

# Bounding twin-width for bounded-treewidth graphs, planar graphs, and bipartite graphs

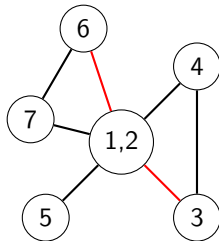
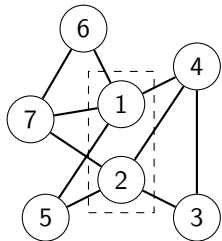
Hugo Jacob

Joint work with Marcin Pilipczuk

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## Twin-width



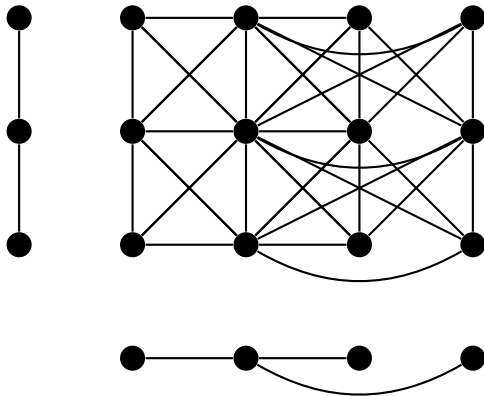
## Strong product

### Definition (Strong product)

Given two graphs  $G, H$ , their strong product is the graph  $G \boxtimes H$  on vertex set  $V(G) \times V(H)$  with edges  $\{(u, u'), (v, v')\}$  such that:

- ▶  $u = v$  or  $uv \in E(G)$ ; and
- ▶  $u' = v'$  or  $u'v' \in E(H)$ .

## Strong product



## Product structure of planar graphs

Using a construction of Pilipczuk and Siebertz [5], Dujmović et al [3] show:

### Theorem

*Every planar graph is a subgraph of:*

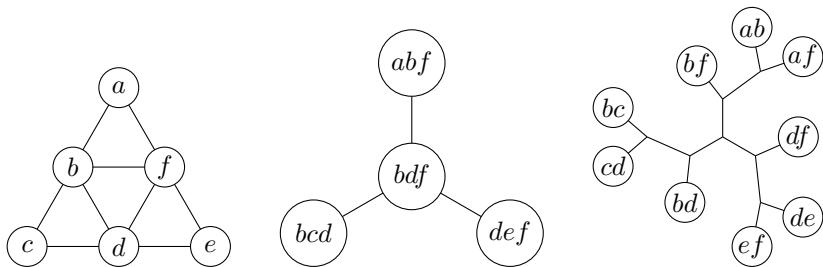
- 1.  $H \boxtimes P$  for some planar graph  $H$  of treewidth at most 8 and some path  $P$ .*
- 2.  $H \boxtimes P \boxtimes K_3$ , for some planar graph  $H$  of treewidth at most 3 and some path  $P$ .*

Ueckerdt et al [6] improved the first result with:

### Theorem

*If  $G$  is planar then  $G$  is a subgraph of  $H \boxtimes P$ , where  $P$  is a path and  $H$  is a minor of  $G$  of treewidth at most 6.*

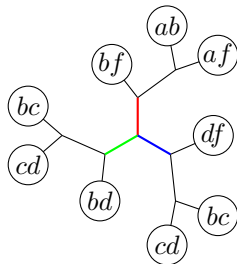
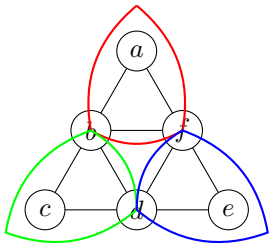
## Treewidth and branchwidth



Theorem (Robertson and Seymour 1991)

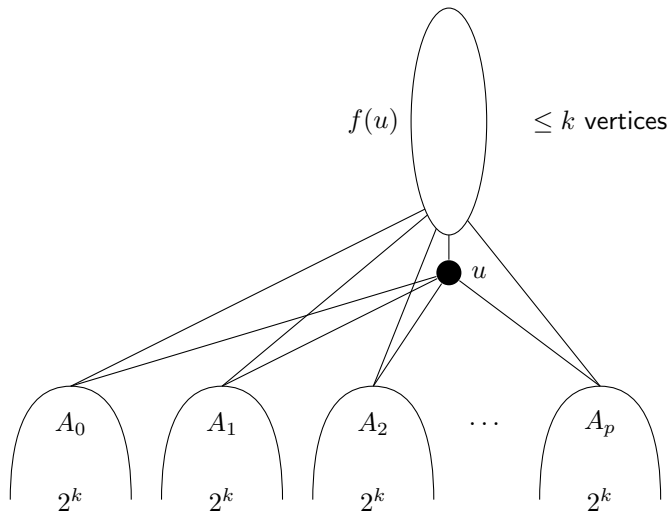
$$bw \leq tw + 1 \leq \lfloor \frac{3}{2}bw \rfloor$$

## Sphere-cut decompositions



# The case of bounded treewidth

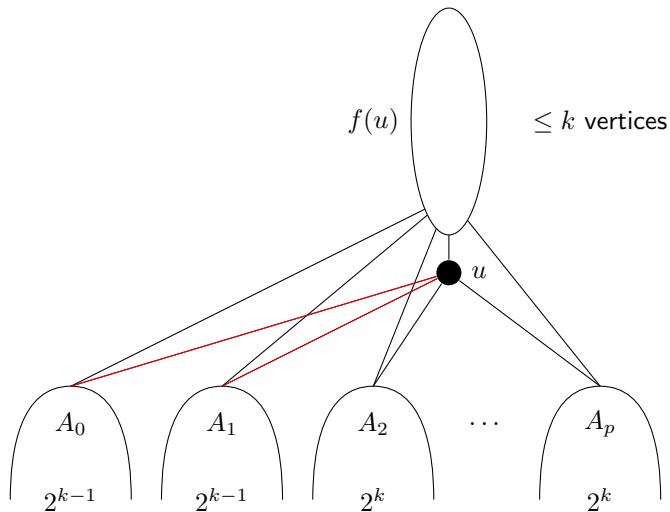
## Avoiding red edges upwards





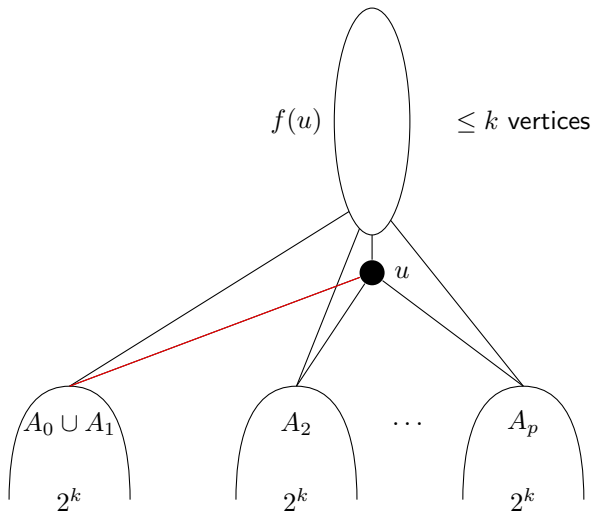
# The case of bounded treewidth

## Avoiding red edges upwards



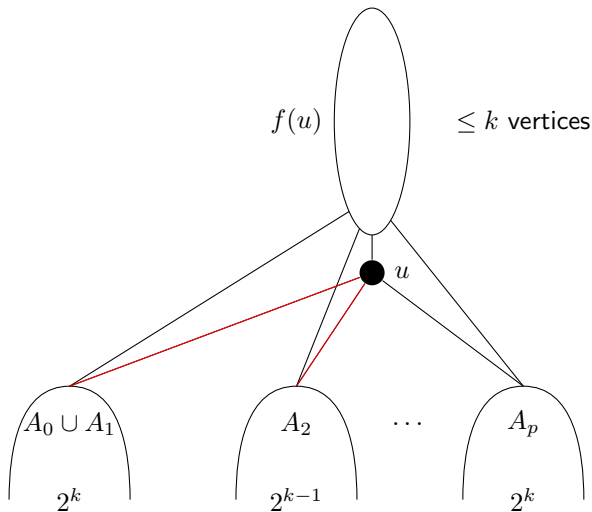
# The case of bounded treewidth

## Avoiding red edges upwards



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## Avoiding red edges upwards



## Concluding with treewidth

The previous construction gives:

### Lemma

$$\text{tw}(G) \leq \text{ctw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$$

Bonnet and Déprés [2] show:

### Lemma

*There are graphs  $G$  with  $\text{tw}(G) \geq 2^{\text{fvs}(G)} \geq 2^{\text{tw}(G)-1}$*

## Neighbourhood complexity

Vertices of  $A_i$  with different neighbourhoods in  $f(u)$  must not be contracted.

### Definition (Neighbourhood complexity)

The *neighbourhood classes* with respect to  $Y$  in  $X$  is

$$\Omega(X, Y) = \{N(x) \cap Y : x \in X\}$$

We call *neighbourhood complexity* the number of classes.

In the general setting, we can bound the neighbourhood complexity by  $2^{|Y|}$ .

If we forbid  $K_{s,s}$  as an induced subgraph and  $|Y| > s$ , we can bound it by  $O(|Y|^s)$ .

## Neighbourhood complexity in planar graphs

We count separately neighbourhoods of sizes less, equal, or more than 2:

$< 2$ :  $|Y|$  singletons, 1 empty set

$= 2$ : at most  $3|Y| - 6$

$> 2$ : at most  $2|Y| - 4$

$$|\Omega(X, Y)| \leq 6|Y| - 9.$$

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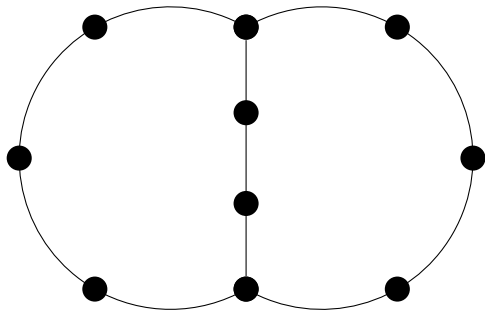
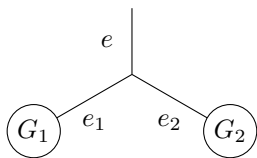
$= 2$ : at most  ~~$3|Y| - 6$~~   $2|Y| - 3$

$> 2$ : at most  ~~$2|Y| - 4$~~   $|Y| - 2$

$|\Omega(X, Y)| \leq$   ~~$6|Y| - 9$~~   $4|Y| - 4$

We now assume that there is a closed curve going through  $Y$  that contains vertices of  $X$ .

## Sphere-cut decompositions





## Sphere-cut decompositions

### Lemma

*For an undirected connected planar graph  $G$  with  $\mathbf{bw}(G) \geq 2$ :*

$$\mathbf{tww}(G) \leq \mathbf{ctww}(G) \leq \max\left(4\mathbf{bw}(G), \frac{9}{2}\mathbf{bw}(G) - 3\right)$$

*An undirected graph with  $\mathbf{bw}(G) \leq 1$  has  $\mathbf{tww}(G) = 0$ .*

## Bounds using previous results

Simple bound on the twin-width of a subgraph of  $H \boxtimes P$ , taking the product of neighbourhood classes:

$$5 (\text{ctww}(H))^3 \leq 115746$$

But we could also use the fact that the subgraph is planar:

$$5(6 \cdot (3 \cdot 7) - 9) \leq 585$$

## Using the embedding

To use our stronger neighbourhood complexity argument, we need an embedding.

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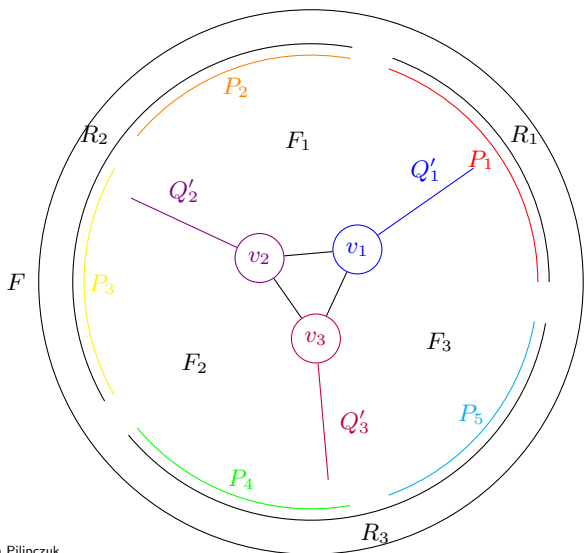
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For every edge  $uv \in E$ ,  $|d(u) - d(v)| \leq 1$

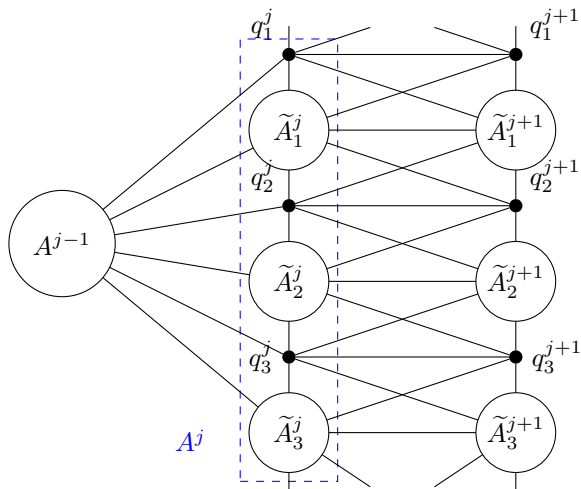
# General planar case

Using the embedding



# General planar case

## Contraction sequence





## Summing up

We can conclude that the twin-width of a planar graph is at most 183.

Our construction can also be translated to a polynomial time algorithm producing a contraction sequence with these guarantees.

### Improving the planar bound

- ▶ Can we improve the product structure theorem further ?
- ▶ Can we improve the lower bound ?
- ▶ Improved to 37 by Bekos et al. [1], then to 9 by Hliněný [4].



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Graph product structure for h-framed graphs.

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É. Bonnet and H. Déprés.

Twin-width can be exponential in treewidth.

*CoRR*, [abs/2204.07670](https://arxiv.org/abs/2204.07670), 2022.



V. Dujmovic, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood.

Planar graphs have bounded queue-number.

In D. Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 862–875. IEEE Computer Society, 2019.



P. Hliněný.

Twin-width of planar graphs is at most 11, 2022.



M. Pilipczuk and S. Siebertz.

Polynomial bounds for centered colorings on proper minor-closed graph classes.

In T. M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1501–1520. SIAM, 2019.



T. Ueckerdt, D. R. Wood, and W. Yi.

An improved planar graph product structure theorem.

*CoRR*, [abs/2108.00198](https://arxiv.org/abs/2108.00198), 2021.

## Universal bipartite graph

