned} -2 \log_b \log_b n &= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) \cdot (\log \log n - \log \log b) \\ &= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log \log n + \log \ln 2 + \frac{2 \log n}{a}) + \mathcal{O}(\log n) \end{aligned}$$

Hence:

$$\begin{aligned} w &= 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n - \log \ln 2 + \log(e/2)) + \mathcal{O}(\log n) \\ &= 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n) \end{aligned}$$

Now, let  $x = 1 - p = 2^{-2 \log n/a}$ . Draw a random partition  $(V_1, V_2)$  where each vertex is put in  $V_1$  independently from the others with probability  $x$ . Let  $(C, S)$  be a pair of clique and stable set of the graph. There are at most  $4^n$  such pairs. The probability that  $S \subseteq V_1$  and  $C \subseteq V_2$  is  $x^\alpha(1-x)^\omega$ . Let us show that  $x^\alpha \geq n^{-(2+2/a+\varepsilon')}$  and  $(1-x)^\omega \geq n^{-(2+\varepsilon')}$  for all  $\varepsilon' > 0$ .

On one hand,

$$\begin{aligned} x^\alpha \geq n^{-(2+2/a+\varepsilon')} &\Leftrightarrow \alpha \log x \geq -(2+2/a+\varepsilon') \log n \\ &\Leftrightarrow 2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2+2/a+\varepsilon') \log n \end{aligned}$$

which is true if  $n$  is large enough.

Note that for  $a \rightarrow_{n \rightarrow \infty} \infty$ , we even have  $2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2+\varepsilon') \log n$  for  $n$  large enough, which implies  $x^\alpha \geq n^{-(2+\varepsilon')}$ .

On the other hand,  $\log(1-x) = \log b$ , thus, using the previous approximations:

$$\begin{aligned} (1-x)^\omega \geq n^{-(2+\varepsilon')} &\Leftrightarrow \omega \log(1-x) \geq -(2+\varepsilon') \log n \\ &\Leftrightarrow \left( 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n) \right) \\ &\quad \cdot \left( \frac{-2^{-2 \log n/a}}{\ln 2} + \mathcal{O}(2^{-4 \log n/a}) \right) \geq -(2+\varepsilon') \log n \\ &\Leftrightarrow 2(\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(1) + \mathcal{O}(2^{-2 \log n/a} \log n) \leq (2+\varepsilon') \log n \end{aligned}$$

which is true if  $n$  is large enough.

This leads us to  $x^\alpha(1-x)^\omega \geq n^{-(5+2\varepsilon')} = n^{-(5+\varepsilon)}$  for  $\varepsilon' = \varepsilon/2$ . Thus  $(C, S)$  is separated by at least  $n^{-(5+\varepsilon)}$  of all the partition. By double counting, there exists a partition that separates at least  $n^{-(5+\varepsilon)}$  of all the pairs. We delete these separated pairs, and it remains at most  $(1 - 1/n^{5+\varepsilon}) \cdot 4^n$  pairs. The same probability for a pair  $(C, S)$  to be cut by a random partition still holds, hence we can iterate the process  $k$  times until  $(1 - 1/n^{5+\varepsilon})^k \cdot 4^n \leq 1$ . This is satisfied for  $k = 2n^{5+\varepsilon}$  which is a polynomial in  $n$ . If  $a \rightarrow_{n \rightarrow \infty} \infty$ , we can do the same and get  $k = 2n^{5+\varepsilon}$ . There is a polynomial family of cuts that separates all the cliques and the stable sets.  $\square$

However, this does not cover the classic configuration where  $p = 1/2$  and  $\alpha \approx 2 \log n$ . Thus we need the following:

**Proposition 3.** *Let  $a(n) = 2c \log n$  (denoted by  $a$  in the following) where  $0 \leq c \leq 1$  is a constant. The random graph  $G(n, p)$  with  $p = 1 - 2^{-c}$  has:*

- independent number  $\alpha = 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)$
- clique number  $\omega = 2c' \log n - 2c' \log \log n + o(\log \log n)$  where  $1/c' = \log(1 - 2^{-1/c})$ .

Moreover, for all  $\varepsilon > 0$ , with high probability there exists a family of  $\mathcal{O}(n^5)$  cuts that separates all the clique and the stable sets.

*Proof.* Let  $p' = 1 - p$ ,  $b = 1/p$  and  $b' = 1/p'$ . We proceed the same way as in the previous proposition. The independence number and clique number of  $G(n, p)$  are given by the following formulas, depending on  $p$  (see [1]):

$$\begin{aligned}\omega &= 2 \log_b(n) - 2 \log_b \log_b n + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1)\end{aligned}$$

Now  $\frac{1}{\log b'} = c$ , thus:

$$\begin{aligned}\alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)\end{aligned}$$

Moreover,  $\log b = \log(1 - 2^{-1/c}) = 1/c'$ , thus, similarly to  $\alpha$ , we get:

$$w = 2c' \log(n) - 2c' \log \log n + 2c'(\log(e/2) - \log c') + 1 + o(1)$$

The same reasoning as for Proposition 2 with the same  $x = 1 - p'$  gives a probability  $x^\alpha(1-x)^\omega \geq n^{-4}$  for a given pair  $(C, S)$  to be separated by a random non-uniform cut  $(V_1, V_2)$ , so there is a family of  $\mathcal{O}(n^5)$  cuts that separate all the clique and the stable sets (see the proof of Proposition 2 for details).  $\square$

## 2.2 Split-free graphs

A geometric generalization of the clique-stable set conjecture was studied by Lovász in [15]. The question is to know if the stable set polytope of a graph is the projection of a polytope in higher dimension, with a polynomial number of facets. **On sait maintenant que c'est faux, cf Fiorini et al.!** He claims that if the answer to this question is positive, then there is a polynomial number of cuts that separates all the cliques and the stable sets. Moreover, he proved it for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and  $t$ -perfect graphs which are a generalization of series-parallel graphs. Let us show the result for split-free graphs, which contain comparability graphs.

A graph  $G = (V, E)$  is *split* if  $V = V_1 \cup V_2$  and the subgraph induced by  $V_1$  is a clique and the subgraph induced by  $V_2$  is a stable set. Let  $H$  be a split graph. We denote by  $\mathcal{C}_H$  the class of graph with no induced  $H$ . Note, for example, that the graph with  $|V_1| = 1$ ,  $|V_2| = 3$  and all possible edges between  $V_1$  and  $V_2$  is a claw, hence  $\mathcal{C}_H$  is the class of claw-free graphs.

**Theorem 3.** *Let  $H$  be a fixed split graph. Then the clique-stable set conjecture is verified on  $\mathcal{C}_H$ .*

Before going to the proof, we need a few additional definitions regarding transversality and Vapnik-Chervonenkis dimension. Let  $G = (V, E)$  be a hypergraph.

**Definition 1.** *The transversality  $\tau$  is the minimum cardinality of a set of vertices intersecting each hyperedge.*

**Definition 2.** *The fractionnal transversality  $\tau^*$  is a relaxation of the transversality : it is a solution to the following linear programming:  $\min \sum_{x \in V} w(x)$  subject to*

- for all  $x \in V$ ,  $0 \leq w(x) \leq 1$ .
- for all  $e \in E$ ,  $\sum_{x \in e} w(x) \geq 1$ .

**Observation 3.** *The constraint  $0 \leq w(x) \leq 1$  for all  $x \in V$  comes naturally from the relaxation of  $\tau$  where each vertex takes value in  $\{0, 1\}$ . However, removing the constraint  $w(x) \leq 1$  for all  $x \in V$  does not change the solution as we search for a minimum: if  $w(y) > 1$  for some  $y \in V$  in a feasible solution  $w$ , getting  $w(y)$  down to 1 will not violate any constraint, and will reduce strictly the objective function.*

**Definition 3.** The Vapnik-Chervonenkis dimension or VC-dimension of a hypergraph  $H = (V, E)$  is the maximum cardinality of a set of vertices  $A \subset V$  such that for every  $B \subset A$  there is an edge  $e \in E$  so that  $e \cap A = B$ .

Let us introduce some dual measure:

**Definition 4.** A set  $T$  of hyperedges forms a complete Venn diagram if for all  $T' \subseteq T$ , there exists a vertex  $v$  such that for all  $e \in T'$ ,  $v \in e$  and for all  $e \notin T'$ ,  $v \notin e$ . In other words, for all subset  $T'$  of hyperedges, there exists a vertex in the intersection of all hyperedges in  $T'$ , and in no other hyperedge.

**Definition 5.** The dual Vapnik-Chervonenkis dimension (dual VC-dimension for short) of  $G$  is the maximum size of a complete Venn diagram in  $G$ .

Since we will use only hypergraphs extracted from neighbourhood of vertices in a given graph, we will work more easily with the dual VC-dimension. In this context, the VC-dimension and the dual VC-dimension of a hypergraph are the same, coming from  $x$  being in  $N(y)$  if and only if  $y \in N(x)$ , for all vertices  $x$  and  $y$ . Consequently, we will abuse notation in the following and say VC-dimension instead of dual VC-dimension.

*Proof of Theorem 3.* Let  $t$  be a constant depending only on  $H$ , to be defined later. Let  $G = (V, E) \in \mathcal{C}_H$  be a  $H$ -free graph, and let  $\mathcal{F}$  be the following family of cuts: for all subsets  $\{x_1, \dots, x_r\}$  of at most  $t$  vertices which is a clique (resp. a stable set), take  $U = \cap_{1 \leq i \leq r} N(x_i)$  (resp.  $U = \cup_{1 \leq i \leq r} N(x_i)$ ). Add  $(U, V \setminus U)$  to  $\mathcal{F}$ . Then  $\mathcal{F}$  has size  $\mathcal{O}(n^t)$ , which is polynomial in  $n$ . Let  $(C, S)$  be a pair of maximal clique and stable set. Let us show that  $(C, S)$  is separated by  $\mathcal{F}$ .

Let us build an oriented graph  $B$  with vertices  $C \cup S$  as shown on Figure 2(b). For all  $x \in C$  and  $y \in S$ , put the oriented edge  $xy$  if  $xy \in E$ , and put the oriented edge  $yx$  otherwise. From the hyperplane separation theorem applied to the cone of positive vectors on one side, and the convex hull of the columns of the adjacency matrix of  $B$  on the other side, we can derive the following lemma:

**Lemma 1.** For all oriented graph  $(V, E)$ , there exists a weight function  $w : V \rightarrow [0, 1]$  such that  $w(V) = 1$  and for all vertex  $x$ ,  $w(N^+(x)) \geq w(N^-(x))$ .

**Corollary 1.** In  $B$ , there exists either:

(i) a weight function  $w : C \rightarrow \mathbb{R}^+$  such that  $w(C) = 2$  and for all vertex  $x \in S$ ,  $w(N^+(x)) \geq 1$ .

or (ii) a weight function  $w : S \rightarrow \mathbb{R}^+$  such that  $w(S) = 2$  and for all vertex  $x \in C$ ,  $w(N^+(x)) \geq 1$ .

*Proof.* Take  $w : V \rightarrow [0, 1]$  as in Lemma 1. Then  $w(V) = 1$  so either  $w(C) > 0$  or  $w(S) > 0$ . Assume  $w(C) > 0$  (the other case is handled symmetrically). Take a new weight function defined by  $w'(x) = 2w(x)/w(C)$  if  $x \in C$ , and 0 otherwise. Then for all  $x \in S$ , on one hand  $w'(N^+(x)) \geq w'(N^-(x))$  by extension of the property of  $w$ , and on the other hand,  $N^+(x) \cup N^-(x) = C$  by construction of  $B$ . Thus  $w'(N^+(x)) \geq w'(C)/2 = 1$  since  $w'(C) = 2$ .  $\square$

In the following, let assume we are in case (i). Case (ii) is handled symmetrically by switching  $C$  (resp. neighbourhood) and  $S$  (resp. non-neighbourhood).

Let us now build  $B'$  an hypergraph with vertices  $C$  and for all  $x \in S$ , build the hyperedge  $C \setminus N_G(x)$ , that is the complementary in  $C$  of the neighbours of  $x$  in the original graph  $G$ . This construction is illustrated on Figure 2(c).

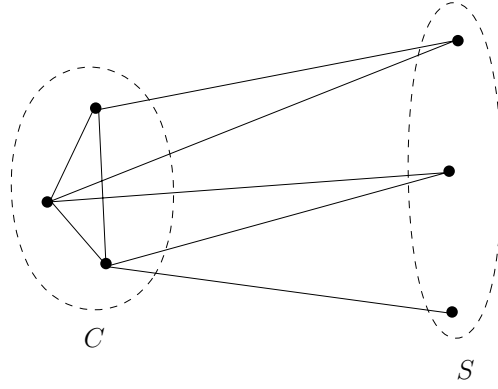
**Lemma 2.** The hypergraph  $B'$  has fractionnal transversality  $\tau^* \leq 2$ .

*Proof.* Let  $w$  be the weight function given by Corollary 1. Let  $h$  be a hyperedge built from the non-neighbourhood of  $x \in S$ . Recall that this non-neighbourhood was precisely  $N^+(x)$  in  $B$ , then we have:

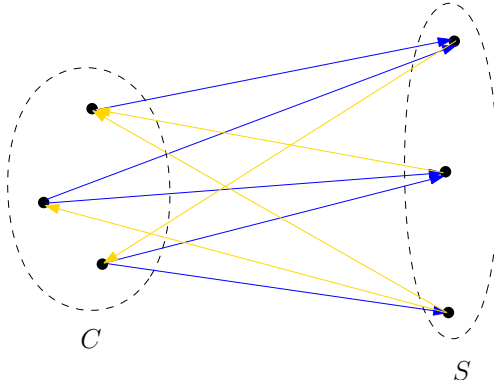
$$\sum_{y \in h} w(y) = w(N^+(x)) \geq 1.$$

Thus  $w$  satisfies the constraints of the extended transversality, and  $w(V) = w(C) \leq 2$ .  $\square$

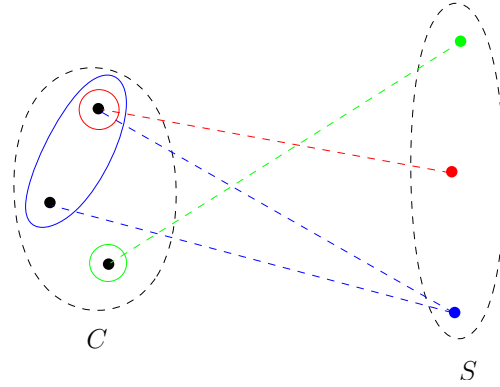
Recall that the vertices of  $H$  are partitionned into  $(V_1, V_2)$  where  $V_1$  is a clique and  $V_2$  is a stable set. Let  $\varphi = \max(|V_1|, |V_2|)$ .



(a) A clique  $C$  and a stable  $S$  in  $G$ .



(b) Graph  $B$  build from  $C$  and  $S$ . Edges in  $G$  are replaced by forward edges (blue edges on the drawing), and non-edges are replaced by backward edges (yellow edges on the drawing).



(c) Hypergraph  $B'$  where hyperedges are built from the non-neighbourhood of vertices from  $S$ .

Figure 2: Illustration of proof of Theorem 3.

**Lemma 3.**  $B'$  has VC-dimension bounded by  $2\varphi - 1$ .

*Proof.* Suppose we have a complete Venn diagram  $\mathcal{D}$  of size  $2\varphi$  in  $B'$ . The aim is to exploit the shuttering to find an induced  $H$ , which builds a contradiction. Let  $s_1, \dots, s_\varphi, t_1, \dots, t_\varphi$  be the hyperedges composing  $\mathcal{D}$ . In the following, we will abuse notation by calling  $s_i$  (resp.  $t_i$ ) both the hyperedge and the vertex of  $S$  whose non-neighborhood is precisely the hyperedge  $s_i$  (resp.  $t_i$ ). We denote by  $y_1, \dots, y_{\varphi'}$  the vertices of  $V_2$  (with  $\varphi' \leq \varphi$ ). Let  $x \in V_1$  and let  $\{y_{i_1}, \dots, y_{i_k}\}$  be the set of its neighbors in  $V_2$ . Since  $\mathcal{D}$  is a complete Venn diagram in  $B'$ , there exists  $x' \in C$  such that  $x'$  is in the intersection of the hyperedges  $s_{i_1}, \dots, s_{i_k}$ , and in no other hyperedge. This means that, in  $G[C \cup \{s_1, \dots, s_\varphi\}]$ ,  $x'$  has exactly the same neighborhood in  $S$  as  $x$  in  $V_2$ . If there is no two vertices  $x_1, x_2 \in V_1$  with the same neighborhood in  $V_2$ , the above statement iterated on the other vertices of  $V_1$  proves that there is an induced  $H$  in  $G[C \cup S]$ , thus in  $G$ , which is absurd.

Now let us handle the case where several vertices in  $H$  have the same neighbourhood. We will artificially use the vertices  $t_1, \dots, t_\varphi$  to ensure that we can find in our setting as many vertices as needed with this particular neighbourhood, to find an induced  $H$ . Assume  $x_1, \dots, x_r \in V_1$  have the same neighborhood  $\{y_{i_1}, \dots, y_{i_k}\}$  in  $V_2$ . Let  $\delta_j = \{s_{i_1}, \dots, s_{i_k}\} \cup \{t_j\}$  for all  $1 \leq j \leq r$ . Then there exists  $x'_j \in C$  such that  $x'_j$  is exactly in the intersection of all hyperedges in  $\delta_j$ , and in no other hyperedge. Thus, in  $G[C \cup \{s_1, \dots, s_\varphi\}]$ ,  $x'_1, \dots, x'_r$  have the same neighborhood  $\{s_{i_1}, \dots, s_{i_k}\}$ , just as  $x_1, \dots, x_r$  in  $V_1$ . In the same way as before, we find an induced  $H$  in  $G[C \cup S]$ , which is absurd.  $\square$

**Observation 4.** In fact, the VC-dimension of  $B'$  is bounded by  $\varphi + \log \varphi$ . Indeed, we need only  $t_1, \dots, t_{\log \varphi}$  in addition to  $s_1, \dots, s_\varphi$ : if  $x_1, \dots, x_r \in V_1$  have the same neighborhood in  $V_2$  with  $r \leq \varphi$ , we need  $r$  different subsets of  $\{t_1, \dots, t_\varphi\}$ , which is already possible with  $t_1, \dots, t_{\log \varphi}$ .

The following bound [9] links the transversality with VC-dimension and extended transversality.

**Lemma 4.** Every hypergraph  $H$  with VC-dimension  $d$  satisfies

$$\tau(H) \leq 16d\tau^*(H) \log(d\tau^*(H)).$$

Applying this lemma to  $B'$ , we obtain

$$\tau(B') \leq 16d\tau^*(B') \log(d\tau^*(B')) \leq 32\varphi(\log(\varphi) + 1).$$

Let us now define  $t = 32\varphi(\log(\varphi) + 1)$  (recall that  $t$  was part of the construction of the family  $\mathcal{F}$  of cuts in the beginning of the proof). As a consequence,  $\tau$  is bounded by  $t$  which depends only on  $H$ . There must be  $x_1, \dots, x_\tau \in C$  such that each hyperedge of  $B'$  contains at least one  $x_i$ . Thus for all  $y \in S$ , there is  $i$  such that  $x_i \in N^C(y)$ . Consequently,  $S \subseteq \cup_{1 \leq i \leq t} N_G^C(x_i)$ . Moreover,  $C \subseteq U = \cap_{1 \leq i \leq t} N_G(x_i)$  as  $x_1, \dots, x_\tau$  are members of the same clique  $C$ . This means that the cut  $(U, V \setminus U) \in \mathcal{F}$  built from the subset  $x_1, \dots, x_\tau$  separates  $C$  and  $S$ .

Let us mention that if case (ii) of Corollary 1 were actually occurring, we would have extracted  $\tau$  vertices  $x_1, \dots, x_\tau \in S$  such that  $S \subseteq \cup_{1 \leq i \leq t} N_G^C(x_i)$  and  $C \subseteq U = \cup_{1 \leq i \leq t} N_G(x_i)$ . The cut  $(U, V \setminus U) \in \mathcal{F}$  built from the subset  $x_1, \dots, x_\tau$  separates  $C$  and  $S$ .  $\square$

## 2.3 Equivalence with the polynomial 2-list coloring of 3-ALCP

The following of this section is devoted to the equivalence between the Clique-Stables separation conjecture and the polynomial 2-list coloring of the 3-ALCP.

Recall that a graph  $G = (V, E)$  is *split* if  $V = V_1 \cup V_2$  and the subgraph induced by  $V_1$  is a clique and the subgraph induced by  $V_2$  is a stable set. A graph with edges colored by  $a$  and  $b$  is *split* if there is a partition of the vertices into a  $a$ -clique and a  $b$ -clique.

**Proposition 4.** Let  $x$  be a vertex. Let  $A$  be the 0-neighbourhood of  $x$ . Let  $U$  be a 12-clique of  $A$ . If  $U$  is not split then there is no solution where  $x$  is colored by 0.



*Proof.* Assume that  $x$  is colored by 0. Then all the vertices of  $U$  are of color 1 or 2 because they are in the 0-neighbourhood of  $x$ . The vertices colored by 1 form a 2-clique and all the vertices colored by 2 form a 1-clique and the union is  $U$ . Hence  $U$  is split.  $\square$

Observe that the same property holds for the 1 and the 2-neighbourhoods. A vertex  $x$  is *really 3-colorable* if for each color  $a$ , each  $bc$ -clique of the  $a$ -neighbourhood of  $x$  is a split graph. If a vertex is not really 3-colorable then, in a solution, it can be colored by at most 2 different colors. Hence, given a 2-list coloring of the graph without  $x$ , there is a natural 2-list coloring of the whole graph by adding the 2-constraint on  $x$ . Hence it is possible to assume that the graph has only really 3-colorable vertices.  $\square$

Let us state the main result of this section. For this, we define two functions and we prove that these two functions are polynomially equivalent. We denote by  $|G|$  the number of vertices of the graph.

$$f_1(n) = \max_{|G|=n} \min_{\mathcal{F}} (|\mathcal{F}| : \mathcal{F} \text{ set of 2-list colorings compatible with all the solutions})$$

$$f_2(n) = \max_{|G|=n} \min_{\mathcal{F}} (|\mathcal{F}| : \mathcal{F} \text{ family of cuts which separates all the cliques and the stable sets})$$

**Theorem 4.** *The functions  $f_1$  and  $f_2$  are polynomially equivalent.*

**Corollary 2.** *The Clique-Stable set separation conjecture is verified iff there exists a polynomial number of 2-list coloring compatible with all the solutions of an instance of 3-ALCP.*

We start the proof of Theorem 4 by the following lemma:

**Lemma 5.**  $f_2(n) \leq n f_1(n)$ .

*Proof.* Let  $G = (V, E)$  be a graph with  $n$  vertices. Consider the graph  $F$  obtained from  $G$  by coloring its edges by 0, and replacing all missing edges by 1-colored edges (formally,  $H$  is an edge-colored graph with the same vertices as  $G$  where  $xy$  is colored by 0 if  $xy \in E$  and colored by 1 otherwise.) By assumption, we have a set  $\mathcal{F}$  of  $f_1(n)$  2-list coloring that is compatible with all solutions of 3-ALCP. The idea is the following: from a pair  $(C, S)$  of clique and stable set of  $G$  which do not intersect, we find a natural solution of 3-ALCP on  $H$ . Hence we have a 2-list coloring in  $\mathcal{F}$  which is compatible with this particular solution. We build from this about  $n$  cuts that depend only on the previous 2-list coloring, and among which there is for sure a cut that separates  $C$  and  $S$ . Let  $C$  be a clique and  $S$  a stable set of  $G$  which do not intersect. Observe that the edges of  $C$  are colored by 0, and those of  $S$  are colored by 1. Then the vertex-coloring 1 for the vertices of  $C$ , 0 for those of  $S$  and 2 for the other vertices is a solution of the 3-ALCP. Left-hand side of Figure 2.3 illustrates the situation.

There is a 2-list coloring  $(X, Y, Z)$  in  $\mathcal{F}$  which is compatible with this solution. The set  $X$  (resp.  $Y, Z$ ) is the set of vertices which have the constraint  $\{0, 1\}$  (resp.  $\{1, 2\}, \{0, 2\}$ ). The vertices of  $C$  cannot be in  $Z$ , and those of  $S$  cannot be in  $Y$  (see right hand-side of Figure 2.3). Then  $C \subseteq X \cup Y$  and  $S \subseteq X \cup Z$ . Since all the edges are of color 0 and 1, and  $X$  can be colored properly, the graph restricted to  $X$  is split (put all the 0-colored vertices on one side, and the 1-colored vertices on the other side to get a 1-clique and a 0-clique). Given a graph, there is a linear number of decompositions into a split graph [6]. Then the number of possible decompositions of  $X$  is linear. Let  $(U_k, V_k)_k$  be these decompositions. The pair  $(C, S)$  is separated by one of the  $(U_k \cup Y, V_k \cup Z)$ . Then  $f_2(n)$  is bounded by a polynomial function of  $f_1(n)$ .  $\square$

Before the next lemma, we need the following observation about the union of two cliques:

**Observation 5.** *If there are  $p$  cuts that separate all the clique from the stable sets, then there are  $p^2$  cuts that separates all the  $C \cup C'$  (where  $C$  and  $C'$  are cliques) from the stable sets. Indeed, if  $(V_1, V_2)$  separates  $C$  from  $S$  and  $(V'_1, V'_2)$  separates  $C'$  from  $S$ , then the new cut  $(V_1 \cup V'_1, V_2 \cap V'_2)$  satisfies  $C \cup C' \subseteq V_1 \cup V'_1$  and  $S \subseteq V_2 \cap V'_2$ .*

**Lemma 6.**  $f_1 \leq \mathcal{O}(f_2^6(n))$ .

*Proof.* Let  $G$  be an instance of 3-ALCP, and  $x$  a vertex of  $G$ . Let us build a set of 2-list colorings of size polynomial in  $f_2$  that is compatible with all the solutions where  $x$  is given color 0. The same can be done for the other colors. As seen before, we can assume that  $x$  is really 3-colorable, otherwise there is a natural 2-constraint on it. First of all, give the constraint  $\{1, 2\}$  to the 0-neighbourhood of  $x$ . Let

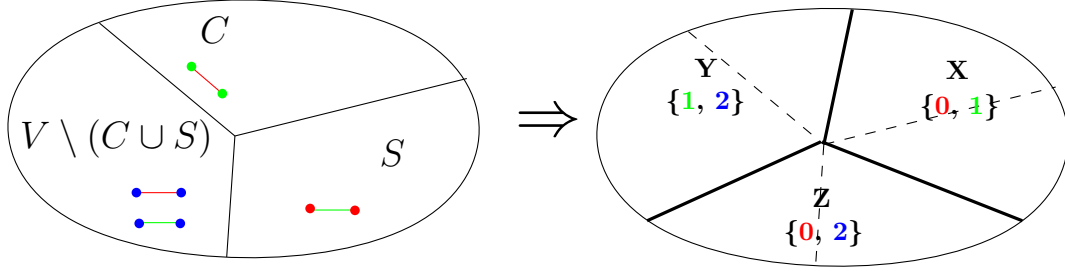


Figure 3: Illustration of proof of lemma 5. On the left hand-side,  $G$  is separated in 3 parts:  $C$ ,  $S$ , and the remaining vertices. Each possible configuration of edge- and vertex-colouring are represented. On the right-hand-side,  $(X, Y, Z)$  is a 2-list coloring compatible with the solution.  $X$  (resp.  $Y, Z$ ) has constraint  $\{0, 1\}$  (resp.  $\{1, 2\}, \{0, 2\}$ ). Color correspondence: 0=red ; 1=green ; 2=blue.

$U$  be the graph obtained from the 1-neighbourhood of  $x$  by removing 0-colored edges. There is a family of  $f_2^2(|U|)$  cuts that separates all the union of two cliques and the stable sets in  $U$  (see Obs 5). For all cut  $(V_1, V_2)$  in this family, give to  $V_2$  the constraint  $\{1, 2\}$ . We iterate this process to get the constraints on  $V_1$ : let  $V$  be the graph obtained from  $V_1$  by keeping only the 2-colored edges. There is a family of  $f_2(|V|)$  cuts that separates all the clique and the stable sets in  $V$ . For all cut  $(V_3, V_4)$ , give to  $V_3$  the constraint  $\{0, 1\}$  and to  $V_4$  the constraint  $\{0, 2\}$ . This gives us  $f_2^2(|U|)f_2(|V|)$  2-list colorings on the 1-neighbourhood of  $x$ . Do the same for the 2-neighbourhood and get  $\mathcal{O}(f_2^6(n))$  2-list colorings.

Take a solution of the 3-ALCP on  $G$  and assume  $x$  is colored by 0. Let us show that one of the 2-list colorings we built is compatible with this solution. Figure 2.3 illustrates the construction. First of all, the 0-neighbourhood can not be colored by 0, because  $x$  already is, so they are colored 1 or 2. Consider the 1-neighbourhood  $N_1(x)$  of  $x$ : the 1-colored vertices in  $N_1(x)$  form a 02-clique, thus it is a split graph (recall that  $x$  is really 3-colorable). Then it is the union of a 0-clique  $A$  and a 2-clique  $B$ . Call  $C$  (resp.  $D$ ) the 12-clique (resp. 01-clique) formed by the 0-colored (resp. 2-colored) vertices of  $N_1(x)$ .  $(A, B, C, D)$  is thus a partition of  $N_1(x)$ . Remove the 0-colored edges to obtain the graph  $U$ . Then  $A$  is a stable set,  $B$  and  $C$  are cliques. Then there exists a cut  $(V_1, V_2)$  with  $B \cup C \subseteq V_1$  and  $A \subseteq V_2$ .  $V_2$  is compatible with the constraint  $\{1, 2\}$  because it is formed by  $A$  and a part  $D'$  of  $D$ . Let  $V$  be the graph obtained from  $V_1$  by keeping only the 2-colored edges. Then  $B$  is a clique and  $D'' = D \setminus D'$  is a stable set, so there is a cut  $(V_3, V_4)$  that separates  $B$  from  $D''$ .  $V_3$  is compatible with the constraint  $\{0, 1\}$  because it is the union of  $B$  and a part of  $C$ .  $V_4$  is compatible with the constraint  $\{0, 2\}$  because it is the union of  $D''$  and a part of  $C$ . We can check the same for the 2-neighbourhood of  $x$ .  $\square$

*Proof of theorem 4.* Lemma 5 and Lemma 6 conclude the proof of the theorem.  $\square$

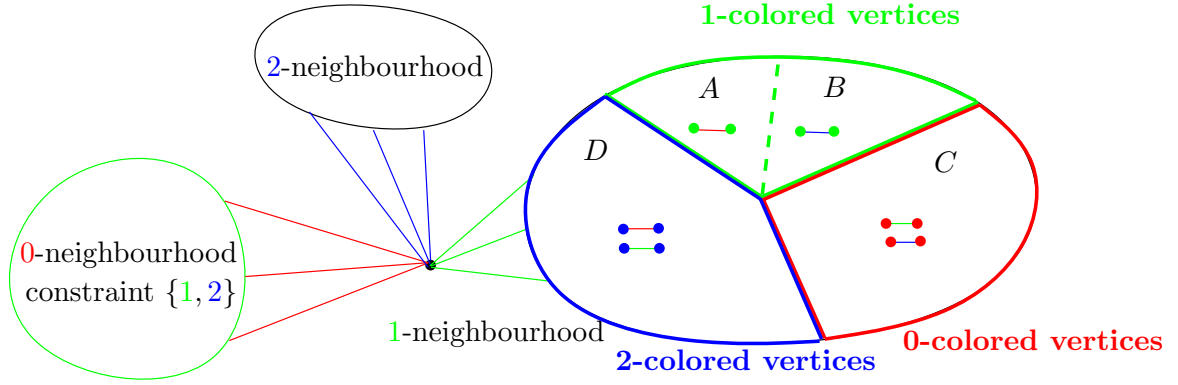
### 3 Link with the oriented Alon-Saks-Seymour conjecture

Given a graph  $G$ , the *chromatic number*  $\chi$  of  $G$  is the minimum number of colors such that, two vertices connected by an edge do not have the same color. The *bipartite packing*  $bp(G)$  of a graph  $G$  is the minimum number of edge disjoint complete bipartite graphs needed to cover the edges of  $G$ . The Alon-Saks-Seymour conjecture states the following.

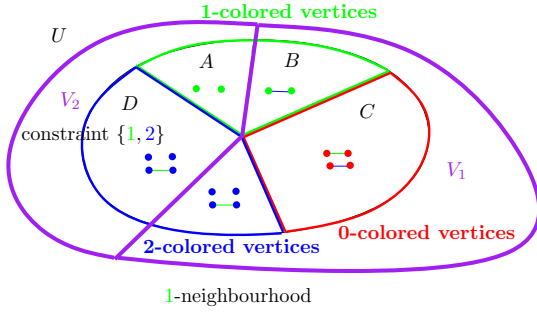
**Conjecture 2.** (*Alon, Saks, Seymour*) If  $bp(G) \leq k$ , then  $\chi \leq k + 1$

The result is a well-known result for complete graphs, due to Graham and Pollak [8]. Indeed,  $n - 1$  edge-disjoint complete bipartite graphs are needed to cover the edges of a complete graph on  $n$  vertices. A beautiful proof of this theorem is due to Tverberg [17]. This conjecture was disproved by Huang and Sudakov in [12] who prove that for some graphs  $\chi \geq k^{6/5}$  using a construction based on Razborov's graphs [16]. Nevertheless the existence of a polynomial bound is still open. In the following we will use an oriented version of the Alon-Saks-Seymour conjecture.

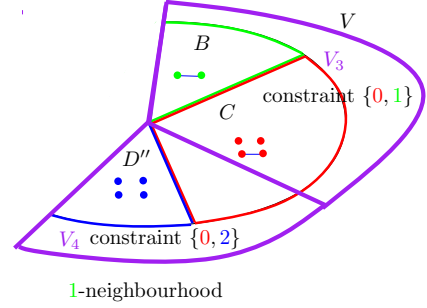
**Conjecture 3.** (*Oriented Alon-Saks-Seymour Conjecture*) There exists a polynomial  $P$  such that if  $G$  is a disjoint union of  $k$  complete oriented bipartite graphs, then the chromatic number  $\chi$  is at most  $P(k)$ .



(a) Vertex  $x$ , its 0-neighbourhood subject to the constraint  $\{1, 2\}$ , and its 1-neighbourhood separated in different parts.



(b) Graph  $U$  built from the 1-neighbourhood by removing 0-edges. There is a partition in  $(V_1, V_2)$  separating  $A$  from  $B \cup C$ , shown by the purple line.  $V_2$  is subject to the 2-constraint  $\{1, 2\}$ .



(c) Graph  $V$  built from  $V_1$  by removing 1-edges. There is a partition in  $(V_3, V_4)$  separating  $B$  from  $D''$ , shown by the purple line.  $V_3$  is subject to the 2-constraint  $\{0, 1\}$  and  $V_4$  is subject to the 2-constraint  $\{0, 2\}$ .

Figure 4: Illustration of proof of lemma 6. Color correspondence: 0=red ; 1=green ; 2=blue.

**Theorem 5.** *The oriented Alon-Saks-Seymour conjecture is verified iff the Clique-Stable set separation conjecture is verified.*

**Lemma 7.** *If the oriented Alon-Saks-Seymour conjecture is verified, then the Clique-Stable set separation conjecture is verified.*

*Proof.* Let  $G$  be a graph on  $n$  vertices. We want to separate all pairs of cliques and stable sets which do not intersect. Consider all the pairs  $(C, S)$  such that the clique  $C$  does not intersect the stable set  $S$ . Construct an auxiliary graph  $H$  as follows. The vertices of  $H$  are the pairs  $(C, S)$  and there is an oriented edge between a pair  $(C, S)$  and a pair  $(C', S')$  iff there is a vertex  $x \in C \cap S'$ . Observe that the number of vertices of this graph is at most  $4^n$ . The bipartite packing of this graph is at most the number  $n$  of vertices of  $G$ . Indeed, let  $H_x$  be the graph  $H$  restricted to the edges for which  $x \in C \cap S'$ . Observe that the union of the graphs  $H_x$  for all  $x$  is the graph  $H$  because if  $(C, S)(C', S')$  is an edge  $C \cap S' \neq \emptyset$ . In addition, the graph  $H_x$  is a complete bipartite graph. Indeed, if there is an edge which starts in  $(C, S)$  and if there is an edge which ends in  $(C', S')$  then  $x \in C$  and  $x \in S'$  and finally there is an oriented edge  $(C, S)(C', S')$ . The graphs  $H_x$  and  $H_y$  cannot share an oriented edge because otherwise the intersection between a clique and a stable set would be at least 2 which is impossible. Hence the bipartite packing of this graph is at most  $n$ .

If the oriented Alon-Saks-Seymour conjecture is verified,  $\chi(H) < P(n)$ . Consider a color of this polynomial coloring. Let  $A$  be the set of vertices of this color. There is no edges between two vertices of  $A$ , then the union of all the first components (cliques) of the vertices of  $A$  do not intersect the union of all the second components (stable sets) of  $A$ . Indeed, if they intersect, there is a clique  $C$  which intersects a stable set  $S$ , hence there is an edge which is impossible.

The union of the cliques and the union of the stable sets do not intersect, hence it defines a cut which separates all the pairs of  $A$ . The same can be done for all the colors. Then we can separate all the pairs  $(C, S)$  by  $\chi(H)$  cuts, which is a polynomial in  $n$  if the Alon-Saks-Seymour conjecture is verified. This achieves the proof of the first implication.  $\square$

**Lemma 8.** *If the Clique-Stable set separation conjecture is verified, then the oriented Alon-Saks-Seymour conjecture is verified.*

*Proof.* Let  $G = (V, E)$  be an oriented graph with bipartite packing  $k$ . Construct an auxiliary graph  $H$  as follows. It has  $k$  vertices which are the oriented complete bipartite graphs that cover the edges of  $G$ . There is an edge between two pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  iff there is a vertex  $x \in A_1 \cap A_2$ . Hence the complete bipartite graphs in which  $x$  appears at the left form a clique of  $H$  (say the clique  $C_x$  associated to  $x$ ) and the complete bipartite graphs for which  $y$  appears at the right form a stable set in  $H$  (say the stable set  $S_y$  associated to  $y$ ). Indeed, it is quite clear for the clique, and it is also true for the stable set because if  $y \in B_1 \cap B_2$  and  $x \in A_1 \cap A_2$  there is an edge covered twice which is impossible, then it is a stable set. Note that a clique or a stable set associated to a vertex can be empty, but this does not trigger any problem.

For each vertex  $x$ , consider a cut  $Cut_x$  that separates  $C_x$  and  $S_x$ . By assumption there is a polynomial number in  $k$  of cuts which separates all the pairs  $(C, S)$ , hence which separate this particular pairs of cliques and stable sets. We color each vertex  $x$  by the color  $Cut_x$ . This coloring is proper: assume there is an edge from  $x$  to  $y$ , and that  $x$  and  $y$  are given the same color. Then there exists a bipartite  $(A, B)$  that cover the edge  $xy$ , hence  $(A, B)$  is in the clique associated to  $x$  and in the stable set associated to  $y$ , which means they intersect: the cut  $Cut_x$  can not separate in the same time  $C_y$  from  $S_y$ , because it would then separate  $C_x$  from  $S_y$ . This is impossible. Then we have a coloring with at most as many colors as cuts needed to separate all the clique and the stable sets, which is a polynomial in  $k$ .  $\square$

**Corollary 3.** *The oriented Alon-Saks-Seymour conjecture is equivalent to the Clique-Stable set separation conjecture. Moreover, they are also equivalent to having a polynomial number of 2-list coloring which are compatible with all the solutions of an instance of 3-ALCP.*

## 4 Relations between the $t$ -biclique covering numbers

(COPIER COLLER SIMPLE. VERIFIER QUE LES NOTATIONS MATCHENT)

definir les  $bp_k(G)$  et les polynomial ASS conjectures

Let  $G$  be a graph and let  $\mathcal{B} = (B_1, \dots, B_n)$  be some bipartite complete graphs which covers the edges of  $G$ . Given an edge  $e$ , the *multiplicity*  $m(e)$  of  $e$  is the number of bipartite which contain the edge  $e$ . Let us denote by  $k = \max_{e \in E} (m(e))$ . We associate a *positive side* and a *negative side* to each bipartite. Given an edge  $xy$ , two *half edges* are associated to  $xy$ :  $(xy, x)$  and  $(xy, y)$ . If  $B_i$  covers the edge  $xy$ , the half edge  $(xy, x)$  is a *positive* (resp. *negative*) *half edge* for  $B_i$  if  $x$  is in the positive (resp. negative) side of the bipartite and  $y$  in the negative (resp. positive) side. The *vertex associated to the half edge*  $(e, x)$  is the vertex  $x$ . A *tag* is a pair consisting into a bipartite of  $\mathcal{B}$  and a sign. The *opposite tag* of a tag is the same bipartite and the opposite sign. A *bag* is a set of tags. Two bags are opposite if for each tag which appears in the first one, the opposite of this tag appears in the second one.

**Theorem 6.** *The polynomial Alon-Saks-Seymour conjecture holds for multiplicity 1 if and only if it holds for all multiplicity  $k$ .  $\tilde{L\tilde{A}}$ , on dirait que savoir la poly ASS pour  $k=5$  par exemple, ne suffit pas  $\tilde{A}$  conclure que c'est bon pour n'importe quel  $k$ , y compris 1. Alors que dans mon souvenir,  $\tilde{A}$  ça marchait. Qu'en penses-tu?*

Let us prove the result by induction on  $k$ . Let  $G$  be a graph and let  $\mathcal{B} = (B_1, \dots, B_n)$  be some bipartite complete graphs. Since each edge is covered by at most  $k$  bipartite, each half edge of the graph can be represented by bags with at most  $k$  tags. Note that there are  $2n$  possible tags, thus the total number of different bags is at most  $(2n)^k$ .

Partition the set of the half edges into  $(2n)^k$  parts  $\mathcal{P}$ , in such a way that two half edges are in the same set if and only if they have exactly the same bag. The vertices *associated to a set*  $X$  of  $\mathcal{P}$  are the vertices associated to the half edges which appear in  $X$ . In a first time we will only consider the edges which are covered exactly  $k$  times. The half edges which do not have  $k$  different tags are not considered for the moment. Let  $X$  and  $Y$  be two opposite sets of  $\mathcal{P}$ . Note first that the vertices associated to  $X$  and  $Y$  are disjoint. Indeed, let  $B_i$  be a bipartite which appears in  $X$ , the vertices of  $X$  are on the positive side of the bipartite  $B_i$  and those of  $Y$  on the negative side (or the reverse). Let  $V$  (resp  $W$ ) be the set of vertices associated to  $A$  (resp.  $B$ ) There is a complete bipartite graph between  $V$  and  $W$  in  $G$ . Indeed, for all bipartite  $B_i \in X$ , all the edges between  $V$  and  $W$  are in  $B_i$ .

Let us now consider the family of complete bipartite graphs  $B'_i$  which links the vertices associated to  $X$  to the vertices associated to opposite tag of  $X$ . All the edges of the graph  $G$  of multiplicity  $k$  are covered by this set of bipartite. Indeed, consider an edge  $xy$  of multiplicity  $k$ . The two half edges  $(xy, x)$  and  $(xy, y)$  have two opposite bags of size  $k$ . Then by construction, they are covered by one of the  $B'_i$ . Let us prove that each edge is covered at most once. Indeed, if  $x$  and  $y$  appears in two opposite bags  $X$  and  $Y$ , then there are already  $k$  bipartite  $B_i$  which cover the edge  $xy$ . If it is also covered for the sets  $X' \neq X$  and  $Y' \neq Y$ , it means that, there is a bipartite  $B_j$  which is not in the bag  $X$  but in the bag  $X'$  (since all the bags have size  $k$ ) which cover the edge  $xy$ . Thus, the edge  $xy$  is covered by at least  $k + 1$  bipartite which contradicts the definition of  $k$ . Thus this set of bipartite cover the edges of multiplicity  $k$  with a multiplicity one.

Hence if the Alon Saks Seymour conjecture holds for  $k = 1$ , it means that the graph restricted to this set of bipartite  $B'_i$  has chromatic number at most  $((2n)^k)^c$  since there are  $(2n)^k$   $B'_i$  satisfying  $k = 1$ . Thus the vertex set can be partitionned into  $((2n)^k)^c$  stable sets. Since all the edges of multiplicity  $k$  are covered it means that the multiplicity of the edges in each stable set is at most  $k - 1$ . Hence by induction hypothesis, it means that the chromatic number of each part is bounded. Thus the chromatic number of the graph is at most  $n^{ck} \cdot n^{f(k-1)}$ . Thus the Alon-Saks-Seymour conjecture holds for multiplicity  $k$  if and only if it holds for multiplicity one.

## 5 Cliques-Stables separation conjecture and Communication complexity

Yannakakis introduced in [18] the following communication complexity problem, called  $CL - IS$ : given a publicly known graph  $\Gamma$  on  $n$  vertices, Alice and Bob agree on a protocol, then a clique is given to Alice and a stable set is given to Bob. They do not know which clique or which stable set was given to the other one, and their goal is to decide whether the clique and the stable set intersect or not, by minimizing the worst-case number of exchanged bits. The intersection can be formalized by a function

$f : X \times Y \rightarrow \{0, 1\}$ , where  $X$  is the set of cliques and  $Y$  the set of stable sets. Note that the intersection of a clique and a stable set is at most one vertex. In the deterministic version, Alice and Bob send alternatively messages one to each other, and the minimization is on  $D(f)$ , the number of bits exchanged between them. In the non-deterministic version, for  $b \in \{0, 1\}$ , an all powerful prover sends a certificate of size  $N^b(f)$  in order to convince both Alice and Bob that the result is  $b$ . Then, Alice and Bob exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize  $N^b(f)$ .

In this particular setting, a certificate proving that  $f(x, y) = 1$  is just the name of a vertex in the intersection of the clique and the stable set. Thus  $N^1(f) \leq \lceil \log_2(n) \rceil$ . Convincing Alice and Bob that the result is 0 is much more complicated. If we have a family of  $m$  cuts that separate all the cliques and the stable sets, then a certificate for  $f(x, y) = 0$  can be a cut that separates  $x$  and  $y$ . Thus  $N^0(f) \leq \lceil \log_2(m) \rceil$ .

It is known in theory of communication complexity [14] that, for general  $f$ ,  $N^b(f) = \lceil \log_2 C^b(f) \rceil$  where  $C^b(f)$  is the minimum number of monochromatic combinatorial rectangles needed to cover the  $b$ -inputs of the communication matrix  $M$  of  $f$ , with  $M_{x,y} = f(x, y)$ . One can observe that a monochromatic combinatorial rectangle that cover some 0-inputs of  $M$  is precisely a cut that separates the cliques and the stable sets involved in the rectangle. Thus finding the minimum  $N^0(f)$  is equivalent to finding the minimum number of cuts needed to separate all the cliques and the stable sets.

The existing bounds for  $CL - IS$  are ([18], [14]):

$$\log_2 m \leq N^0(f) \leq D(f) \leq \mathcal{O}(\log^2(n)).$$

Moreover, Huang and Sudakov improved in [12] the trivial lower bound  $N^0(f) \geq \log_2 n$  thanks to a counterexample to the Alon-Saks-Seymour conjecture: for CL-IS, there exists an infinite collection of graphs such that

$$N^0(f) \geq 6/5 \log(n) - \mathcal{O}(1).$$

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