

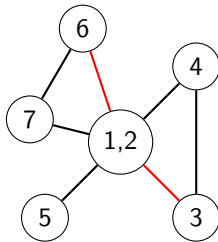
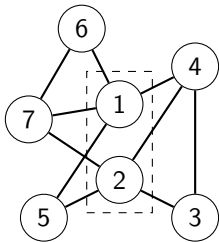
Twin-width One

Hugo Jacob

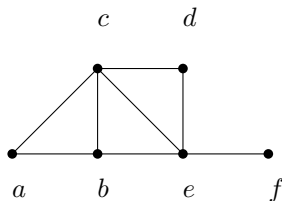
joint work with Junggho Ahn, Noleen Köhler, Christophe Paul,
Amadeus Reinald, and Sebastian Wiederrecht



Twin-width



Twin-width



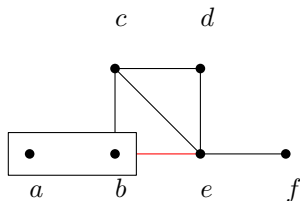
Definition

A d -contraction sequence is a sequence of trigraphs obtained by iterative contraction of pairs of vertices such that the red degree of a vertex is at most d .

Definition

The twin-width of a graph is the minimum d such that the graph admits a d -contraction sequence.

Twin-width



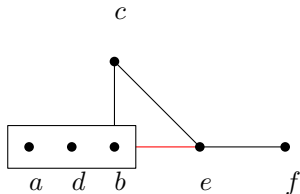
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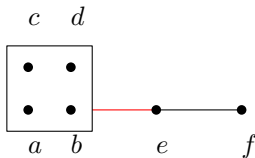
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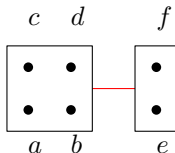
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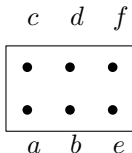
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Definition

The twin-width of a graph is the minimum d such that the graph admits a d -contraction sequence.

Motivations for twin-width

Pros:

- ▶ A partial extension of sparsity theory to classes of dense graphs.
- ▶ Tractability dichotomy for FO in ordered structures.

Cons:

- ▶ Computing the twin-width exactly is NP-hard (already for $d = 4$).
- ▶ There is no known approximation algorithm

Small values of twin-width

- ▶ $d = 0$: Cographs = P_4 -free graphs, linear time recognition.
- ▶ $d \leq 1$: Polynomial time recognition, have a linear structure.

This talk:

Theorem

Twin-width 1 graphs can be recognized in linear time.

Previous algorithm

Main observation:

Lemma

If a graph admits a 1-contraction sequence, it admits a 1-contraction sequence with at most one red edge per trigraph.

Algorithm sketch:

```
while the graph is not fully contracted:  
    Guess a pair of vertices to contract.  
    if a red edge appears:  
        keep contracting greedily on the red edge,  
        then contract the red edge.
```

Cographs and modular decompositions

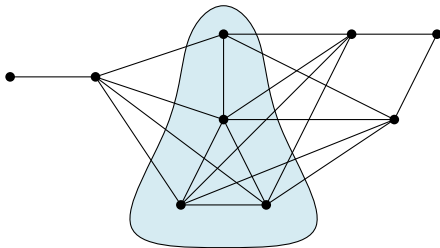
Cographs can be characterised in two ways:

- ▶ A cograph is a single vertex graph or a graph G obtained from two cographs H_1 and H_2 by either the **disjoint union** of H_1 and H_2 or adding all edges between H_1 and H_2 (**join**).
- ▶ A cograph is a graph that admits a **twin**-elimination ordering.

Both characterisations follow from a top-down or bottom-up description of the cotree of the cograph

Definition (Module)

A module M is a subset of vertices such that every vertex not in M is adjacent either to all vertices of M , or no vertex of M .



Modules

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A module M is a subset of vertices such that every vertex not in M is adjacent either to all vertices of M , or no vertex of M .

A graph is prime if it has only trivial modules.

Observation

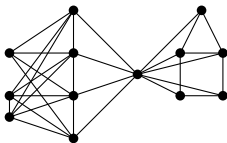
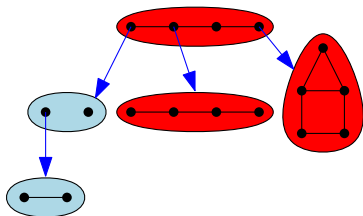
A contraction in a module creates **red** edges only within the module.

Definition (Splitter)

A splitter with respect to M , is a vertex v such that $vx \in E(G), vy \notin E(G)$ for $x, y \in M$. It certifies that M is not a module.

Modular decomposition

For every graph, its modular decomposition is a canonical object that recursively decomposes it into **prime** graphs, **disjoint unions**, and **joins**.



Lemma

The twin-width of a graph is the maximum twin-width over the prime nodes of its modular decomposition.

Construct the contraction sequence by following the modular decomposition bottom-up. We iteratively contract **twins**, and contract **prime** nodes using their optimal sequence.

Complementation

Observation

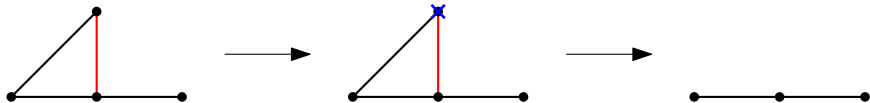
Twin-width is invariant under complementation.

A contraction applied to G or to \overline{G} produces the same red edges.

Induced subgraphs

Observation

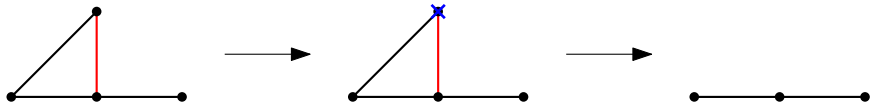
Twin-width is not increased by deletion of vertices.



Induced subgraphs

Observation

Twin-width is not increased by deletion of vertices.

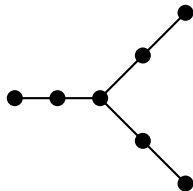
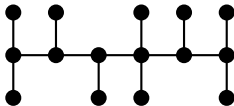
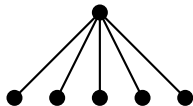


The class of graphs of twin-width at most c is hereditary.

Restriction to trees

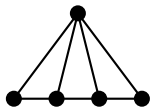
A tree has:

- ▶ twin-width 0 if and only if it is a star.
- ▶ twin-width at most 1 if and only if it is a caterpillar.
- ▶ twin-width at most 2.

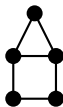


Some examples of twin-width 1 graphs

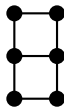
Small examples



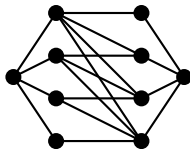
Gem



House



Domino



Perfect graphs

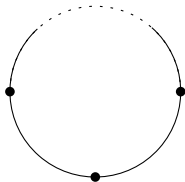
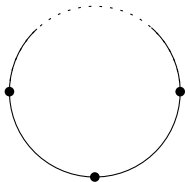
Theorem (Strong perfect graph theorem)

Perfect graphs are exactly the graphs that do not contain odd holes nor odd antiholes as induced subgraphs.

Perfect graphs

Theorem (Strong perfect graph theorem)

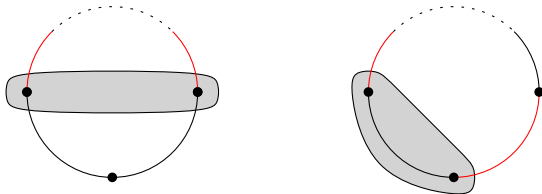
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Observation

Holes and antiholes have twin-width **2**.

Perfect graphs

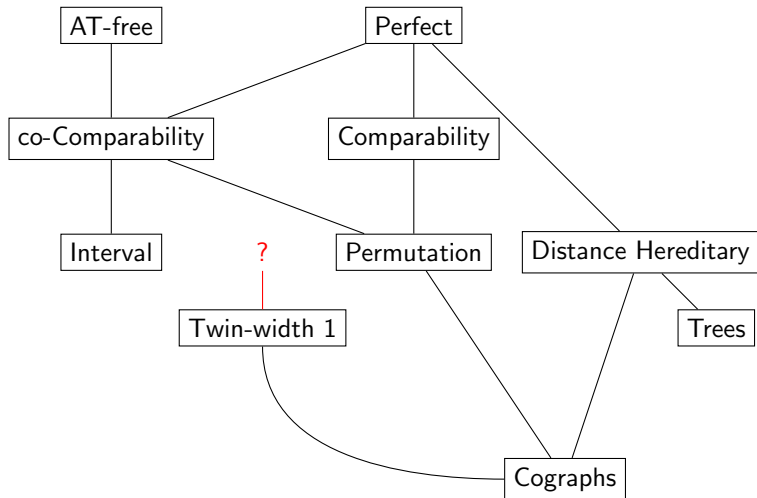
Theorem (Strong perfect graph theorem)

Perfect graphs are exactly the graphs that do not contain odd holes nor odd antiholes as induced subgraphs.

Observation

Holes and antiholes have twin-width 2.
In particular, twin-width 1 graphs are perfect.

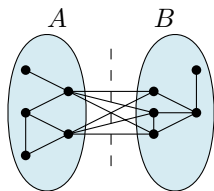
Classes of perfect graphs



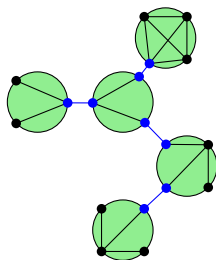
Characterisations of DH

- ▶ (House,Hole,Domino,Gem)-free graphs
- ▶ Distances are preserved in every connected induced subgraph.
- ▶ Admit a sequential elimination of twins and pendant vertices.
- ▶ Fully decomposable with respect to the split decomposition.

Split decomposition



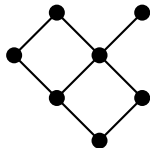
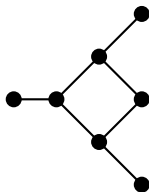
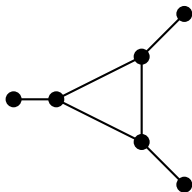
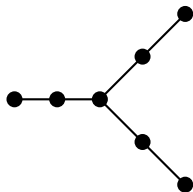
$$|A|, |B| \geq 2$$



Asteroidal triple

Definition

An asteroidal triple is a triple of vertices such that any pair is connected by a path that does not dominate the third.



Twin-width and asteroidal triples

Observation

For trees, asteroidal triples characterise twin-width 2.

Observation

ATs imply twin-width at least 2.

Can be checked by computing the twin-width of all minimal graphs containing an AT.

Lemma

Graphs that are AT-free and DH have twin-width at most 1.

Easy induction on split decomposition tree.

Lemma

DH graphs have twin-width at most 2.

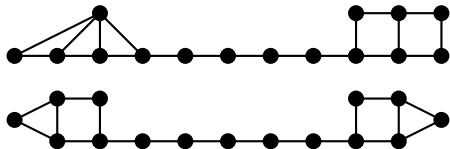
Follows the proof for trees, thanks to the split decomposition tree.

Theorem

We can characterise the twin-width of a DH graph by the structure of its split decomposition tree. In particular, asteroidal triples characterise twin-width 2.

Distance-Hereditary graphs

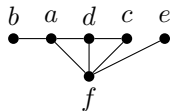
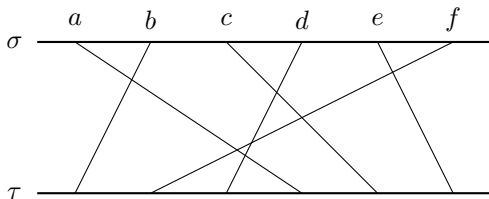
Some obstructions to twin-width 1



Definition

Definition

Given a permutation $\pi : [n] \rightarrow [n]$, we define the graph $G[\pi] = ([n], E_\pi)$ where E_π is the set of inversions of π (i.e. $xy \in E_\pi$ iff $x < y$ and $\pi(x) > \pi(y)$).



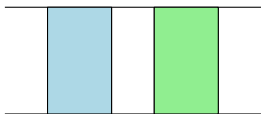
Warm up: Cographs

Lemma

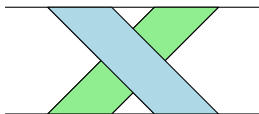
Cographs are permutation graphs.

Induction on the cotree with two cases:

Disjoint union



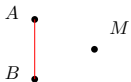
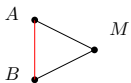
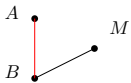
Join



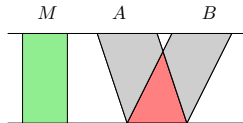
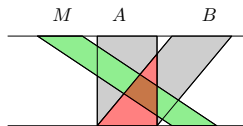
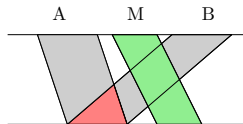
Twin-width 1 graphs are permutation graphs

Induction on the contraction sequence:

3-vertex trigraph



Permutation diagram



Twin-width 1 graphs are permutation graphs

Vertices contracted together are consecutive in the diagram.

Lemma

If G admits a 1-contraction sequence G_n, \dots, G_1 , then there exists a realiser (σ, τ) such that:

- ▶ *if x is a vertex of G_i , then vertices of G that are in x form an interval of σ or τ .*
- ▶ *if xy is a red edge of G_i , then vertices of G that are in xy form an interval of σ or τ*

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Corollary

If the graph is prime, an extremal vertex of the diagram can be contracted last. There are only 4 vertices to try!

Exploiting arbitrary realisers

- ▶ For prime graphs, the realiser is unique (up to symmetry) so all realisers have the consecutivity property.
- ▶ For cographs, we can always find a pair of consecutive twins.

Corollary

For any realiser (σ, τ) of a twin-width 1 graph, there exists a 1-contraction sequence that contracts only vertices consecutive in σ or τ .

We do not need to find a specific permutation diagram!

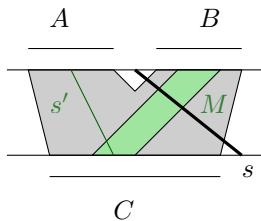
High level description

Algorithm sketch:

```
Compute permutation diagram, reject when not permutation graph
Compute modular decomposition
for each prime node of the decomposition:
    Guess last vertex  $s$  to be contracted
    Try to decompose recursively using  $s$  as splitter
```


The algorithm

Prime case description



```
while an endpoint of  $C$  is also an endpoint of  $A$  or  $B$ :  
    Remove it and move indices correspondingly  
if  $|A| > 1$  and  $|B| > 1$ : Reject  
else if  $|A| > 1$  or  $|B| > 1$ :  
    Deduce  $(M, s')$  from  $(A, B)$   
    Recurse on  $(M, s')$  with  $s'$  as splitter  
else: Accept
```

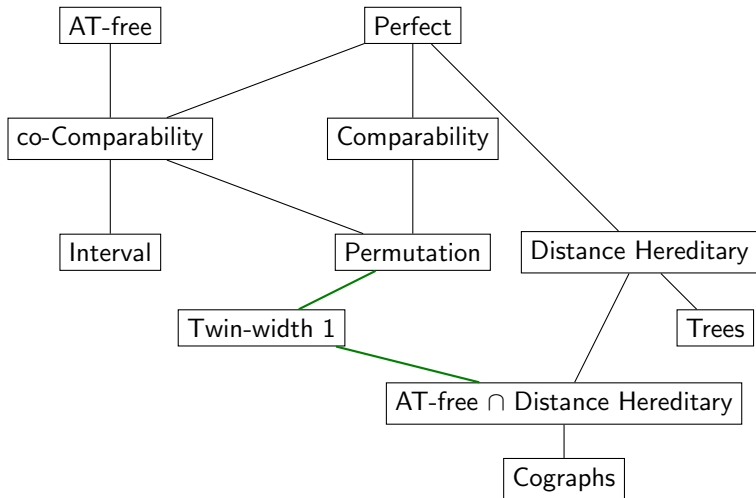
Structural analysis

We consider separately the modular decompositions of $G[A]$ and $G[B]$.

- ▶ Only one side can have a prime node.
- ▶ There cannot be two disjoint modules.
- ▶ There is at most one splitter.

We conclude that the algorithm can always decompose twin-width 1 graphs.

Our results



Going further

Our results:

- ▶ Linear time recognition algorithm for twin-width **1**
- ▶ Permutation diagram and consecutivity properties

Open questions:

- ▶ Can we recognize twin-width **2** graphs in polynomial time?
- ▶ Is recognition of twin-width **3** NP-hard?
- ▶ Can we approximate twin-width in XP time?
- ▶ Can we compute the twin-width of permutation graphs in polynomial time?