# On a tree-based variant of bandwidth and forbidding simple topological minors

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#### joint work with William Lochet and Christophe Paul



# Graph layouts



Cutwidth: Maximum cut of a layout prefix.

Relation and hardness

- Both problems are NP-complete.
- ▶ CUTWIDTH is FPT but BANDWIDTH is W[t]-hard for every *t*.
- ► CUTWIDTH approximates TOPOLOGICAL BANDWIDTH

# Graph tree-layouts



# Different definitions for treebandwidth

Several closely related ways to define treebandwidth:

- bandwidth of a tree-layout
- $\blacktriangleright$  maximum clique size (-1) of a minimum proper chordal completion
- minimum k such that G is a subgraph of the underlying graph of the k-th power of an in-arborescence
- minimum height of a backedge in a DFS of a supergraph
- minimum occupation time of a vertex by searchers in a monotone search against a visible agile fugitive

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# Question

What is the structure of graphs that have bounded treebandwidth?

# Topological minors



# Topological minors



Subcubic minors are topological minors.

# Treewidth

# Theorem (Grid Minor Theorem)

A graph has large treewidth if and only if it contains a large grid minor.



#### Useful vocabulary:



# Treewidth

# Theorem (Grid Minor Theorem)

A graph has large treewidth if and only if it contains a large grid minor.



#### Useful vocabulary:



Treewidth can also be characterised in terms of tree-layouts.

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# Generalising layouts

Layout problems such as  ${\rm CUTWIDTH}$  and  ${\rm BANDWIDTH}$  can be seen as path embedding problems.

In particular, one can generalise bandwidth and cutwidth as embedding problems in trees.

# Definitions





# Domino treewidth

#### Theorem (Bienstock '90, Ding and Oporowski '95)

The following parameters are equivalent:

- $\blacktriangleright \ \mathbf{tw}(G) + \Delta(G)$
- the congestion of an embedding in a bounded degree tree.
- the stretch of an embedding in a bounded degree tree.
- domino treewidth

The parameter treespan derived from graph searching is also equivalent.

# Domino treewidth

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#### Definition (Domino tree decomposition)

A tree decomposition such that each vertex appears in at most two bags.

# Tree-partition-width



An example of tree-partition.

Tree-partition-width is also known as strong treewidth.

# Tree-partition-width

# Theorem (Ding and Oporowski)

A graph has bounded tree-partition-width if and only if there exists k such that it contains no k-wall, k-fan, k-multiple of k-star, nor k-multiple of k-path as a topological minor.



# Comparing the parameters

Graphs of bounded domino treewidth have bounded tree-partition-width.



# Comparing the parameters

Graphs of bounded tree-partition-width have bounded treebandwidth.



# Treebandwidth

#### Theorem



# Treebandwidth

### Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k-wall nor k-fan as a topological minor.



# Parameter hierarchy



# Parameter hierarchy



# Treebandwidth

#### Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k-wall nor k-fan as a topological minor.



# Treebandwidth

#### Theorem

# A graph has bounded treebandwidth if and only if there exists k such that it contains no k-wall nor k-fan as a topological minor.

Walls  $(\mathcal{W})$  admit only tree-layouts with large separators. The figure below illustrates how large separators imply large bandwidth. For simplicity, the drawn graph is a grid instead of a wall.



# Treebandwidth

#### Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k-wall nor k-fan as a topological minor.

Paths  $(\mathcal{P})$  admit no tree-layout of constant depth. Fans  $(\mathcal{F})$  admit no tree-layout of small bandwidth



# Treebandwidth

#### Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k-wall nor k-fan as a topological minor.

What are graphs which do not contain a k-fan?

# Fan models





# Fan models

The size of the largest fan model is functionally equivalent to the 'neighbourhood treedepth':

 $\max_{v \in V(G)} \operatorname{td}(G, N[v])$ 

It is easy to build elimination trees certifying a bound on  $\mathbf{td}(G, N[v])$  given the tree decomposition from the structure theorem.

Extracting structure from forbidden fans

Structure of sets of bounded treedepth in good tree decompositions of bounded adhesion:



- Bounded number of green vertices per bag
- Green tree decomposition has bounded diameter.
- Green bags have bounded size or bounded degree in green tree decomposition.

# Exploiting structure from forbidden fans

Neighbourhoods have an underlying 'small diameter' structure in good tree decompositions.

We will translate this into an explicit diameter bound by <u>folding</u> the tree decomposition.

# Exploiting structure from forbidden fans

#### Folding a tree.



# Exploiting structure from forbidden fans

Folding a tree decomposition.



# About algorithms

# Deducing an approximation algorithm

#### Observation

Folding preserves boundedness of bags.

Bounded treewidth		Decomposition of bounded width
and	$\rightarrow$	and
forbidden fan		bounded 'vertex diameter'

#### Theorem

In time  $f(k) \cdot n$ , we can either deduce the existence of a large fan/wall or construct a tree-layout of bandwidth at most g(k).

# Conclusion

# Open questions

- ▶ A  $k^{O(1)}$  approximation ratio for TREEBANDWIDTH?
- Are there natural problems that become FPT when parameterised by treebandwidth?
- Can TREEBANDWIDTH be computed in FPT (or even polynomial) time on planar graphs?
- Is it necessary to ask for monotonicity in the graph searching characterisation for treebandwidth?

# Structure theorem

#### Theorem

If a graph G excludes  $F_k$  as a topological minor, then there exist integers a, b, c depending only on k and a tree decomposition  $(T, \beta)$  of G such that:

- ▶ adhesion size is at most *a*,
- $\blacktriangleright$  each vertex v of G has at most b neighbours in the torso at each bag, and

• the bags containing v induce a subtree of T of diameter at most c. Such a decomposition can be computed in time  $f(k) \cdot n$ .

Conversely, if such a decomposition exists, G excludes  $F_{k'}$  as a topological minor for some k' depending only on a, b, c.

# About equivalent characterisations

# Chordal classes



# About equivalent characterisations

# Graph searching

	Cop number	Occupation time
Agile invisible	Pathwidth	Bandwidth
Lazy invisible	Treewidth	Treespan
Agile visible	Treewidth	Treebandwidth

Agile: The fugitive can always move.

Lazy: The fugitive moves only when about to be captured.