

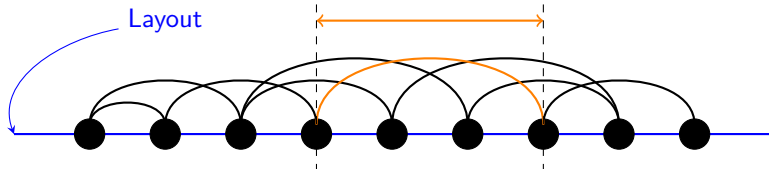
On a tree-based variant of bandwidth and forbidding simple topological minors

Hugo Jacob

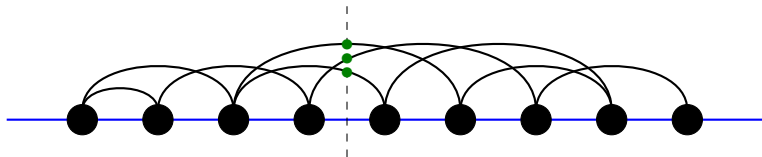
joint work with William Lochet and Christophe Paul



Graph layouts



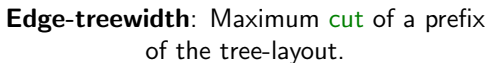
Bandwidth: Maximum **distance** in the (linear) layout between adjacent vertices.



Cutwidth: Maximum **cut** of a layout prefix.

Relation and hardness

- ▶ Both problems are NP-complete.
- ▶ CUTWIDTH is FPT but BANDWIDTH is $W[t]$ -hard for every t .
- ▶ CUTWIDTH approximates TOPOLOGICAL BANDWIDTH



Different definitions for treebandwidth

Several closely related ways to define treebandwidth:

- ▶ bandwidth of a tree-layout
- ▶ maximum clique size (-1) of a minimum proper chordal completion
- ▶ minimum k such that G is a subgraph of the underlying graph of the k -th power of an in-arborescence
- ▶ minimum height of a backedge in a DFS of a supergraph
- ▶ minimum occupation time of a vertex by searchers in a monotone search against a visible agile fugitive

Different definitions for treebandwidth

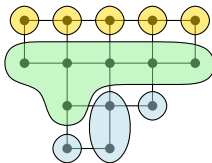
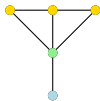
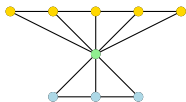
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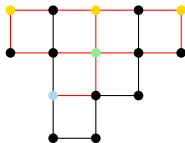
Question

What is the structure of graphs that have bounded treebandwidth?

Topological minors

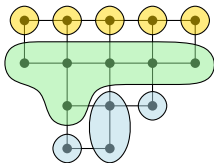
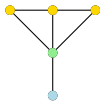
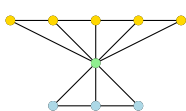


Minor

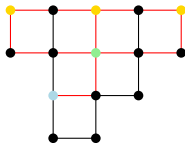


Topological minor

Topological minors



Minor



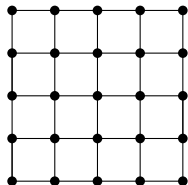
Topological minor

Subcubic minors are topological minors.

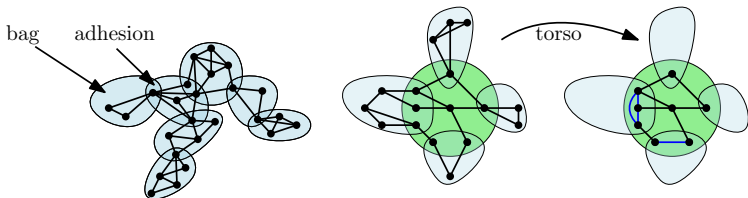
Treewidth

Theorem (Grid Minor Theorem)

A graph has large treewidth if and only if it contains a large grid minor.



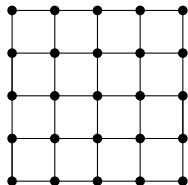
Useful vocabulary:



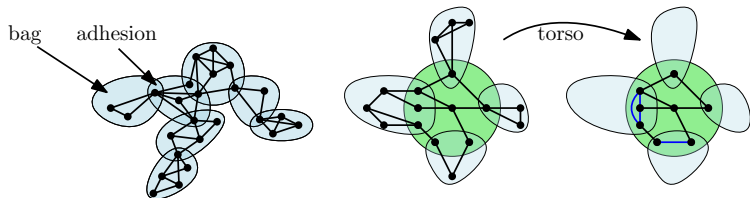
Treewidth

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Useful vocabulary:



Treewidth can also be characterised in terms of tree-layouts.

Tree-layouts and tree decompositions

From tree decomposition to tree-layout:

- ▶ Root the tree decomposition arbitrarily.
- ▶ Vertices introduced in current bag are linearly ordered arbitrarily.
- ▶ Branch according to tree decomposition.

From tree-layout to tree decomposition:

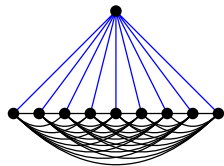
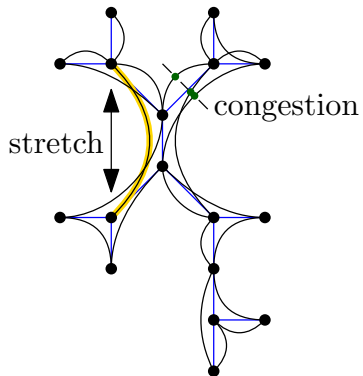
- ▶ One bag per vertex of the tree-layout:
- ▶ This vertex and its ancestors which are adjacent to vertices in its subtree.

Generalising layouts

Layout problems such as **CUTWIDTH** and **BANDWIDTH** can be seen as path embedding problems.

In particular, one can generalise bandwidth and cutwidth as embedding problems in trees.

Definitions



Domino treewidth

Theorem (Bienstock '90, Ding and Oporowski '95)

The following parameters are equivalent:

- ▶ $\mathbf{tw}(G) + \Delta(G)$
- ▶ *the congestion of an embedding in a bounded degree tree.*
- ▶ *the stretch of an embedding in a bounded degree tree.*
- ▶ *domino treewidth*

The parameter treewidth derived from graph searching is also equivalent.

Domino treewidth

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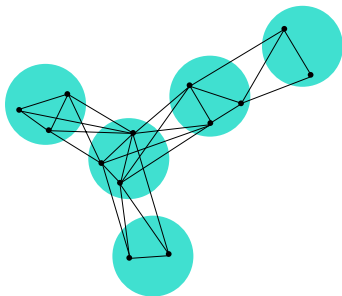
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Definition (Domino tree decomposition)

A tree decomposition such that each vertex appears in at most two bags.

Tree-partition-width



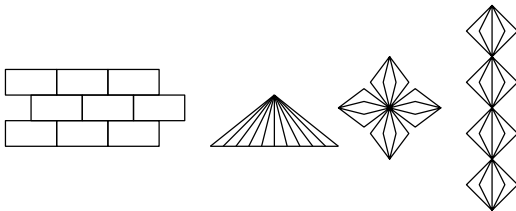
An example of tree-partition.

Tree-partition-width is also known as strong treewidth.

Tree-partition-width

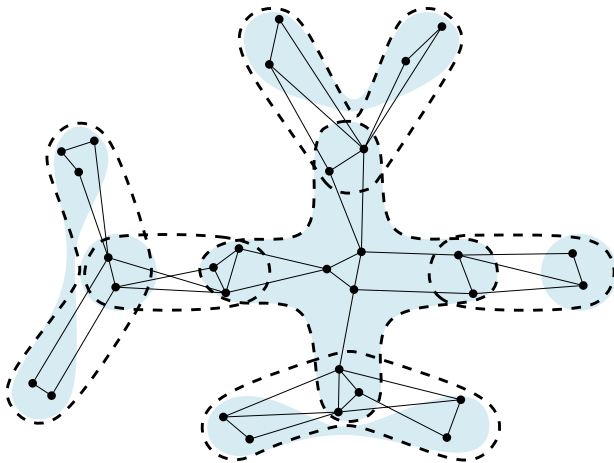
Theorem (Ding and Oporowski)

A graph has bounded tree-partition-width if and only if there exists k such that it contains no k -wall, k -fan, k -multiple of k -star, nor k -multiple of k -path as a topological minor.



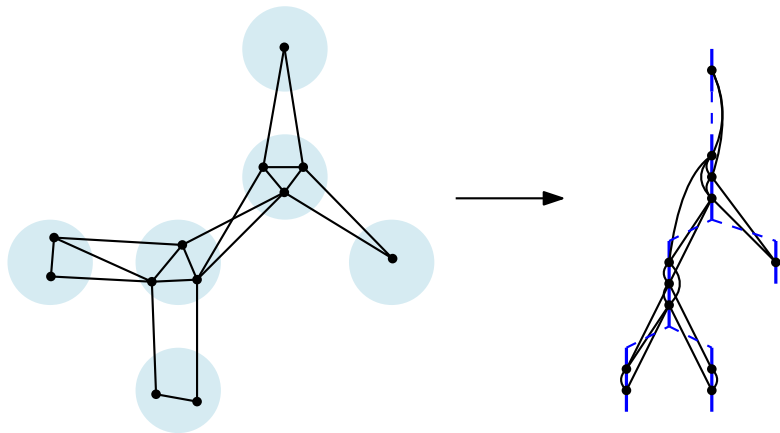
Comparing the parameters

Graphs of bounded domino treewidth have bounded tree-partition-width.



Comparing the parameters

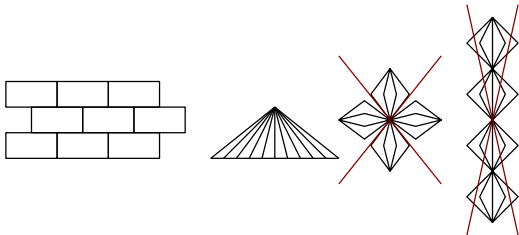
Graphs of bounded tree-partition-width have bounded treebandwidth.



Treebandwidth

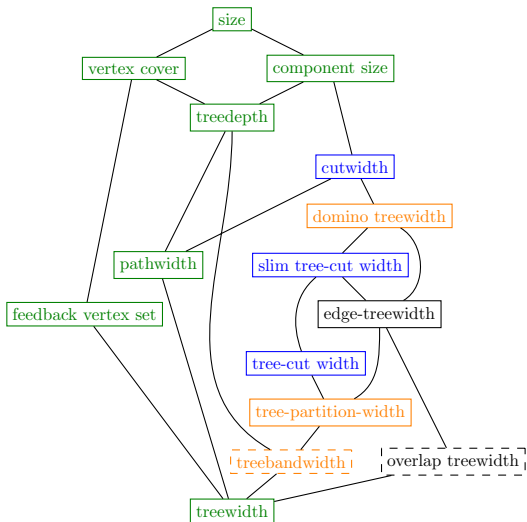
Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k -wall nor k -fan as a topological minor.



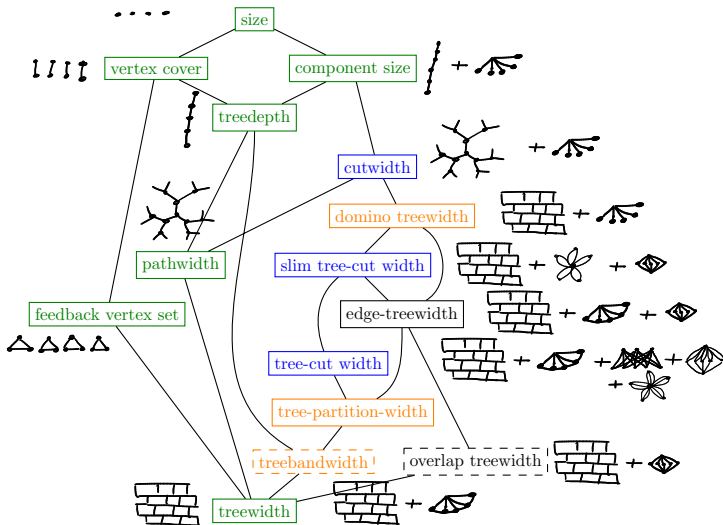
Embedding in trees

Parameter hierarchy



Embedding in trees

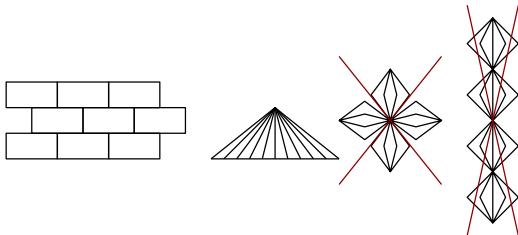
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Treebandwidth

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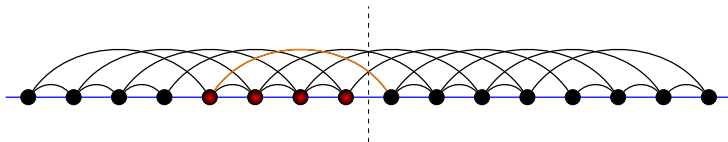


Treebandwidth

Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k -wall nor k -fan as a topological minor.

Walls (\mathcal{W}) admit only tree-layouts with large separators.
 The figure below illustrates how large **separators** imply large **bandwidth**.
 For simplicity, the drawn graph is a grid instead of a wall.

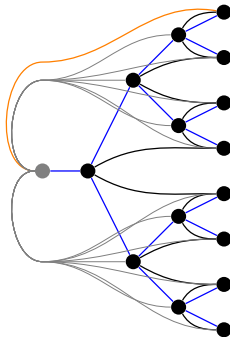


Treebandwidth

Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k -wall nor k -fan as a topological minor.

Paths (\mathcal{P}) admit no tree-layout of constant depth.
Fans (\mathcal{F}) admit no tree-layout of small bandwidth



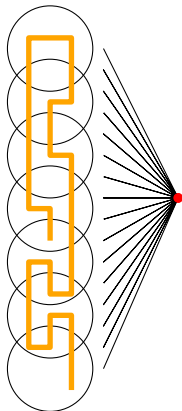
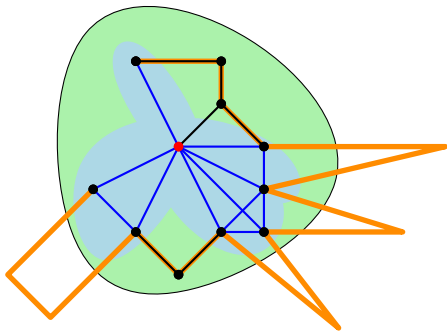
Treebandwidth

Theorem

A graph has bounded treebandwidth if and only if there exists k such that it contains no k -wall nor k -fan as a topological minor.

What are graphs which do not contain a k -fan?

Fan models



Structure theorem

Theorem

If a graph G excludes F_k as a topological minor, then there exist integers a, b, c depending only on k and a tree decomposition (T, β) of G such that:

- ▶ *adhesion size is at most a ,*
- ▶ *each vertex v of G has at most b neighbours in the torso at each bag, and*
- ▶ *the bags containing v induce a subtree of T of diameter at most c .*

Such a decomposition can be computed in time $f(k) \cdot n$.

Conversely, if such a decomposition exists, G excludes $F_{k'}$ as a topological minor for some k' depending only on a, b, c .

Fan models

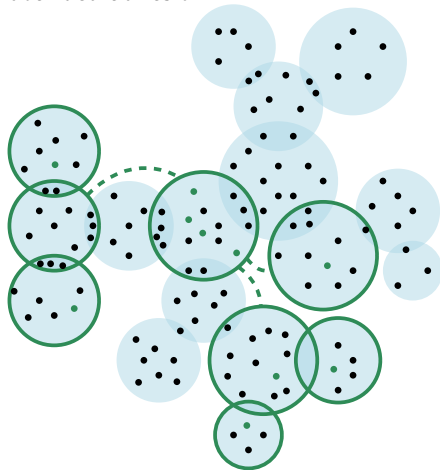
The size of the largest fan model is functionally equivalent to the ‘neighbourhood treedepth’:

$$\max_{v \in V(G)} \mathbf{td}(G, N[v])$$

It is easy to build elimination trees certifying a bound on $\mathbf{td}(G, N[v])$ given the tree decomposition from the structure theorem.

Extracting structure from forbidden fans

Structure of sets of bounded treedepth in good tree decompositions of bounded adhesion:



- ▶ Bounded number of green vertices per bag
- ▶ Green tree decomposition has bounded diameter.
- ▶ Green bags have bounded size or bounded degree in green tree decomposition.

Exploiting structure from forbidden fans

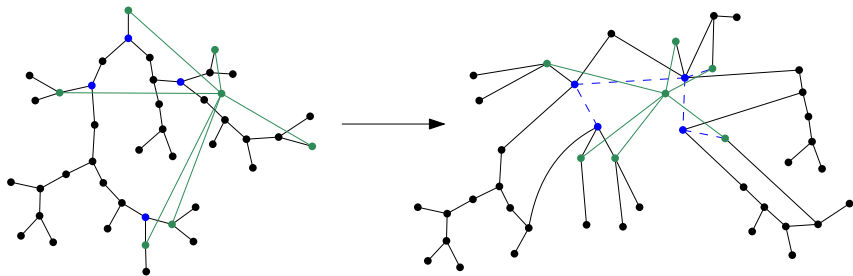
Neighbourhoods have an underlying ‘small diameter’ structure in good tree decompositions.

We will translate this into an explicit diameter bound by folding the tree decomposition.

Forbidden fan

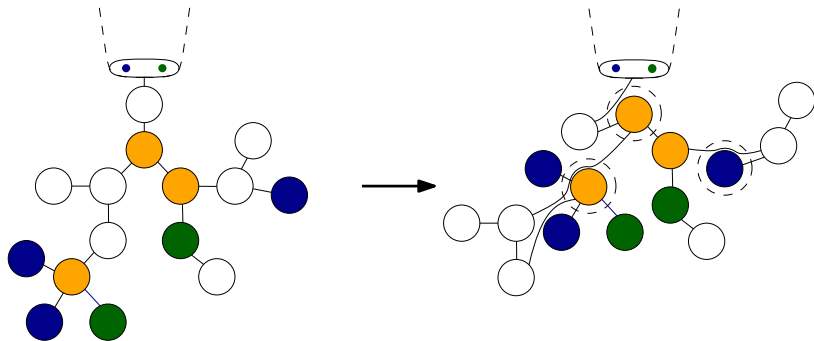
Exploiting structure from forbidden fans

Folding a tree.



Exploiting structure from forbidden fans

Folding a tree decomposition.



Deducing an approximation algorithm

Observation

Folding preserves boundedness of bags.

Bounded treewidth and forbidden fan	→	Decomposition of bounded width and bounded 'vertex diameter'
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Theorem

In time $f(k) \cdot n$, we can either deduce the existence of a large fan/wall or construct a tree-layout of bandwidth at most $g(k)$.

Hardness of exact algorithm

BANDWIDTH is XNLP-complete. The tree-like analogue of BANDWIDTH is complete for the tree-like analogue of XNLP:

Theorem

TREEBANDWIDTH *is* XALP-complete.

This implies a deterministic algorithm running in XP time and XP space which is likely to be qualitatively optimal.

Hardness holds even when parameterised by treewidth plus degree.

A linear bound on p -centered colourings

Definition

A p -centered colouring of G is a colouring $\lambda : V(G) \rightarrow \mathbb{N}$ such that for every connected subgraph H of G , either λ uses more than p colours on H (i.e. $|\lambda(V(H))| > p$), or there is a colour assigned to a unique vertex in H .

An example of application:

Theorem (Pilipczuk and Siebertz)

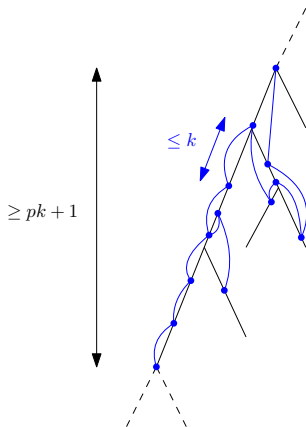
SUBGRAPH ISOMORPHISM *can be solved in time $O(c^p \cdot p^{O(p)} \cdot n^{O(1)})$ and space $n^{O(1)}$ with p the size of the subgraph H , c the number of colours of a p -centered colouring.*

For treewidth at most k , there is a p -centered colouring using $O(p^k)$ colours and a matching lower bound.

A linear bound on p -centered colourings

Theorem

If a graph G has treebandwidth at most k , it has p -centered colourings with $pk + 1$ colours.

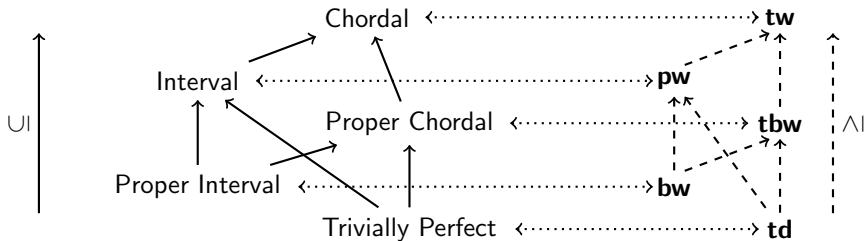


Open questions

- ▶ A $k^{O(1)}$ approximation ratio for TREEBANDWIDTH?
- ▶ Are there natural problems that become FPT when parameterised by treebandwidth?
- ▶ Can TREEBANDWIDTH be computed in FPT (or even polynomial) time on planar graphs?
- ▶ Is it necessary to ask for monotonicity in the graph searching characterisation for treebandwidth?

About equivalent characterisations

Chordal classes



Graph searching

	Cop number	Occupation time
Agile invisible	Pathwidth	Bandwidth
Lazy invisible	Treewidth	Treespan
Agile visible	Treewidth	Treebandwidth

Agile: The fugitive can always move.

Lazy: The fugitive moves only when about to be captured.