

Master 2 Internship report

Alon-Saks-Seymour conjecture, Clique-Stable
separation and the stubborn problem

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1 Introduction

Yannakakis introduced in [25] the following communication complexity problem, called *CL-IS*: given a publicly known graph Γ on n vertices, Alice and Bob agree on a protocol, then Alice is given a clique and Bob is given a stable set. They do not know which clique or which stable set was given to the other one, and their goal is to decide whether the clique and the stable set intersect or not, by minimizing the worst-case number of exchanged bits. The intersection can be formalized by a function $f : X \times Y \rightarrow \{0, 1\}$, where X is the set of cliques and Y the set of stable sets. It can also be represented by a $|X| \times |Y|$ matrix M with $M_{x,y} = f(x, y)$. Note that the intersection of a clique and a stable set is at most one vertex. In the deterministic version, Alice and Bob send alternatively messages one to each other, and the minimization is on $D(f)$, the number of bits exchanged between them. In the non-deterministic version, for $b \in \{0, 1\}$, an all powerful prover sends a certificate of size $N^b(f)$ in order to convince both Alice and Bob that the result is b . Then, Alice and Bob exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize $N^b(f)$.

In this particular setting, a certificate proving that $f(x, y) = 1$ is just the name of a vertex in the intersection of the clique and the stable set. Such a certificate has clearly a logarithmic size. Thus $N^1(f) \leq \lceil \log_2(n) \rceil$. Convincing Alice and Bob that the result is 0 is much more complicated. If we have a family of m cuts that separate all the cliques and the stable sets, then a certificate for $f(x, y) = 0$ can be a cut that separates x and y . Thus $N^0(f) \leq \lceil \log_2(m) \rceil$. Hence if there is a polynomial number of cuts separating all the cliques and the stable sets (such a set of cuts is called a CS-separator), one can ensure that the non-deterministic certificate has size $\mathcal{O}(\log n)$.

It is known in theory of communication complexity [18] that, for general f , $N^b(f) = \lceil \log_2 C^b(f) \rceil$ where $C^b(f)$ is the minimum number of monochromatic combinatorial rectangles needed to cover the b -inputs of the communication matrix M (a monochromatic combinatorial rectangle is a set of rows and columns such that the matrix restricted to this set of rows and this set of columns is entirely filled by b). One can observe that a monochromatic combinatorial rectangle that cover some 0-inputs of M is precisely a cut that separates the cliques and the stable sets involved in the rectangle. Thus finding the minimum $N^0(f)$ is equivalent to finding the minimum number of cuts needed to separate all the cliques and the stable sets.

The existing bounds for *CL-IS* are ([25], [18]):

$$\log_2 n \leq N^0(f) \leq D(f) \leq \mathcal{O}(\log^2(n)).$$

There is a $\mathcal{O}(\log n)$ certificate for the *CL-IS* problem if and only if there is a CS-separator of polynomial size. The existence of such a CS-separator is called in the following clique-stable set separation problem. A geometric generalization of the clique-stable set conjecture was studied by Lovász in [19]. He wanted to determine if the stable set polytope of a graph is the projection of a polytope in higher dimension, with a polynomial number of facets. He proved that if the answer to this question is positive, then there is a polynomial number of cuts that separate all the cliques and the stable sets. Moreover, he proved it for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and t -perfect graphs which are a generalization of series-parallel graphs. The question on stable set polytope for general graphs has recently been answered negatively by Fiorini et al. [10].

We prove that there is a polynomial CS-separator for two main classes of graphs: random graphs and split-free graphs. The proof for random graphs is based on random cuts. For split-free graphs, it is based on VC-dimension. These proofs are really different since VC-dimension is a certificate of non-randomness of the graph. Hence on a randomness scale, our results are opposite.

Alon and Haviv [1] observed that this problem is deeply linked with another conjecture known as the Alon-Saks-Seymour conjecture [13] which states that if a graph can be covered with k edge-disjoint complete bipartite graphs, then its chromatic number is at most $k + 1$. It is inspired from the Graham Pollak theorem [12] and has interested several authors ([22],[11]). Huang and Sudakov found in [15] a counterexample to the Alon-Saks-Seymour conjecture, twenty years after its statement. Actually they proved that there are some graphs for which at least $\mathcal{O}(k^{6/5})$ colors are necessary. And they remark that it implies, for $CL - IS$ that there exists an infinite collection of graphs such that:

$$N^0(f) \geq 6/5 \log(n) - \mathcal{O}(1)$$

which improves the trivial (and only known) lower bound $N^0(f) \geq \log_2 n$. In communication complexity it is a long standing open problem to prove a $\mathcal{O}(\log^2 n)$ lower bound for the $CL - IS$ problem.

Given a graph G , the bipartite packing, denoted by bp , is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of the graph. We denote by bp_k the minimum number of complete bipartite graphs needed to cover the edges of the graph such that each edge is covered at most k times. In this report we prove that for every integer k , the chromatic number χ can be bounded polynomially on bp if and only if it can be bounded polynomially on bp_k . In particular, it gives that a polynomial gap for the Alon-Saks-Seymour conjecture is equivalent to $N^0(f) = \mathcal{O}(\log n)$ (as observed in [15]). This question was left as open in [15]. We also give another direct proof of the equivalence between the oriented version of the Alon Saks Seymour conjecture and $N^0(f) = \mathcal{O}(\log n)$.

Surprisingly, we also find some link between the Alon-Saks-Seymour conjecture and some constraint satisfaction problem. The complexity of so-called *list-M partition problems* has been widely studied in the last decades (see [23] for an overview). M stands for a fixed $k \times k$ symmetric matrix filled with 0, 1 and * as illustrated on Fig. 1. The input is a graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, \dots, A_k\})$ and the question is to determine whether the vertices of G can be partitioned into k sets A_1, \dots, A_k respecting two types of requirements. The first one is given by the list assignments, that is to say v can be put in A_i only if $A_i \in \mathcal{L}(v)$. The second one is described in M , namely: if $M_{i,i} = 0$ (resp. $M_{i,i} = 1$), then A_i is a stable set (resp. a clique), with no restriction if $M_{i,i} = *$; and if $M_{i,j} = 0$ (resp. $M_{i,j} = 1$), then A_i and A_j are completely non-adjacent (resp. completely adjacent), with no restrictions if $M_{i,j} = *$.

Feder et al. ([7], [9]) proved a *quasi-dichotomy theorem*. The list-M partition problems are classified between NP-complete and quasi-polynomial time solvable (i.e. time $\mathcal{O}(n^{c \log n})$ where c is a constant). Moreover, many investigations have been made about small matrices M ($k \leq 4$) to get a *dichotomy theorem*, meaning a classification of the list-M partition problems between polynomial time solvable and NP-complete. Cameron et al. [4] reached such a dichotomy for $k \leq 4$, except for one special case (and its complement) then called the *stubborn problem* (see Fig. 1 for the corresponding matrix), which remained only quasi-polynomial time solvable. Cygan et al. [5] closed the question by finding a polynomial time algorithm solving the stubborn problem. More precisely, they found a polynomial time algorithm for 3-COMPATIBLE COLORING, which was introduced in [6] and said to be no easier than the stubborn problem. 3-COMPATIBLE COLORING has also been introduced and studied in [17] under the name ADAPTED LIST COLORING, and was proved to be a model for some strong scheduling problems.

These two problems under study are defined in the following way:

$$\begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}$$

Figure 1: Matrix M for the stubborn problem.

3-COMPATIBLE COLORING PROBLEM (3-CCP)

Input: An edge coloring f_E of the complete graph on n vertices with 3 colors $\{A, B, C\}$.

Question: Is there a coloring of the vertices with the same three colors, such that no edge has the same color as both its endpoints?

STUBBORN PROBLEM

Input: A graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.

Question: Can V be partitioned into four sets A_1, \dots, A_4 such that A_4 is a clique, both A_1 and A_2 are stable sets, there is no edge between A_1 and A_3 , and each vertex v belongs to A_i only if $A_i \in \mathcal{L}(v)$?

Given a partition problem, a 2-list assignment is an assignment of at most two sets X, Y , to each vertex, meaning that this vertex can belong either to X or to Y in the partition. Links we highlight between the clique-stable set conjecture and both these problems are not concerned with the existence of a solution, but with covering all the solutions of 3-CCP and of the stubborn problem with a polynomial family of 2-list assignments. We prove that the existence of such a polynomial covering family for the stubborn problem (under some small restrictions on the solutions) is equivalent to the existence of such a polynomial covering family for 3-CCP, which in turn is equivalent to the existence of a polynomial CS-separator.

We may observe that the existence of a polynomial family of 2-list assignments that cover all the edges does not imply the polynomial solvability of the problem: indeed, such a family may not be computable in polynomial time.

2 Definitions

Let $G = (V, E)$ be a graph and k be an integer. An edge $uv \in E$ links its two *endpoints* u and v . The *neighborhood* $N(x)$ of x is the set of vertices y such that $xy \in E$. The *non-neighborhood* $N^C(x)$ of x is $V \setminus N(x)$. For oriented graphs, $N^+(x)$ (resp. $N^-(x)$) denote the outgoing (resp. incoming) neighborhood of x , i.e. the set of vertices y such that $xy \in E$ (resp. $yx \in E$). The subgraph *induced by* $X \subseteq V$ denoted by $G[X]$ is the graph with vertex set X and edge set $E \cap (X \times X)$. A *clique of size* n , denoted by K_n , is a complete induced subgraph. A *stable set* is an induced subgraph with no edge. Note that a clique and a stable set intersect on at most one vertex. Two subsets of vertices $X, Y \subseteq V$ are *completely adjacent* if for all $x \in X$, $y \in Y$, $xy \in E$. They are *completely non-adjacent* if there are no edge between them. A graph $G = (V, E)$ is *split* if $V = V_1 \cup V_2$ and the subgraph induced by V_1 is a clique and the subgraph induced by V_2 is a stable set.

A graph G is *bipartite* if V can be partitioned into (U, W) such that both U and W are stable sets. Moreover, G is *complete* if U and W are completely adjacent. An *oriented bipartite graph* is a bipartite graph together with an edge orientation such that all the edges go from U to W . A *hypergraph* $H = (V, E)$ is a generalization of a graph and is composed of a set of vertices V and a set of *hyperedges* $E \subseteq \mathcal{P}(V)$.

A *vertex-coloring* (resp. *edge-coloring*) of G with a set COL of k colors is a function f_V (resp. f_E) from V (resp. E) to COL . For edge-colored graphs, an $\alpha_1, \dots, \alpha_k$ -*clique* is a clique for which every edge has a color in $\{\alpha_1, \dots, \alpha_k\}$. The α -*neighborhood* of x is the set of vertices y such that xy is an α -*edge*, i.e. an edge colored with α . The *majority color* of $x \in V$ is the color which appears the most often for the edges with endpoint x (in case of ties, we arbitrarily cut them).

3 Cliques-Stable Sets separation conjecture

A *cut* is a pair (A, B) such that $A \cup B = V$ and $A \cap B = \emptyset$. It *separates* a clique C and a stable set S if $C \subseteq A$ and $S \subseteq B$. Note that a clique and a stable set can be separated if and only if they do not intersect. Let \mathcal{C}_G be the set of cliques of G and \mathcal{S}_G be the set of stable sets of G . We say that a family \mathcal{F} of cuts *is a CS-separator* if for all $(C, S) \in \mathcal{C}_G \times \mathcal{S}_G$ which do not intersect, there exists a cut in \mathcal{F} that separates C and S .

Conjecture 1. (*Clique-Stable Set separation Conjecture*) *There is a polynomial Q , such that for every graph G on n vertices, there is a CS-separator of size at most $Q(n)$.*

Proposition 2. *Conjecture 1 holds if and only if a polynomial family \mathcal{F} of cuts separates all the maximal (in sense of inclusion) cliques from the maximal stable sets that do not intersect.*

Proof. First note that one direction is direct. Let us prove the other one. Assume \mathcal{F} is a polynomial family that separates all the maximal cliques from the maximal stable sets that do not intersect. Let $\text{Cut}_{1,x}$ be the cut $(N(x) \cup x, V \setminus \{N(x), x\})$ and $\text{Cut}_{2,x}$ be the cut $(N(x), V \setminus N(x))$. Let us prove that \mathcal{F}' which is the family \mathcal{F} plus the families $\text{Cut}_{1,x}$ and $\text{Cut}_{2,x}$ for all x is a CS-separator.

Let (C, S) be a pair of clique and stable set. Grow up C and S by adding vertices to get a maximal clique C' and a maximal stable set S' . Either C' and S' do not intersect, and there is a cut in \mathcal{F} that separate C' from S' (thus C from S). Or C' and S' intersect in x (recall that a clique and a stable set intersect on at most one vertex): if $x \in C$, then $\text{Cut}_{1,x}$ separates C from S , otherwise $\text{Cut}_{2,x}$ does. □

Consequently, in the following, we will only focus on separating the maximal cliques from the maximal stable sets.

In the following, we first verify in Section 3.1 that Conjecture 1 holds for random graphs. The proof is based on random cuts and we prove that random cuts separate an important part of cliques and stable sets when the probability of taking the vertex is equal to the probability of an edge.

Then, we verify Conjecture 1 for split-free graphs. The proof is in some sense exactly the opposite of Section 3.1, since when a bipartite graph is forbidden, there are no random-like structures. Then, via an argument based on VC-dimension, one can find a polynomial number of cuts defined locally which form a CS-separator.

3.1 Random graphs

Let n be a positive integer and $p \in [0, 1]$. Several models of random graphs have been studied [3], and we will work on the Erdős-Rényi model. The random graph $G(n, p)$ is a probability space over the set of graphs on the vertex set $\{1, \dots, n\}$ determined by $\Pr[\{i, j\} \in G] = p$, with these events mutually independent. We say that $G(n, p)$ has clique number ω if ω satisfies $\mathbb{E}(\text{number of cliques of size } \omega) = 1$. We define similarly the independent number of $G(n, p)$.

Theorem 3. *There exists a polynomial P , such that for almost every graph in $G(n, p)$, there exists a CS-separator of size $P(n)$.*

The claim is detailed by the following propositions, in which \ln denotes the logarithm to the base e , \log_b denotes the logarithm to the base b , and \log denotes the logarithm to the base 2.

Proposition 4. *Let $a(n)$ (denoted by a in the following) be a function of n such that $2 \leq a(n) \ll \log n$. The random graph $G(n, p)$ with $p = 1 - n^{-2/a} = 1 - 2^{-2 \log n/a}$ has:*

- independent number $\alpha = a - \frac{a \log a}{\log n} + 1 + o(1)$
- clique number $\omega = 2 \ln 2 \cdot 2^{2 \log n/a} \cdot \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n)$.

Moreover, for all $\varepsilon > 0$, with high probability there exists a family of $\mathcal{O}(n^{6+\varepsilon})$ cuts (more precisely $\mathcal{O}(n^{5+2/a+\varepsilon})$ if a is a constant, and $\mathcal{O}(n^{5+\varepsilon})$ if $a(n) \rightarrow_{n \rightarrow \infty} \infty$) that separates all the cliques and the stable sets.

Proof. The interested reader can find details of computation in Appendix A. Let $p' = 1 - p$, $b = 1/p$ and $b' = 1/p'$. The independence number and clique number of $G(n, p)$ are given by the following formulas, depending on p (see [3]):

$$\begin{aligned} \omega &= 2 \log_b(n) - 2 \log_b(\log_b n) + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'}(\log_{b'} n) + 2 \log_{b'}(e/2) + 1 + o(1) \end{aligned}$$

Using Taylor series, we get:

- $\alpha = a + 1 + o(1)$
- $\omega = 2 \ln 2 \cdot n^{2/a} \left(\frac{a-2}{a} \log n - \log \log n \right) + \mathcal{O}(\log n)$.

Now, let $x = 1 - p = 2^{-2 \log n/a}$. Draw a random partition (V_1, V_2) where each vertex is put in V_1 independently from the others with probability x . Let (C, S) be a pair of clique and stable set of the graph. There are at most 4^n such pairs. The probability that $S \subseteq V_1$ and $C \subseteq V_2$ is $x^\alpha (1-x)^\omega$. Let us show that $x^\alpha \geq n^{-(2+2/a+\varepsilon')}$ and $(1-x)^\omega \geq n^{-(2+\varepsilon')}$ for all $\varepsilon' > 0$.

On one hand,

$$\begin{aligned} x^\alpha \geq n^{-(2+2/a+\varepsilon')} &\Leftrightarrow \alpha \log x \geq -(2 + 2/a + \varepsilon') \log n \\ &\Leftrightarrow 2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2 + 2/a + \varepsilon') \log n \end{aligned}$$

which is true if n is large enough.

Note that for $a \rightarrow_{n \rightarrow \infty} \infty$, we even have $2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2 + \varepsilon') \log n$ for n large enough, which implies $x^\alpha \geq n^{-(2+\varepsilon')}$.

On the other hand, $\log(1-x) = \log b$, thus, using again Taylor series:

$$\begin{aligned} (1-x)^\omega \geq n^{-(2+\varepsilon')} &\Leftrightarrow \omega \log(1-x) \geq -(2 + \varepsilon') \log n \\ &\Leftrightarrow 2 \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(1) + \mathcal{O}(2^{-2 \log n/a} \log n) \leq (2 + \varepsilon') \log n \end{aligned}$$

which is true if n is large enough.

This leads us to $x^\alpha (1-x)^\omega \geq n^{-(5+2\varepsilon')} = n^{-(5+\varepsilon)}$ for $\varepsilon' = \varepsilon/2$. Thus (C, S) is separated by at least $n^{-(5+\varepsilon)}$ of all the partition. By double counting, there exists a partition that separates at least $n^{-(5+\varepsilon)}$ of all the pairs. We delete these separated pairs, and there remains at most $(1 - 1/n^{5+\varepsilon}) \cdot 4^n$ pairs. The same probability for a pair (C, S) to be cut by a random partition

still holds, hence we can iterate the process k times until $(1 - 1/n^{5+\varepsilon})^k \cdot 4^n \leq 1$. This is satisfied for $k = 2n^{6+\varepsilon}$ which is a polynomial in n . If $a \rightarrow_{n \rightarrow \infty} \infty$, we can do the same and get $k = 2n^{5+\varepsilon}$. There is a polynomial family of cuts that separates all the cliques and the stable sets. \square

However, this does not cover the classic configuration where $p = 1/2$ and $\alpha \approx 2 \log n$. Thus we need the following:

Proposition 5. *Let $a(n) = 2c \log n$ (denoted by a in the following) where $0 \leq c \leq 1$ is a constant. The random graph $G(n, p)$ with $p = 1 - 2^{-c}$ has:*

- independent number $\alpha = 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)$
- clique number $\omega = 2c' \log n - 2c' \log \log n + o(\log \log n)$ where $1/c' = \log(1 - 2^{-1/c})$.

Moreover, for all $\varepsilon > 0$, with high probability there exists a family of $\mathcal{O}(n^5)$ cuts that separates all the cliques and the stable sets.

Proof. Let $p' = 1 - p$, $b = 1/p$ and $b' = 1/p'$. We proceed the same way as in the previous proposition. The independence number and clique number of $G(n, p)$ are given by the following formulas, depending on p (see [3]):

$$\begin{aligned}\omega &= 2 \log_b(n) - 2 \log_b \log_b n + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1)\end{aligned}$$

Now $\frac{1}{\log b'} = c$, thus:

$$\begin{aligned}\alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= 2c \log(n) - 2c \log \log n + 2c(\log(e/2) - \log c) + 1 + o(1)\end{aligned}$$

Moreover, $\log b = \log(1 - 2^{-1/c}) = 1/c'$, thus, similarly to α , we get:

$$w = 2c' \log(n) - 2c' \log \log n + 2c'(\log(e/2) - \log c') + 1 + o(1)$$

The same reasoning as for Proposition 4 with the same $x = 1 - p'$ gives a probability $x^\alpha(1-x)^\omega \geq n^{-4}$ for a given pair (C, S) to be separated by a random non-uniform cut (V_1, V_2) , so there is a family of $\mathcal{O}(n^5)$ cuts that separates all the cliques and the stable sets (see the proof of Proposition 4 for details). \square

3.2 Split-free graphs

Let us first state some definitions concerning hypergraphs and VC-dimension. Let $H = (V, E)$ be a hypergraph. The *transversality* $\tau(H)$ is the minimum cardinality of a subset of vertices intersecting each hyperedge. The transversality corresponds to an optimal solution of the following integer linear programming:

$$\text{Objective function:} \quad \min \sum_{x \in V} w(x)$$

Subject to:

- for all $x \in V$, $w(x) \in \{0, 1\}$.
- for all $e \in E$, $\sum_{x \in e} w(x) \geq 1$.

The *fractional transversality* τ^* is a relaxation of the above linear programming. It means that we look for real solutions. The first condition becomes: for all $x \in V$, $0 \leq w(x) \leq 1$. Note that removing the constraint $w(x) \leq 1$ for all $x \in V$ does not change the solution as we want to minimize the objective function. Indeed if $w(y) > 1$ for some $y \in V$ in a feasible solution w , getting $w(y)$ down to 1 will not violate any constraint, and will reduce strictly the objective function.

The *Vapnik-Chervonenkis dimension* or *VC-dimension* of a hypergraph $H = (V, E)$ is the maximum cardinality of a set of vertices $A \subseteq V$ such that for every $B \subseteq A$ there is an edge $e \in E$ so that $e \cap A = B$. The following bound due to Haussler and Welzl [14] links the transversality, the VC-dimension and the fractional transversality.

Lemma 6. *Every hypergraph H with VC-dimension d satisfies*

$$\tau(H) \leq 16d\tau^*(H) \log(d\tau^*(H)).$$

Let us introduce the dual measure of VC-dimension. A set T of hyperedges forms a *complete Venn diagram* if for all $T' \subseteq T$, there exists a vertex v such that $v \in e$ if and only if $e \in T'$. The *dual Vapnik-Chervonenkis dimension* (*dual VC-dimension* for short) of H is the maximum size of a complete Venn diagram in H . In the remaining of the paper, we only consider hypergraphs which are neighborhood hypergraphs, i.e. hypergraphs so that the hyperedges are the neighborhoods of vertices in a given graph G . In this context, the VC-dimension and the dual VC-dimension of a hypergraph coincide, since for all vertices x and y , $x \in N(y)$ if and only if $y \in N(x)$. Consequently, for simplicity, in the following the dual VC-dimension will be called VC-dimension.

We denote by \mathcal{C}_Γ the class of graph with no induced Γ .

Theorem 7. *Let Γ be a fixed split graph. Then the clique-stable set conjecture is verified on \mathcal{C}_Γ .*

Proof. The vertices of Γ are partitionned into (V_1, V_2) where V_1 is a clique and V_2 is a stable set. Let $\varphi = \max(|V_1|, |V_2|)$. Let t be the constant $32\varphi(\log(\varphi) + 1)$. Let $G = (V, E) \in \mathcal{C}_\Gamma$ and let \mathcal{F} be the following family of cuts. For every subset $\{x_1, \dots, x_r\}$ of at most t vertices which is a clique (resp. a stable set), take $U = \bigcap_{1 \leq i \leq r} N(x_i)$ (resp. $U = \bigcup_{1 \leq i \leq r} N(x_i)$), and put $(U, V \setminus U)$ in \mathcal{F} . Since each member of \mathcal{F} is defined with a set of at most t vertices, the size of \mathcal{F} is at most $\mathcal{O}(n^t)$. Let us now prove that \mathcal{F} is a CS-separator.

Let (C, S) be a pair of maximal clique and stable set. Let us prove that (C, S) is separated by \mathcal{F} . Let us build an oriented graph B with vertex set $C \cup S$. For all $x \in C$ and $y \in S$, put the arc xy if $xy \in E$, and put the arc yx otherwise (see Fig. 2(b)). We use the following variant of Farkas' lemma, from which we derive Lemma 9:

Lemma 8 (Farkas' lemma [20]). *Let A be a $m \times n$ matrix and $b \in \mathbb{R}^m$. Then either:*

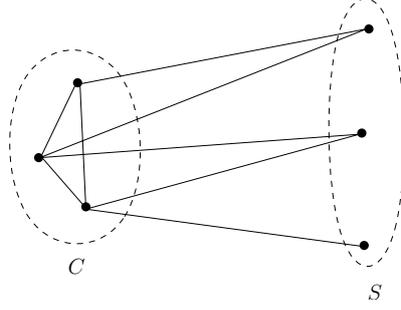
1. *There is a $w \in \mathbb{R}^n$ such that $Aw \leq b$,*

or 2. *There is a $y \in \mathbb{R}^m$ such that $y \geq 0$, $yA = 0$ and $yb < 0$.*

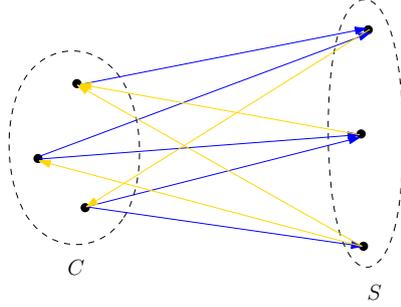
Lemma 9. *For all oriented graph $G = (V, E)$, there exists a weight function $w : V \rightarrow [0, 1]$ such that $w(V) = 1$ and for all vertex x , $w(N^+(x)) \geq w(N^-(x))$.*

Proof. Let us define a $(2n + 1) \times n$ matrix A obtained from the vertical concatenation of first the transpose ${}^t\text{Adj}(G)$ of the adjacency matrix of G , second the matrix $-\text{Id}_n$, and finally the line vector $(-1, \dots, -1) \in \mathbb{R}^n$. Define $b = {}^t(0, \dots, 0, -1) \in \mathbb{R}^{2n+1}$.

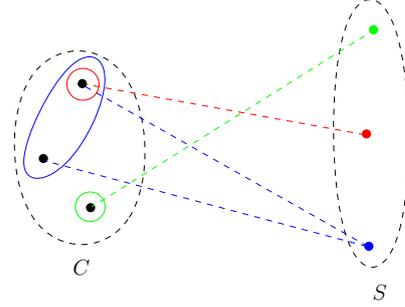
Then apply lemma 8: either case one occurs and then $Aw \leq b$: thanks to ${}^t\text{Adj}(G)$, we get $w(N^+(x)) \geq w(N^-(x))$ for all $x \in V$; thanks to the other lines, we get $w(x) \geq 0$ for all x and $w(V) \geq 1$. We conclude by rescaling the weight function with a factor $1/w(V)$.



(a) A clique C and a stable S in G .



(b) Graph B build from C and S . Edges in G are replaced by forward edges, and non-edges are replaced by backward edges.



(c) Hypergraph H where hyperedges are built from the non-neighborhood of vertices from S .

Figure 2: Illustration of proof of Theorem 7. For more visibility in 2(b), forward edges are drawn in blue and backward edges in yellow.

Otherwise, case two occurs and there is $y \in \mathbb{R}^{2n+1}$: let $w \in \mathbb{R}^n$ be the projection of y on the first n coordinates. Then for all $1 \leq j \leq 2n+1$, $y_j \geq 0$ so $w(x) \geq 0$ for all x ; moreover, since $yA = 0$ then for all $x_i \in V$, $w(N^+(x)) - w(N^-(x)) - y_{n+i} - y_{2n+1} = 0$ so $w(N^+(x)) \geq w(N^-(x))$. Since $yb < 0$, then $y_{2n+1} > 0$ which ensures $w(V) > 0$: indeed, otherwise for all x , $w(x) = 0$ and the previous equality becomes $y_{2n+1} = -y_{n+i} \leq 0$, which is a contradiction. We conclude as before by rescaling with a factor $1/w(V)$. \square

Corollary 10. *In B , there exists either:*

(i) *a weight function $w : C \rightarrow \mathbb{R}^+$ such that $w(C) = 2$ and for all vertex $x \in S$, $w(N^+(x)) \geq 1$.*

or (ii) *a weight function $w : S \rightarrow \mathbb{R}^+$ such that $w(S) = 2$ and for all vertex $x \in C$, $w(N^+(x)) \geq 1$.*

Proof. Let $w : V \rightarrow [0, 1]$ be a weight function satisfying conditions of Lemma 9. Since $w(V) = 1$, either $w(C) > 0$ or $w(S) > 0$. Assume $w(C) > 0$ (the other case is handled symmetrically). Take a new weight function defined by $w'(x) = 2w(x)/w(C)$ if $x \in C$, and 0 otherwise. Then for all $x \in S$, on one hand $w'(N^+(x)) \geq w'(N^-(x))$ by extension of the property of w , and on the other hand, $N^+(x) \cup N^-(x) = C$ by construction of B . Thus $w'(N^+(x)) \geq w'(C)/2 = 1$ since $w'(C) = 2$. \square

In the following, let assume we are in case (i). Case (ii) is handled symmetrically by switching C (resp. neighborhood) and S (resp. non-neighborhood).

Let us now build H an hypergraph with vertex set C . For all $x \in S$, build the hyperedge $C \setminus N_G(x)$, that is the complementary in C of the neighbors of x (see Fig. 2(c)).

Lemma 11. *The hypergraph H has fractionnal transversality $\tau^* \leq 2$.*

Proof. Let w be the weight function given by Corollary 10. Let h be a hyperedge built from the non-neighborhood of $x \in S$. Recall that this non-neighborhood is precisely $N^+(x)$ in B , then we have:

$$\sum_{y \in h} w(y) = w(N^+(x)) \geq 1.$$

Thus w satisfies the constraints of the fractionnal transversality, and $w(C) \leq 2$, i.e. $\tau^* \leq 2$. \square

Lemma 12. *H has VC-dimension bounded by $2\varphi - 1$.*

Proof. Assume that there is a complete Venn-diagram \mathcal{D} of size 2φ in H . The aim is to exploit the shuttering to find an induced H , which builds a contradiction. Let $s_1, \dots, s_\varphi, t_1, \dots, t_\varphi$ be the hyperedges composing \mathcal{D} . In the following, we will abuse notation by calling s_i (resp. t_i) both the hyperedge and the vertex of S whose non-neighborhood is precisely the hyperedge s_i (resp. t_i). Recall that the forbidden split graph Γ is the union of a clique $V_1 = \{x_1, \dots, x_r\}$ and a stable set $V_2 = \{y_1, \dots, y_{r'}\}$ (with $r, r' \leq \varphi$). Let $x_i \in V_1$ and let $\{y_{i_1}, \dots, y_{i_k}\} = N_\Gamma(x_i) \cap V_2$ be the set of its neighbors in V_2 .

Consider $\mathcal{S} = \{s_{i_1}, \dots, s_{i_k}\} \cup \{t_i\}$ (possible because $|V_1|, |V_2| \leq \varphi$). As \mathcal{D} is a complete Venn diagram, there exists $x'_i \in C$ such that $x'_i \in \cap_{s \in \mathcal{S}} s$ and $x'_i \notin \cup_{s' \notin \mathcal{S}} s'$, meaning that the set of hyperedges containing x'_i is precisely \mathcal{S} . Now, forget about the existence of t_1, \dots, t_φ , and look at the subgraph of G induced by x'_1, \dots, x'_r and s_1, \dots, s_φ : x'_i has exactly the same shape of neighborhood in $\{s_1, \dots, s_\varphi\}$ as the neighborhood of x_i in V_2 . Thus we have found an induced Γ , which is impossible. \square

Note that the presence of t_1, \dots, t_φ is useful in case where two vertices of V_1 are twins with respect to V_2 , meaning that their neighborhoods intersected with V_2 are the same. Then, we could not have ensured thanks to the complete Venn diagram to have **two** vertices contained in exactly the set of hyperedges $\mathcal{S} = \{s_{i_1}, \dots, s_{i_k}\}$, and no more. In fact, this remark lead us to assert that the VC-dimension of H is bounded by $\varphi + \log \varphi$. Indeed, we need only $t_1, \dots, t_{\log \varphi}$ in addition to s_1, \dots, s_φ : for $x_i \in V_1$, code i in binary over $\log \varphi$ bits and define \mathcal{S} to be the union of $\{s_{i_1}, \dots, s_{i_k}\}$ with the set of t_j such that the j -th bit is one. This ensures that no two $x_i, x_{i'}$ have the same \mathcal{S} .

Applying Lemma 6 to H , we obtain

$$\tau(H) \leq 16d\tau^*(H) \log(d\tau^*(H)) \leq 32\varphi(\log(\varphi) + 1).$$

Since $t = 32\varphi(\log(\varphi) + 1)$, τ is bounded by t which only depends on H . There must be $x_1, \dots, x_\tau \in C$ such that each hyperedge of H contains at least one x_i . Thus for all $y \in S$, there is an i such that $x_i \in N^C(y)$. Consequently, $S \subseteq \cup_{1 \leq i \leq t} N_G^C(x_i)$. Moreover, $C \subseteq U = \cap_{1 \leq i \leq t} N_G(x_i) \cup \{x_i\}$ since x_1, \dots, x_τ are in the same clique C . This means that the cut $(U, V \setminus U) \in \mathcal{F}$ built from the subclique x_1, \dots, x_τ separates C and S .

When case (ii) of Corollary 10 occurs, there are τ vertices $x_1, \dots, x_\tau \in S$ such that for all $y \in C$, there exists $x_i \in N(y)$. Thus $C \subseteq U = \cup_{1 \leq i \leq t} N_G(x_i)$ and $S \subseteq \cup_{1 \leq i \leq t} N_G^C(x_i)$. The cut $(U, V \setminus U) \in \mathcal{F}$ built from the stable set x_1, \dots, x_τ separates C and S . \square

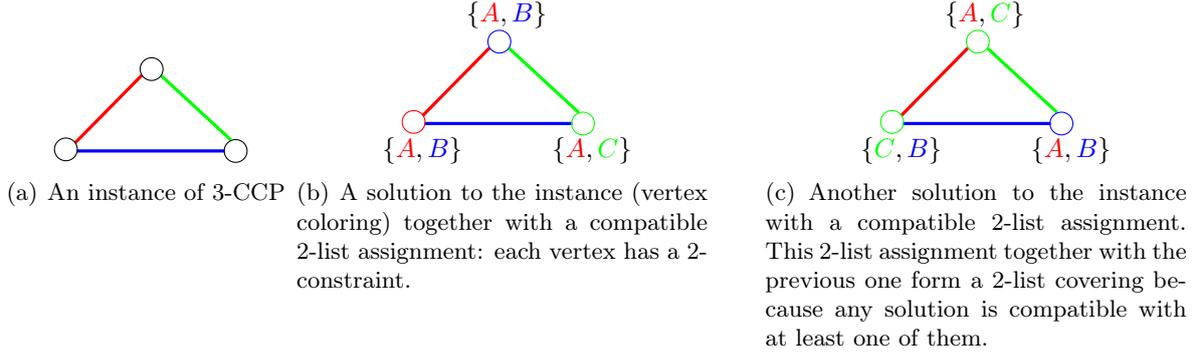


Figure 3: Illustration of definitions. Color correspondence: A=red ; B=blue ; C=green.

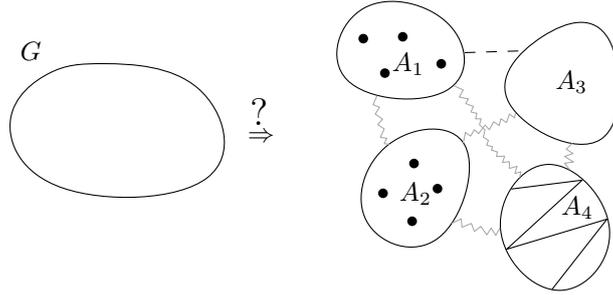


Figure 4: Diagram representing the stubborn problem. Cliques are represented by hatched sets, stable sets by dotted sets. Completely non-adjacent sets are linked by a dashed edge. Grey lines represent edges that may or may not appear in the graph.

4 Link with 3-CCP and the stubborn problem

The following definitions are illustrated on Figure 3 and are relative with list coloring. Formally, given a graph G and a set COL of k colors, we can associate to each vertex a set of possible colors of size at most k . A set of possible colors, called *constraint*, is associated to each vertex. A vertex has an l -constraint if its set of possible colors has size l . An l -list assignment is a function $\mathcal{L} : V \rightarrow \mathcal{P}(\text{COL})$ that give each vertex an l -constraint. A solution \mathcal{S} is a coloring of the vertices $S : V \rightarrow \text{COL}$. For small k , we may equivalently consider \mathcal{S} as a partition $\mathcal{S} = (A_1, \dots, A_k)$ with $x \in A_i$ if and only if $\mathcal{S}(x) = A_i$. An l -list assignment \mathcal{L} is *compatible* with a solution \mathcal{S} if for each vertex x , $S(x) \in \mathcal{L}(x)$. A set of l -list assignment *covers* a solution \mathcal{S} if at least one of the l -list assignment is compatible with \mathcal{S} .

We recall the definitions of 3-CCP and the stubborn problem:

3-COMPATIBLE COLORING PROBLEM (3-CCP)

Input: An edge coloring f_E of the complete graph on n vertices with 3 colors $\{A, B, C\}$.

Question: Is there a coloring of the vertices with the same three colors, such that no edge has the same color as both its extremities?

STUBBORN PROBLEM (see Fig. 4)

Input: A graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.

Question: Can V be partitionned into four sets A_1, \dots, A_4 such that A_4 is a clique, both A_1 and A_2 are stable sets, A_1 and A_3 are completely non-adjacent, and the partition is compatible with \mathcal{L} ?

Given an edge-coloring f_E on K_n , we say that a set of 2-list assignment is a *2-list covering*

for 3-CCP on (K_n, f_E) if it covers all the solutions of 3-CCP on this instance. Moreover, it is called a *polynomial 2-list covering* if its size is bounded by a polynomial in n . Symmetrically, we want to define a *2-list covering for the stubborn problem*. However, we will not be able to cover all the solutions, so we need a notion of maximal solutions. A solution (A_1, A_2, A_3, A_4) of the stubborn problem on (G, \mathcal{L}) is a *maximal* solution if no member of A_1 satisfies $A_3 \in \mathcal{L}(v)$. This notion is extracted from the notion of dominating set (here A_3 dominates A_1) in the language of general list- M partition problem (see [7]). Intuitively, if $\mathcal{L}(v)$ contains both A_1 and A_3 and v belongs to A_1 in some solution \mathcal{S} , we can build a simpler solution by putting $v \in A_3$ and leaving everything else unchanged. We may note that if A_3 is contained in every $\mathcal{L}(v)$ for $v \in V$, then every maximal solutions of the stubborn problem on (G, \mathcal{L}) let A_1 empty. Now, a set of 2-list assignment is a *2-list covering for the stubborn problem on (G, \mathcal{L})* if it covers all the **maximal** solutions on this instance. Moreover, it is called a *polynomial 2-list covering* if its size is bounded by a polynomial in the number of vertices in G .

In this section, we prove that the existence of a polynomial 2-list covering for the stubborn problem is equivalent to the existence of one for 3-CCP, which in turn is equivalent to the existence of a polynomial CS-separator.

We start by justifying the interest of 2-list covering and observing that we can always find a quasi-polynomial 2-list covering.

Observation 13. *Given a polynomial number of 2-list assignment, it is possible to decide in polynomial time if there exists a solution covered by them.*

Indeed, each 2-list assignment can be translated into an instance of 2-SAT.

Theorem 14. [8] *There exists an algorithm in time $O(n^{\log n})$ which solves 3-CCP. This algorithm gives a 2-list covering of size $O(n^{\log n})$.*

Proof. We give the proof for self completeness. This algorithm consists in testing $n^{\log(n)}$ 2-list assignment which cover all the solutions. Consider the vertex x , without loss of generality we can assume that the majority color of x is A . If there is a solution in which x is colored by A then all its A -neighbors cannot have the color A , hence all its A -neighbors has 2-constraints. And the number of remaining vertices without 2-constraint is at most $2n/3$ (recall that the graph is complete). It is possible to make the same for the remaining vertices. Hence consider the following algorithm, for each vertex x try to find a solution for which x is colored by its majority color. If there is no such solution for all x , then all the vertices have 2-constraints. This algorithm is representable by a research-tree of degree $n + 1$ (one child for the case where no vertex can be colored by its majority color, and one child for each vertex that can be first chosen to be colored by its majority color) and depth $\log(n)$ (we remove 1/3 of the vertices at each node) in which each of the $n^{\log(n)}$ leaves is a 2-list assignment, and the set of all leaves covers all the solutions. \square

The main result of this section is the following theorem:

Theorem 15. *The following are equivalent:*

1. *For all graph G and all list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on (G, \mathcal{L}) .*
2. *For all n and all edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f) .*
3. *For all graph G , there is a polynomial CS-separator.*

We decompose the proof into three lemmas, each of which describing one implication.

Lemma 16. (1 \Rightarrow 2): *Suppose for all graph G and all list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on (G, \mathcal{L}) . Then for all graph n and all edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f) .*

Proof. Let $n \in \mathbb{N}$, (K_n, f) be an instance of 3-CCP, and x a vertex of K_n . Let us build a polynomial number of 2-list assignment that cover all the solutions where x is given color A . Since all the colors are symmetric, we just have to multiply the number of 2-list assignment by 3 to cover all the solutions. Let (A, B, C) be a solution of 3-CCP where $x \in A$.

Claim 17. *Let x be a vertex and u, v, w be the three different colors. Let U be the u -neighborhood of x . If there is a vw -clique Z of U which is not split, then there is no solution where x is colored with u .*

Proof. Indeed, consider a solution in which x is colored with u . All the vertices of Z are of color v or w because they are in the u -neighborhood of x . The vertices colored with v form a w -clique, those colored by w form a v -clique. Hence Z is split. \square

A vertex x is *really 3-colorable* if for each color u , every vw -clique of the u -neighborhood of x is a split graph. If a vertex is not really 3-colorable then, in a solution, it can be colored by at most 2 different colors. Hence if $K_n(V \setminus x)$ has a polynomial 2-list covering, the same holds for K_n by assigning the only two possible colors to x in each 2-list assignment.

Thus we can assume that x is really 3-colorable, otherwise there is a natural 2-constraint on it. Since we assume that the color of x is A , we can consider that in all the following 2-list assignment, the constraint $\{B, C\}$ is given to the A -neighborhood of x . Let us abuse notation and still denote by (A, B, C) the partition of the C -neighborhood of x , induced by the solution (A, B, C) . As x is really 3-colorable, C is a split graph $C' \uplus C''$ with C' a B -clique and C'' a A -clique. The situation is described in Fig. 6(a). Let H be the non-colored graph where the vertices are the vertices of the C -neighborhood of x and there is an edge e if and only if $f(e) = B$ or $f(e) = C$ (see Fig. 6(b)). Moreover, let H' be the non-colored graph where the vertices are also the vertices of the C -neighborhood of x and there is an edge e if and only if $f(e) = B$ (see Fig. 6(c)). We consider (H, \mathcal{L}_0) and (H', \mathcal{L}_0) as two instances of the stubborn problem, where \mathcal{L}_0 is the trivial list assignment that give each vertex the constraint $\{A_1, A_2, A_3, A_4\}$.

By assumption, there exists \mathcal{F} (resp. \mathcal{F}') a polynomial 2-list covering for the stubborn problem on (H, \mathcal{L}_0) (resp. (H', \mathcal{L}_0)). We construct \mathcal{F}'' the set of 2-list assignment f'' built from all the pairs $(f, f') \in \mathcal{F} \times \mathcal{F}'$ according to the rules described in Figure 5. \mathcal{F}'' aims at being a polynomial 2-list covering for 3-CCP on the C -neighborhood of x .

The following is illustrated on Fig. 6(b) and 6(c). Let \mathcal{S} be the partition defined by $A_1 = \emptyset$, $A_2 = C''$, $A_3 = B \cup C'$ and $A_4 = A$. We can check that A_2 is a stable set and A_4 is a clique (the others restrictions are trivially satisfied by A_1 being empty and \mathcal{L}_0 being trivial). In parallel, let \mathcal{S}' be the partition defined by $A'_1 = \emptyset$, $A'_2 = B$, $A'_3 = A \cup C''$ and $A'_4 = C'$. We can also check that A'_2 is a stable set and A'_4 is a clique. Thus \mathcal{S} (resp. \mathcal{S}') is a maximal solution for the stubborn problem on (H, \mathcal{L}_0) (resp. (H', \mathcal{L}_0)) inherited from the solution $(A, B, C = C' \uplus C'')$ for 3-CCP.

Let $f \in \mathcal{F}$ (resp. $f' \in \mathcal{F}'$) be a 2-list assignment compatible with \mathcal{S} (resp. \mathcal{S}'). Then $f'' \in \mathcal{F}''$ built from (f, f') is a 2-list assignment compatible with (A, B, C) .

Doing so for the B -neighborhood of x and pulling everything back together gives a polynomial 2-list covering for 3-CCP on (K_n, f) . \square

$f(v)$	$f'(v)$	$f''(v)$
A_2 or A_1, A_2	*	C
A_3 or A_1, A_3	*	B, C
A_4 or A_1, A_4	*	A
A_2, A_4	*	A, C
A_2, A_3	*	B, C
A_3, A_4	A'_2 or A'_1, A'_2	B
A_3, A_4	A'_3 or A'_1, A'_3	A, C
A_3, A_4	A'_4 or A'_1, A'_4	C
A_3, A_4	A'_2, A'_4	B, C
A_3, A_4	A'_2, A'_3	A, B
A_3, A_4	A'_3, A'_4	A, C

Figure 5: This table describes the rules used in proof of lemma 16 to built a 2-list assignment f'' for 3-CCP from a pair (f, f') of 2-list assignment for two instances of the stubborn problem. Symbol * stands for any constraint. For simplicity, we write X, Y (resp. X) instead of $\{X, Y\}$ (resp. $\{X\}$).

Lemma 18. ($2 \Rightarrow 3$): *Suppose for all n and all edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f) . Then for all graph G , there is a polynomial CS-separator.*

Proof. Let $G = (V, E)$ be a graph on n vertices. Let f be the coloring on K_n defined by $f(e) = A$ if $e \in E$ and $f(e) = B$ otherwise. In the following (K_n, f) is considered as an instance of 3-CCP. By hypothesis, there is a polynomial 2-list covering \mathcal{F} for 3-CCP on (K_n, f) . Let us prove that we can derive from \mathcal{F} a polynomial CS-separator \mathcal{C} .

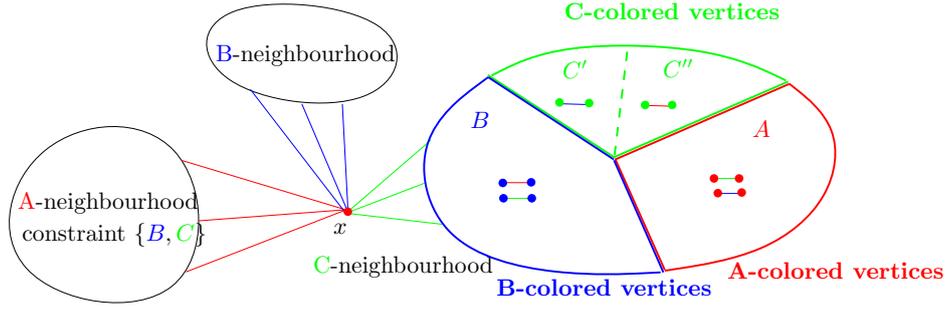
Let $\mathcal{L} \in \mathcal{F}$ be a 2-list assignment. Define X (resp. Y, Z) the set of vertices which have the constraint $\{A, B\}$ (resp. $\{B, C\}, \{A, C\}$). Since no edge has color C , X is split. Indeed, the vertices of color A form a B -clique and conversely. Given a graph, there is a linear number of decompositions into a split graph [7]. Thus there are a linear number of decomposition $(U_k, V_k)_{k \leq cn}$ of X into a split graph where U_k is a B -clique. For all k , the cut $(U_k \cup Y, V_k \cup Z)$ is added in \mathcal{C} . For each 2-list assignment we create a linear number of separators.

Let K be a clique and S a stable set of G which do not intersect. The edges of K are colored by A , and those of S are colored by B . Then the coloration $\mathcal{S}(x) = B$ if $x \in K$, $\mathcal{S}(x) = A$ if $x \in S$ and $\mathcal{S}(x) = C$ otherwise is a solution of (K_n, f) . Left-hand side of Fig. 7 illustrates the situation. There is a 2-list assignment \mathcal{L} in \mathcal{F} which is compatible with this solution. As before, let X (resp. Y, Z) be the set of vertices which have the constraint $\{A, B\}$ (resp. $\{B, C\}, \{A, C\}$). Since the vertices of K are colored B , we have $K \subseteq X \cup Y$ (see right hand-side of Fig. 7). Likewise, $S \subseteq X \cup Z$. Then $(K \cap X, S \cap X)$ forms a split partition of X . So, by construction, there is a cut $((K \cap X) \cup Y, (S \cap X) \cup Z) \in \mathcal{C}$ which ensures that (K, S) is separated by \mathcal{C} . \square

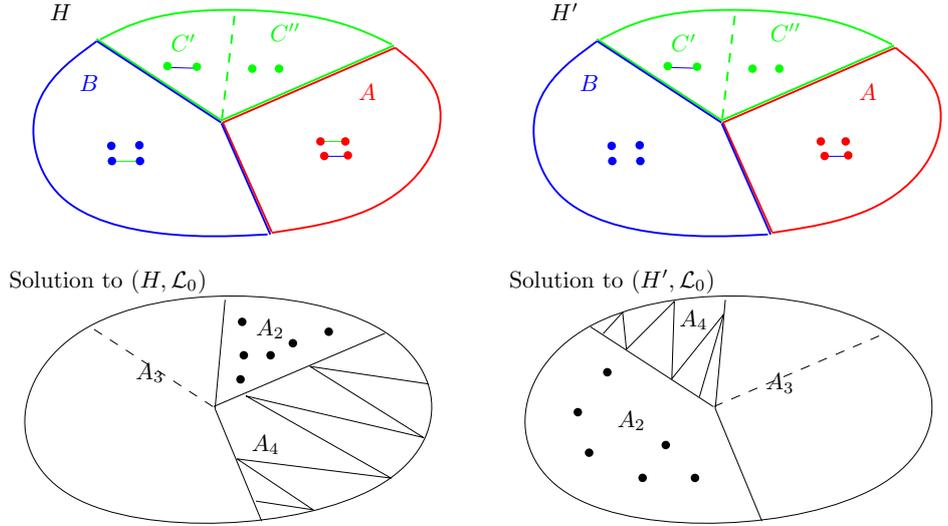
Lemma 19. ($3 \Rightarrow 1$): *Suppose for all graph G , there is a polynomial CS-separator. Then for all graph G and all list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on (G, \mathcal{L}) .*

Proof. Let (G, \mathcal{L}) be an instance of the stubborn problem. By assumption, there is a polynomial CS-separator for G .

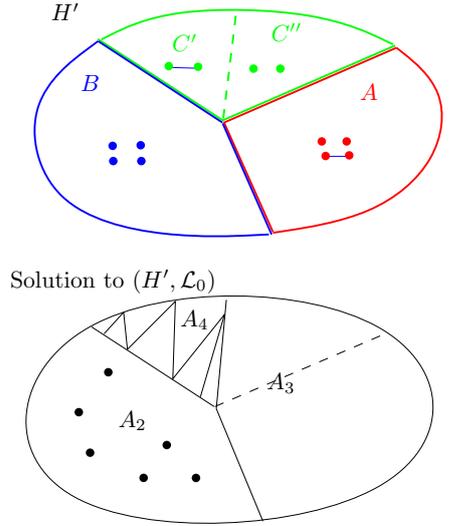
Claim 20. *If there are p cuts that separate all the cliques from the stable sets, then there are p^2 cuts that separates all the cliques from the unions $S \cup S'$ (where S and S' are stable sets).*



(a) Vertex x , its A-neighborhood subject to the constraint $\{B, C\}$, and its C-neighborhood separated in different parts.



(b) Above, the graph H obtained from the C-neighborhood by keeping only B-edges and C-edges. Below, the solution of the stubborn problem.



(c) Above, the graph H' obtained from the C-neighborhood by keeping only B-edges. Below, the solution of the stubborn problem.

Figure 6: Illustration of proof of lemma 16. Color correspondence: A =red ; B =blue ; C =green. As before, cliques are represented by hatched sets, stable sets by dotted sets.

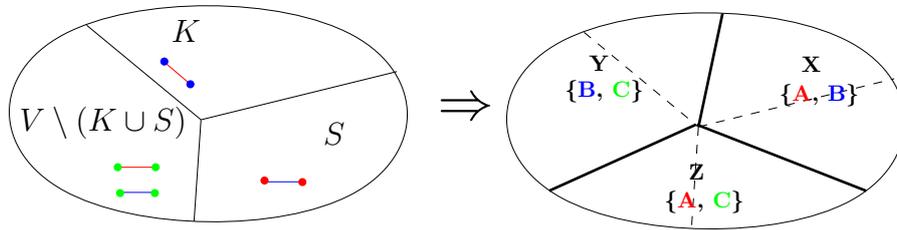


Figure 7: Illustration of proof of lemma 18. On the left hand-side, G is separated in 3 parts: K , S , and the remaining vertices. Each possible configuration of edge- and vertex-coloring are represented. On the right-hand-side, (X, Y, Z) is a 2-list assignment compatible with the solution. X (resp. Y, Z) has constraint $\{A, B\}$ (resp. $\{B, C\}, \{A, C\}$). Color correspondence: A =red ; B =blue ; C =green.

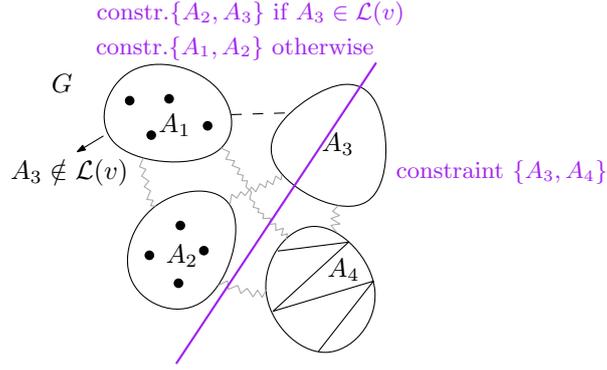


Figure 8: Illustration of proof of lemma 19. A solution to the stubborn problem together with the cut that separates A_4 from $A_1 \cup A_2$. The 2-list assignment built from this cut is indicated in purple.

Proof. Indeed, if (V_1, V_2) separates C from S and (V'_1, V'_2) separates C from S' , then the new cut $(V_1 \cap V'_1, V_2 \cup V'_2)$ satisfies $C \subseteq V_1 \cap V'_1$ and $S \cup S' \subseteq V_2 \cup V'_2$. \square

Let \mathcal{F}_2 be a polynomial family of cuts that separate all the cliques from union of two stable sets, which exists by Claim 20. Then for all $(U, W) \in \mathcal{F}_2$, we build the following 2-list assignment \mathcal{L}' :

1. If $v \in U$, let $\mathcal{L}'(v) = \{A_3, A_4\}$.
2. If $v \in W$ and $A_3 \in \mathcal{L}(v)$, then let $\mathcal{L}'(v) = \{A_2, A_3\}$.
3. Otherwise, $v \in W$ and $A_3 \notin \mathcal{L}(v)$, let $\mathcal{L}'(v) = \{A_1, A_2\}$.

Now the set \mathcal{F}' of such 2-list assignment \mathcal{L}' is a 2-list covering for the stubborn problem on (G, \mathcal{L}) : let $\mathcal{S} = (A_1, A_2, A_3, A_4)$ be a maximal solution of the stubborn problem on this instance. Then A_4 is a clique and A_1, A_2 are stable sets, so there is a separator $(U, W) \in \mathcal{F}_2$ such that $A_4 \subseteq U$ and $A_1 \cup A_2 \subseteq W$ (see Fig. 8), and there is a corresponding 2-list assignment $\mathcal{L}' \in \mathcal{F}'$. Consequently, the 2-constraint $\mathcal{L}'(v)$ built from rules 1 and 3 are compatible with \mathcal{S} . Finally, as \mathcal{S} is maximal, there is no $v \in A_1$ such that $A_3 \in \mathcal{L}(v)$: the 2-constraints built from rule 2 are also compatible with \mathcal{S} . \square

Proof of theorem 15. Lemmas 16, 18 and 19 concludes the proof of Theorem 15. \square

5 Link with the oriented Alon-Saks-Seymour conjecture

Given a graph G , the *chromatic number* χ of G is the minimum number of colors needed to color the vertices such that, two vertices connected by an edge do not have the same color. The *bipartite packing* $bp(G)$ of a graph G is the minimum number of edge-disjoint complete bipartite graphs needed to cover the edges of G . The Alon-Saks-Seymour conjecture states the following.

Conjecture 21. (*Alon, Saks, Seymour*) *If $bp(G) \leq k$, then $\chi \leq k + 1$.*

The result is a well-known result for complete graphs, due to Graham and Pollak [12]. Indeed, $n - 1$ edge-disjoint complete bipartite graphs are needed to cover the edges of a complete graph on n vertices. A beautiful proof of this theorem is due to Tverberg [24]. This conjecture

was disproved by Huang and Sudakov in [15] who prove that for some graphs $\chi \geq k^{6/5}$ using a construction based on Razborov's graphs [21]. Nevertheless the existence of a polynomial bound is still open. In the following we will use an oriented version of the Alon-Saks-Seymour conjecture.

Conjecture 22. (*Oriented Alon-Saks-Seymour Conjecture*) *There exists a polynomial P such that if G is a disjoint union of k complete oriented bipartite graphs, then the chromatic number χ is at most $P(k)$.*

Theorem 23. *The oriented Alon-Saks-Seymour conjecture is verified if and only if the Clique-Stable set separation conjecture is verified.*

Lemma 24. *If the oriented Alon-Saks-Seymour conjecture is verified, then the Clique-Stable set separation conjecture is verified.*

Proof. Let G be a graph on n vertices. We want to separate all the pairs of cliques and stable sets which do not intersect. Consider all the pairs (C, S) such that the clique C does not intersect the stable set S . Construct an auxiliary graph H as follows. The vertices of H are the pairs (C, S) and there is an oriented edge between a pair (C, S) and a pair (C', S') if and only if there is a vertex $x \in S \cap C'$. Observe that the number of vertices of this graph is at most 4^n . The bipartite packing of this graph is at most the number n of vertices of G . Indeed, let H_x be the graph H restricted to the edges for which $x \in S \cap C'$. Observe that the union of the graphs H_x for all x is the graph H because if $(C, S)(C', S')$ is an edge $S \cap C' \neq \emptyset$. In addition, the graph H_x is a complete bipartite graph: if there is an edge which starts in (C, S) and if there is an edge which ends in (C', S') then $x \in S$ and $x \in C'$ and finally there is an oriented edge $(C, S)(C', S')$. The graphs H_x and H_y cannot share an oriented edge because otherwise the intersection between a clique and a stable set would be at least 2 which is impossible. Hence the bipartite packing of this graph is at most n .

If the oriented Alon-Saks-Seymour conjecture is verified, $\chi(H) < P(n)$. Consider a color of this polynomial coloring. Let A be the set of vertices of this color. There is no edge between two vertices of A , then the union of all the second components (stable sets) of the vertices of A do not intersect the union of all the first components (cliques) of A . Indeed, if they intersect, there is a clique C which intersects a stable set S , hence there is an edge which is impossible.

The union of the cliques and the union of the stable sets do not intersect, hence it defines a cut which separates all the pairs of A . The same can be done for every colors. Then we can separate all the pairs (C, S) by $\chi(H)$ cuts, which is a polynomial in n if the Alon-Saks-Seymour conjecture is verified. This achieves the proof of the first implication. \square

Lemma 25. *If the Clique-Stable set separation conjecture is verified, then the oriented Alon-Saks-Seymour conjecture is verified.*

Proof. Let $G = (V, E)$ be an oriented graph with bipartite packing k . Construct an auxiliary graph H as follows. It has k vertices which are the oriented complete bipartite graphs that cover the edges of G . There is an edge between two pairs (A_1, B_1) and (A_2, B_2) if and only if there is a vertex $x \in A_1 \cap A_2$. Hence the complete bipartite graphs in which x appears at the left form a clique of H (say the clique C_x associated to x) and the complete bipartite graphs for which y appears at the right form a stable set in H (say the stable set S_y associated to y). Indeed, it is quite clear for the clique, and it is also true for the stable set because if $y \in B_1 \cap B_2$ and there is an edge resulting from $x \in A_1 \cap A_2$, then the edge xy is covered twice which is impossible. Note that a clique or a stable set associated to a vertex can be empty, but this does not trigger any problem.

By assumption there are $P(k)$ (with P a polynomial) cuts which separate all the pairs (C, S) , in particular which separate all the pairs (C_x, S_x) for $x \in V$. We color each vertex x by the color of the cut separating (C_x, S_x) . This coloring is proper: assume there is an edge from x to y , and that x and y are given the same color. Then there exists a bipartite graph (A, B) that cover the edge xy , hence (A, B) is in the clique associated to x and in the stable set associated to y , which means they intersect: no cut can separate at the same time both C_x from S_x and C_y from S_y , because it would then separate C_x from S_y . This is impossible. Then we have a coloring with at most $P(k)$ colors, which is a polynomial in k . \square

Corollary 26. *The oriented Alon-Saks-Seymour conjecture is equivalent to the Clique-Stable set separation conjecture. Moreover, they are also equivalent to having a polynomial 2-list covering for 3-CCP, and to having a polynomial 2-list covering for the stubborn problem.*

6 Relations between the t -biclique covering numbers

We study here a natural generalization of the Alon-Saks-Seymour conjecture, introduced by Huang and Sudakov in [15]. While the Alon-Saks-Seymour conjecture deals with partitioning the edges, we relax here to a covering of the edges by complete bipartite graphs, meaning that an edge can be covered several times. Formally, a t -biclique covering of an undirected graph G is a collection of bicliques that cover every edge of G at least once and at most t times. The minimum size of such a covering is called the t -biclique covering number, and is denoted by $\mathbf{bp}_t(G)$. In particular, $\mathbf{bp}_1(G)$ is the usual biclique partition number $\mathbf{bp}(G)$.

In addition to being an interesting parameter to study in its own right, the t -biclique covering number of complete graphs is also closely related to a question in combinatorial geometry about neighborly families of boxes. It was studied by Zaks [16] and then by Alon [2], who proved that \mathbb{R}^d has a t -neighborly family of k standard boxes if and only if the complete graph K_k has a t -biclique covering of size d . Alon gives also asymptotic bounds for $\mathbf{bp}_t(K_k)$:

$$(1 + o(1))(t!/2^t)^{1/t} k^{1/t} \leq \mathbf{bp}_t(K_k) \leq (1 + o(1))tk^{1/t} .$$

Our results are concerned not only with K_k but for every graph G . It is natural to ask the same question for $\mathbf{bp}_t(G)$ as for $\mathbf{bp}(G)$, namely:

Conjecture 27 (Generalized Alon-Saks-Seymour conjecture of order t). *There exists a polynomial P such that for all graphs G , $\chi(G) \leq P(\mathbf{bp}_t(G))$.*

Observation 28. *Since a t -biclique covering is a fortiori a t' -biclique covering for all $t' \geq t$, we have the following inequalities:*

$$\dots \leq \mathbf{bp}_{t+1}(G) \leq \mathbf{bp}_t(G) \leq \mathbf{bp}_{t-1}(G) \leq \dots \mathbf{bp}_1(G) .$$

In particular, if the generalized Alon-Saks-Seymour conjecture of order t holds, then $\chi(G)$ is bounded by a polynomial in $\mathbf{bp}_t(G)$ and thus by a polynomial in $\mathbf{bp}_1(G)$, so the generalized Alon-Saks-Seymour of order 1 holds.

We prove that the reverse is also true.

Theorem 29. *Let $t \in \mathbb{N}^*$. The generalized Alon-Saks-Seymour conjecture of order t holds if and only if it holds for order 1.*

Before going to the proof, we need a few definitions: let G be a graph and let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a family of bipartite complete graphs which covers the edges of G . Given an edge e , the *multiplicity* $m(e)$ of e is the number of bipartite graphs which contain the edge e . We associate a *positive side* and a *negative side* to each bipartite graph, meaning that $B_i = (B_i^+, B_i^-)$ with B_i^+ (resp. B_i^-) being a stable set and referred to as the positive (resp. negative) side of B_i . Given an edge xy , two *half edges* are associated to xy : (xy, x) and (xy, y) . If B_i covers the edge xy , the half edge (xy, x) is a *positive (resp. negative) half edge* for B_i if x is in the positive (resp. negative) side of the bipartite graph and y in the other side. The *vertex associated to the half edge* (e, x) is the vertex x . A *tag* is a pair consisting into a bipartite graph $B_i \in \mathcal{B}$ and a sign (+ or -). The *opposite tag* of a tag is the same bipartite graph and the opposite sign. A *bag* is a set of tags and two bags X and Y are *opposite* if for each tag in X , the opposite tag is in Y .

Proof of Theorem 29. As Observation 28 proves one direction, we focus on the other, and assume that the generalized Alon-Saks-Seymour conjecture of order 1 holds. Let us prove the result by induction on t , initialization for $t = 1$ being obvious. Let G be a graph and let $\mathcal{B} = (B_1, \dots, B_k)$ be a t -blique covering. Since each edge is covered by at most t bipartite graphs, each half edge (e, x) of the graph can be represented by a bag with at most t tags, containing all the $(B_i, +)$ such that (e, x) is a positive half edge in B_i , and all the $(B_j, -)$ such that (e, x) is a negative half edge in B_j . Note that there are $2k$ possible tags, thus the total number of different bags is at most $(2k)^t$.

Partition the set of half edges into $\{P_1, \dots, P_m\}$ with $m = (2k)^t$, in such a way that two half edges are in the same P_j if and only if they have exactly the same bag. The vertices *associated to* P_j are the vertices associated to the half edges which appear in P_j . In a first time we will only consider the edges which are covered exactly t times. Let P_j and $P_{j'}$ be two parts such that their induced bags X and \bar{X} are opposite and of size t , and let U and U' be their set of associated vertices, respectively. Observe that the bipartite graphs appearing in X and in \bar{X} are the same, and that the vertices of U and U' appear in all these bipartite graphs. Note first that U and U' are disjoint: indeed, if $x \in U \cap U'$, then there exists $(e, x) \in P_j$ and $(e', x) \in P_{j'}$. For all tag (B_i, \pm) in X , (B_i, \mp) is in \bar{X} , thus x has to be on both side of B_i , which is impossible. Moreover, there is a complete bipartite graph between U and U' in G . Indeed, for all $x \in U$, $x' \in U'$ and for all bipartite graph B_i in X , x is on one side on B_i and x' on the other so the edge xx' is in B_i .

Select one bipartite graph (it can be any of those appearing in X) and call it B'_j . It covers all the edges between U and U' . We do the same for all pair (X, \bar{X}) of opposite bags of size t and get a family \mathcal{B}' of complete bipartite graphs. All the edges of the graph G of multiplicity t are covered by this set of bipartite graphs. Indeed, consider an edge xy of multiplicity t , the two half edges (xy, x) and (xy, y) have two opposite bags of size t . Then by construction, they are covered by one of the B'_j . Let us prove that each edge is covered at most once. Indeed, if xy is covered by B'_j , then the two half-edges are appearing in two opposite bags (X, \bar{X}) of size t , thus there are already t bipartite graphs B_i which cover xy . If it is also covered by another bipartite graph $B'_{j'}$ selected from a different pair of opposite bags (Y, \bar{Y}) , it means that, there is a bipartite graph $B_{i'}$ which is not in the bag X but in the bag Y (since all the bags have size t) which covers the edge xy . Thus, the edge xy is covered by at least $t + 1$ bipartite graphs which contradicts the hypothesis on t . Thus this set of bipartite graphs cover the edges of multiplicity t with a multiplicity one.

As we assumed that the generalized Alon-Saks-Seymour conjecture of order 1 holds, it means that the graph restricted to this set of bipartite graphs \mathcal{B}' has chromatic number at most $P((2k)^t)$ (with P a polynomial) since \mathcal{B}' ensures $\mathbf{bp}_1(G) \leq (2k)^t$. Thus the vertex set can be partitionned into $P((2k)^t)$ stable sets $S_1, \dots, S_{P((2k)^t)}$. Since all the edges of multiplicity t in \mathcal{B} are covered, it

means that the multiplicity of the edges of G in each part S_i is at most $t - 1$. Hence by induction hypothesis, it means that the chromatic number of each S_i is bounded by a polynomial Q in $\mathbf{bp}_{t-1}(S_i)$. As S_i contains no edge of multiplicity t , \mathcal{B} restricted to the vertices of S_i ensures that $\mathbf{bp}_{t-1}(S_i) \leq (2k)^t$. Thus the chromatic number of G seen as a product graph is at most $P \cdot Q(2k^t)$, which is a polynomial in k . Thus the generalized Alon-Saks-Seymour conjecture of order t holds if and only if it holds for order one. \square

7 Conclusion

These results are interesting due to the link they make between some distant areas of theoretical computer science such as communication complexity, graph theory, constraint satisfaction problem, and even polytope geometrics, via an equivalence between long-standing open problems in each area. It has been somehow fascinating to explore such a wide range of domains and to see links appearing between them. The main question is now of course to prove or disprove one of these equivalent problems. Our results are a step forward, enabling anyone to chose his favorite domain between the three involved ones.

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Appendices

A Detailed proof on random graphs

We recall Proposition 4 and give a detailed proof with all the computations and Taylor series.

Proposition 30. *Let $a(n)$ (denoted by a in the following) be a function of n such that $2 \leq a(n) \ll \log n$. The random graph $G(n, p)$ with $p = 1 - n^{-2/a} = 1 - 2^{-2 \log n/a}$ has:*

- independent number $\alpha = a - \frac{a \log a}{\log n} + 1 + o(1)$
- clique number $\omega = 2 \ln 2 \cdot 2^{2 \log n/a} \cdot \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n)$.

Moreover, for all $\varepsilon > 0$, with high probability there exists a family of $\mathcal{O}(n^{6+\varepsilon})$ cuts (more precisely $\mathcal{O}(n^{5+2/a+\varepsilon})$ if a is a constant, and $\mathcal{O}(n^{5+\varepsilon})$ if $a(n) \rightarrow_{n \rightarrow \infty} \infty$) that separates all the cliques and the stable sets.

Proof. Let $p' = 1 - p$, $b = 1/p$ and $b' = 1/p'$. The independence number and clique number of $G(n, p)$ are given by the following formulas, depending on p (see [3]):

$$\begin{aligned}\omega &= 2 \log_b(n) - 2 \log_b \log_b n + 2 \log_b(e/2) + 1 + o(1) \\ \alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1)\end{aligned}$$

Now $\frac{1}{\log b'} = \frac{a}{2 \log n}$, thus:

$$\begin{aligned}\alpha &= 2 \log_{b'}(n) - 2 \log_{b'} \log_{b'} n + 2 \log_{b'}(e/2) + 1 + o(1) \\ &= a - \frac{a}{\log n} \log \left(\frac{a}{2} \right) + 1 + o(1) \\ &= a - \frac{a \log a}{\log n} + 1 + o(1)\end{aligned}$$

Moreover, thanks to Taylor series we get:

$$\begin{aligned}\frac{1}{\log b} &= \frac{-1}{\log(1 - 2^{-2 \log n/a})} && \text{by definition of } b \\ &= \frac{-\ln 2}{-2^{-2 \log n/a} + \mathcal{O}(2^{-4 \log n/a})} && \text{using } \ln(1+x) = x + \mathcal{O}(x^2) \\ &= \frac{\ln 2 \cdot 2^{2 \log n/a}}{1 + \mathcal{O}(2^{-2 \log n/a})} && \text{by factorization} \\ &= \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) && \text{using } \frac{1}{1-x} = 1 + \mathcal{O}(x)\end{aligned}$$

Thus, let us look at the different terms in the approximation of ω :

- $2 \log_b n = 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) \cdot \log n$
 $= 2 \ln 2 \cdot 2^{2 \log n/a} \log n + \mathcal{O}(\log n)$

- $-2 \log_b \log_b n = -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a})) \cdot (\log \log n - \log \log b)$

by substitution of $\log b$

$$= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a}))$$

$$\cdot (\log \log n + \log \ln 2 - \log(2^{-2 \log n/a}(1 + \mathcal{O}(2^{-2 \log n/a}))))$$

by previous computation

$$= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (1 + \mathcal{O}(2^{-2 \log n/a}))$$

$$\cdot (\log \log n + \log \ln 2 + \frac{2 \log n}{a} + \mathcal{O}(2^{-2 \log n/a}))$$

using $\ln(1+x) = x + \mathcal{O}(x^2)$

$$= -2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log \log n + \log \ln 2 + \frac{2 \log n}{a}) + \mathcal{O}(\log n)$$

by developping.

- $2 \log_b(e/2) + 1 + o(1) = 2 \log(e/2) \ln 2 \cdot 2^{2 \log n/a} + \mathcal{O}(1)$

Hence:

$$w = 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n - \log \ln 2 + \log(e/2)) + \mathcal{O}(\log n)$$

$$= 2 \ln 2 \cdot 2^{2 \log n/a} \cdot (\log n - \frac{2 \log n}{a} - \log \log n) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n)$$

Now, let $x = 1 - p = 2^{-2 \log n/a}$. Draw a random partition (V_1, V_2) where each vertex is put in V_1 independently from the others with probability x . Let (C, S) be a pair of clique and stable set of the graph. There are at most 4^n such pairs. The probability that $S \subseteq V_1$ and $C \subseteq V_2$ is $x^\alpha(1-x)^\omega$. Let us show that $x^\alpha \geq n^{-(2+2/a+\varepsilon')}$ and $(1-x)^\omega \geq n^{-(2+\varepsilon')}$ for all $\varepsilon' > 0$.

On one hand,

$$\begin{aligned} x^\alpha \geq n^{-(2+2/a+\varepsilon')} &\Leftrightarrow \alpha \log x \geq -(2 + 2/a + \varepsilon') \log n \\ &\Leftrightarrow (a - \frac{a \log a}{\log n} + 1 + o(1)) \cdot \frac{-2 \log n}{a} \geq -(2 + 2/a + \varepsilon') \log n \\ &\Leftrightarrow 2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2 + 2/a + \varepsilon') \log n \end{aligned}$$

which is true if n is large enough.

Note that for $a \rightarrow_{n \rightarrow \infty} \infty$, we even have $2 \log n + \frac{2 \log n}{a} + o(\log n) \leq (2 + \varepsilon') \log n$ for n large enough, which implies $x^\alpha \geq n^{-(2+\varepsilon')}$.

On the other hand, $\log(1-x) = \log b$, thus, using the previous approximations:

$$\begin{aligned}
(1-x)^\omega \geq n^{-(2+\varepsilon)} &\Leftrightarrow \omega \log(1-x) \geq -(2+\varepsilon) \log n \\
&\Leftrightarrow \left(2 \ln 2 \cdot 2^{2 \log n/a} \cdot \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(2^{2 \log n/a}) + \mathcal{O}(\log n) \right) \\
&\quad \cdot \left(\frac{-2^{-2 \log n/a}}{\ln 2} + \mathcal{O}(2^{-4 \log n/a}) \right) \geq -(2+\varepsilon) \log n \\
&\Leftrightarrow 2 \left(\log n - \frac{2 \log n}{a} - \log \log n \right) + \mathcal{O}(1) + \mathcal{O}(2^{-2 \log n/a} \log n) \leq (2+\varepsilon) \log n
\end{aligned}$$

which is true if n is large enough.

This leads us to $x^\alpha (1-x)^\omega \geq n^{-(5+2\varepsilon')} = n^{-(5+\varepsilon)}$ for $\varepsilon' = \varepsilon/2$. Thus (C, S) is separate by at least $n^{-(5+\varepsilon)}$ of all the partitions. By double counting, there exists a partition that separates at least $n^{-(5+\varepsilon)}$ of all the pairs. We delete these separated pairs, and there remains at most $(1 - 1/n^{5+\varepsilon}) \cdot 4^n$ pairs. The same probability for a pair (C, S) to be cut by a random partition still holds, hence we can iterate the process k times until $(1 - 1/n^{5+\varepsilon})^k \cdot 4^n \leq 1$. This is satisfied by $k = 2n^{6+\varepsilon}$ which is a polynomial in n . If $a \rightarrow_{n \rightarrow \infty} \infty$, we can do the same and get $k = 2n^{5+\varepsilon}$. As a conclusion, we have a polynomial family of random cuts that separates all the cliques and the stable sets. \square