

Maximum Independent Set of Rectangles

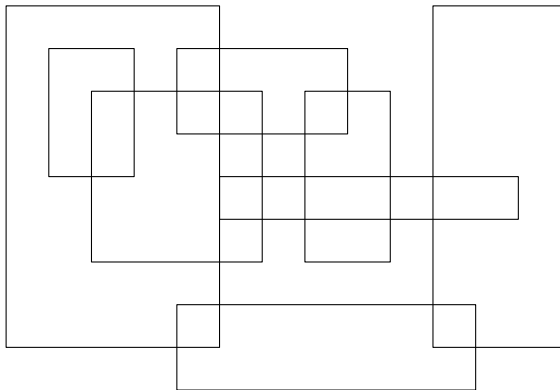
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Supervised by William Lochet and Christophe Paul, LIRMM

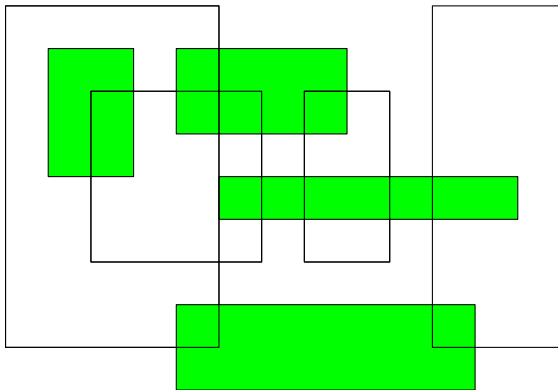
école ———
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The studied problem

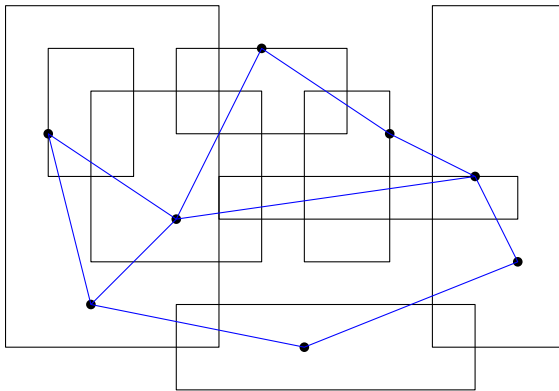


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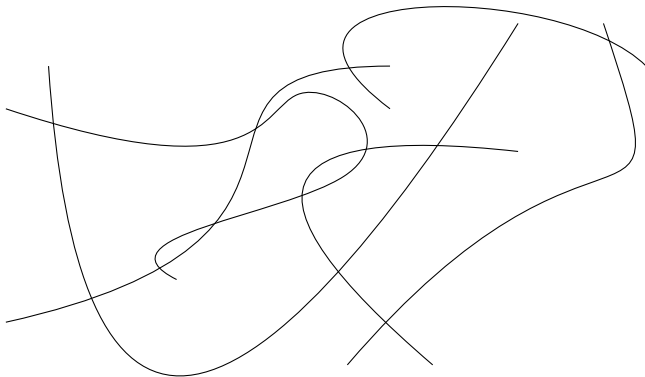
Preliminaries

The problem from a graph perspective



We must pick as many vertices as possible such that they are pairwise nonadjacent. This problem is called *Maximum Independent Set (MIS)*.

Intersection graphs



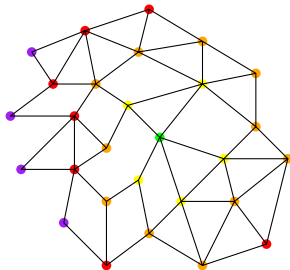
When curves are segments: d -DIR.
1-DIR = interval graph.

Preliminaries

Greedy (Earliest Deadline First)



Baker's shifting technique



Baker's shifting technique

Theorem (Baker)

MIS can be $(1 + \varepsilon)$ -approximated in polynomial time for planar graphs.

Generalizes to:

- ▶ Minor-closed graph classes
- ▶ Intersection graphs of 'fat' objects in the plane

But **not** intersection graphs of rectangles !

Terminology of approximation algorithms

A **ρ -approximation algorithm** is an algorithm that produces a solution such that $x^* \leq \rho \cdot x$.

A **PTAS** is an algorithm that produces a $(1 + \varepsilon)$ -approximation algorithm in polynomial time for any $\varepsilon > 0$.

- ▶ **QPTAS**: quasi-polynomial time
- ▶ **EPTAS**: runs in time $f(\varepsilon) \cdot n^c$

A QPTAS is expected to imply a PTAS (nonconstructive).

State of the art

- ▶ Adamaszek and Wiese gave a QPTAS in 2013.
- ▶ Mitchell gave the first constant approximation in polynomial time (FOCS'21)
- ▶ Gálvez et al. gave a $(2 + \varepsilon)$ -approximation in polynomial time.
- ▶ Trivial 2-approx for 2-DIRs

PTAS is still open.

What is a kernel ?

A *kernel* is a 'smaller' equivalent instance that can be obtained in polynomial time. The size must be bounded by a parameter of the instance.

There is a known polynomial kernel for segments.
There should not be a kernel for squares.

A simple quadratic kernel

Reducing # segments: Greedy algorithm

A simple quadratic kernel

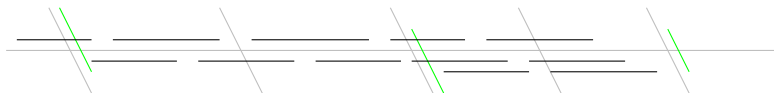
Reducing # segments: Greedy algorithm

Property: Each prefix of the greedy solution minimizes the last endpoint among solutions of maximal size.

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Sparse case

Theorem (Lee)

There is a polynomial time algorithm that produces a balanced separator of size $O(\sqrt{m})$ in string graphs.

If the input graph is sufficiently sparse (e.g. $K_{t,t}$ -free), we can obtain a PTAS by local search.

Unit length case

Two cases:

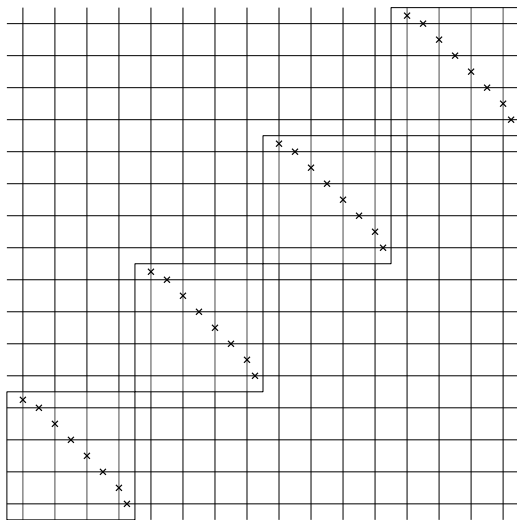
- ▶ Unit longest side
- ▶ Unit height

In both cases, first **shifting technique** then **DP** on disconnected parts.

LP relaxation

$$\begin{aligned} & \max \sum_{v \in V} x_v \text{ s.t.} \\ & \sum_{v \in C} x_v \leq 1 \text{ for each clique } C \\ & x \in [0, 1]^V \end{aligned}$$

LP relaxation and integrality gap



Adding constraints to the relaxation?

If we add constraints on odd cycles, we can avoid the previous example.
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Still unclear how to approach rounding.

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We show that in this case:

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- ▶ Integral values of LP solution are 'correct'

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What is the structure of irreducible LP instances?

Conjectures on structural properties of 2-DIRs

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What are the structures responsible for the integrality gap?

If only odd cycles are responsible, we could use a linear Erdős-Pósa property to find a good approximation when there are few odd cycles and guarantee a better ratio when there are many odd cycles.

Conjecture

Triangle-free 2-DIRs satisfy a linear Erdős-Pósa property for odd cycles.

In fact, the triangle-free assumption can easily be lifted.

This could lead to a $(2 - \varepsilon)$ -approximation algorithm.