

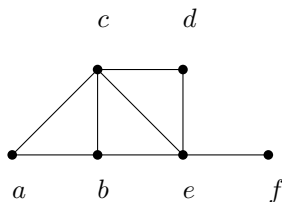
Twin-width One

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joint work with Junggho Ahn, Noleen Köhler, Christophe Paul,
Amadeus Reinald, and Sebastian Wiederrecht



Twin-width



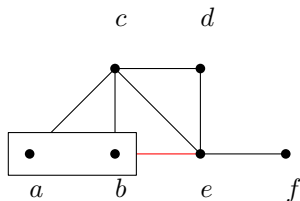
Definition

A d -contraction sequence is a sequence of trigraphs obtained by iterative contraction of pairs of vertices such that the **red** degree of a vertex is at most d .

Definition

The twin-width of a graph is the minimum d such that the graph admits a d -contraction sequence.

Twin-width



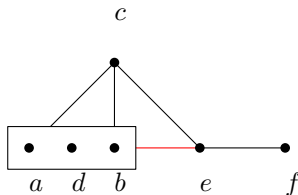
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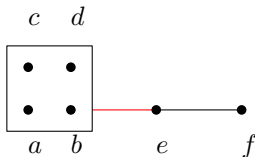
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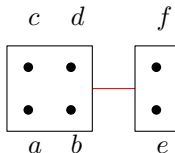
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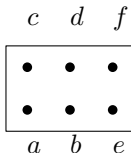
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Motivations for twin-width

Pros:

- ▶ A partial extension of sparsity theory to classes of dense graphs.
- ▶ Tractability dichotomy for FO in ordered structures.

Cons:

- ▶ Computing the twin-width exactly is NP-hard (already for $d = 4$).
- ▶ There is no known approximation algorithm

Small values of twin-width

- ▶ $d = 0$: Cographs = P_4 -free graphs, linear time recognition.
- ▶ $d \leq 1$: Polynomial time recognition, have a linear structure.

This talk:

Theorem

Twin-width 1 graphs can be recognized in linear time.

Previous algorithm

Main observation:

Lemma

If a graph admits a 1-contraction sequence, it admits a 1-contraction sequence with at most one red edge per trigraph.

Algorithm sketch:

```
while the graph is not fully contracted:  
    Guess a pair of vertices to contract.  
    if a red edge appears:  
        keep contracting greedily on it,  
        then contract it.
```

Perfect graphs

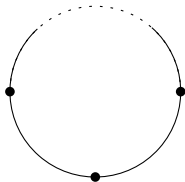
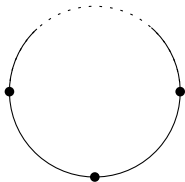
Theorem (Strong perfect graph theorem)

Perfect graphs are exactly the graphs that do not contain odd holes or odd antiholes as induced subgraphs.

Perfect graphs

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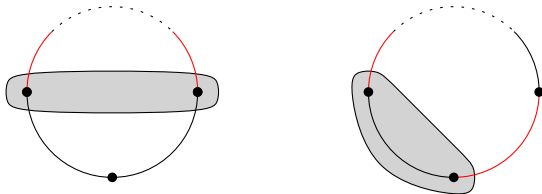
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Observation

Holes and antiholes have twin-width **2**.

Perfect graphs

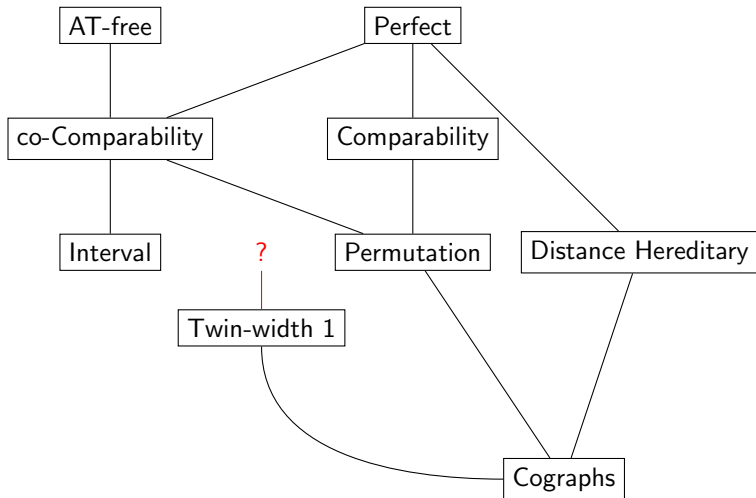
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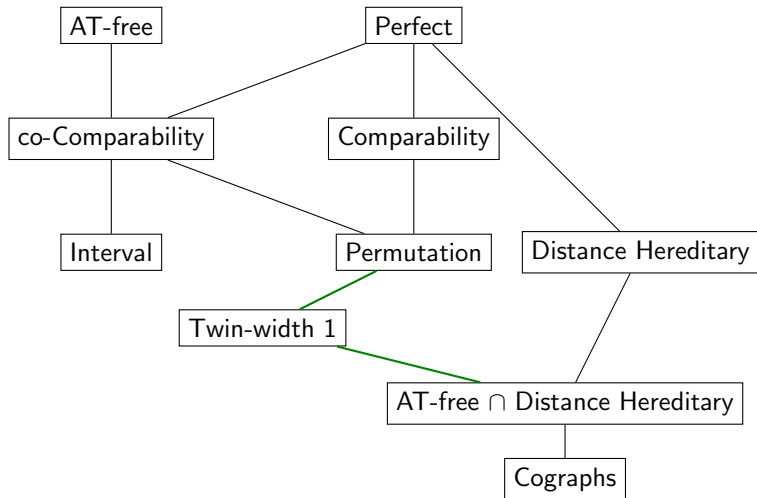
Observation

Holes and antiholes have twin-width 2 .
In particular, twin-width 1 graphs are perfect.

Classes of perfect graphs



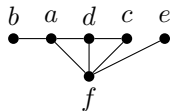
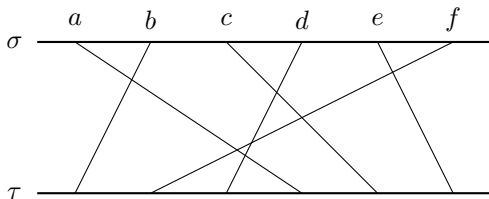
Classes of perfect graphs



Definition

Definition

Given a permutation $\pi : [n] \rightarrow [n]$, we define the graph $G[\pi] = ([n], E_\pi)$ where E_π is the set of inversions of π (i.e. $xy \in E_\pi$ iff $x < y$ and $\pi(x) > \pi(y)$).



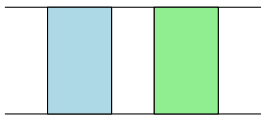
Warm up: Cographs

Lemma

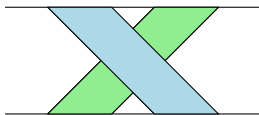
Cographs are permutation graphs.

Induction on the cotree with two cases:

Disjoint union



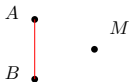
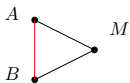
Join



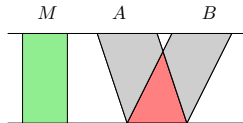
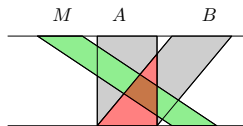
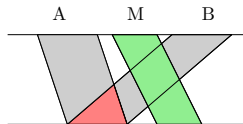
Twin-width 1 graphs are permutation graphs

Induction on the contraction sequence:

3-vertex trigraph



Permutation diagram



Twin-width 1 graphs are permutation graphs

Vertices contracted together are consecutive in the diagram.

Lemma

If G admits a 1-contraction sequence G_n, \dots, G_1 , then there exists a realizer (σ, τ) such that:

- ▶ *if x is a vertex of G_i , then vertices of G that are in x form an interval of σ or τ .*
- ▶ *if xy is a red edge of G_i , then vertices of G that are in xy form an interval of σ or τ*

Twin-width 1 graphs are permutation graphs

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Corollary

If the graph is prime, an extremal vertex of the diagram can be contracted last. There are only 4 vertices to try!

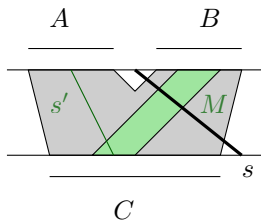
High level description

Algorithm sketch:

```
Compute permutation diagram, reject when not permutation graph
Compute modular decomposition
for each prime node of the decomposition:
    Guess last vertex  $s$  to be contracted
    Try to decompose recursively using  $s$  as splitter
```

The algorithm

Prime case description



```
while an endpoint of  $C$  is also an endpoint of  $A$  or  $B$ :  
    Remove it and move indices correspondingly  
if  $|A| > 1$  and  $|B| > 1$ : Reject  
else if  $|A| > 1$  or  $|B| > 1$ :  
    Deduce  $(M, s')$  from  $(A, B)$   
    Recurse on  $(M, s')$  with  $s'$  as splitter  
else: Accept
```


Going further

Our results:

- ▶ Linear time recognition algorithm for twin-width **1**
- ▶ Permutation diagram and consecutivity properties

Open questions:

- ▶ Can we recognize twin-width **2** graphs in polynomial time?
- ▶ Is recognition of twin-width **3** NP-hard?
- ▶ Can we approximate twin-width in XP time?
- ▶ Can we compute the twin-width of permutation graphs in polynomial time?

Cographs and modular decompositions

Cographs can be characterised in two ways:

- ▶ A cograph is a single vertex graph or a graph G obtained from two cographs H_1 and H_2 by either the **disjoint union** of H_1 and H_2 or adding all edges between H_1 and H_2 (**join**).
- ▶ A cograph is a graph that admits a **twin**-elimination ordering.

Both characterisations follow from a top-down or bottom-up description of the cotree of the cograph

Definition (Module)

A module M is a subset of vertices such that every vertex not in M is adjacent either to all vertices of M , or no vertex of M .

A graph is prime if it has only trivial modules.

Observation

A contraction in a module creates **red** edges only within the module.

For every graph, its modular decomposition is a canonical object that recursively decomposes it into **prime** graphs, **disjoint unions**, and **joins**.

Lemma

The twin-width of a graph is the maximum twin-width over the prime nodes of its modular decomposition.

Construct the contraction sequence by following the modular decomposition bottom-up. We iteratively contract **twins**, and contract **prime** nodes using their optimal sequence.