

Close relatives (of Feedback Vertex Set), revisited

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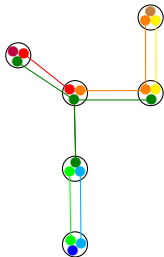
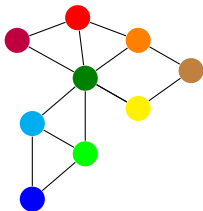
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Treewidth

A *tree decomposition* of graph G is a tree T with bags $\{X_t\}$ such that:

- ▶ For each edge uv of G , there is a node t s.t. $u, v \in X_t$
- ▶ For each vertex v of G , the nodes whose bags contain v form a non-empty subtree.



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The *width* of a tree decomposition is $\max_{t \in V(T)} \{|X_t| - 1\}$.

The *treewidth* of G is the minimum width over all tree decompositions of G , we denote it **tw**.

Restricting the tree to a path, we obtain the notion of *path decomposition* and *pathwidth* (denoted **pw**).

Clique-width

A graph $G = (V, E)$ together with a labeling function $\gamma : V \rightarrow \{1, \dots, k\}$ is called a *k-labeled graph*.

We build *k-labeled graphs* using the following operations:

- ▶ Compute disjoint union of two graphs.
- ▶ Add edges between vertices labeled i and vertices labeled j .
- ▶ Change the label of vertices labeled i to j .

A *k-expression* is an expression constructing a graph using these operations starting from single-vertex graphs.

The *clique-width* of a graph is the minimum k such that the graph can be constructed by a *k-expression*.

Definition (FEEDBACK VERTEX SET)

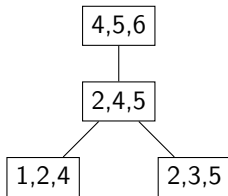
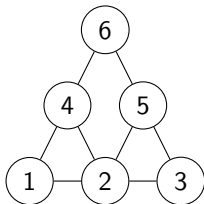
Find the size of a minimum subset of vertices X s.t. $G - X$ is cycle-free.

Variants:

- ▶ Hitting cycles of some parity: ODD/EVEN CYCLE TRANSVERSAL
- ▶ Hitting cycles containing vertices of some subset S : SUBSET FEEDBACK VERTEX SET
- ▶ Both: SUBSET ODD/EVEN CYCLE TRANSVERSAL

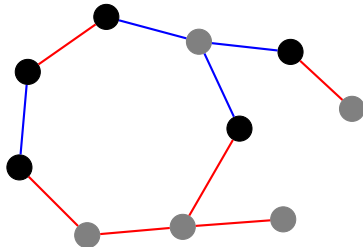
Designing the dynamic programming

Simple example : independent set



Intuition

Consider all possible graphs obtained by vertex deletion that contain no forbidden cycle.



We need to be able to check whether a union contains a forbidden cycle.

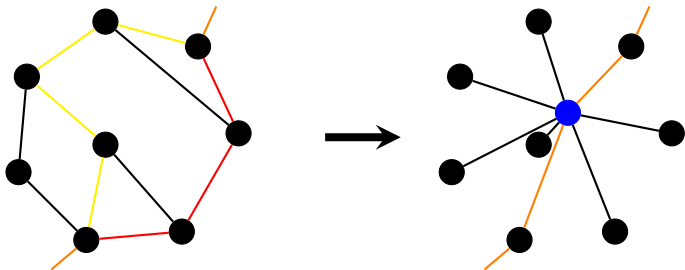
Forbidden cycle checking

To perform checks, we use reduced representations of the graph. They are forests with $\mathcal{O}(k)$ vertices for k the width of our decomposition.

A state of the dynamic programming will consist of such a forest. This amounts to $2^{\mathcal{O}(k \log k)}$ states per node of the decomposition.

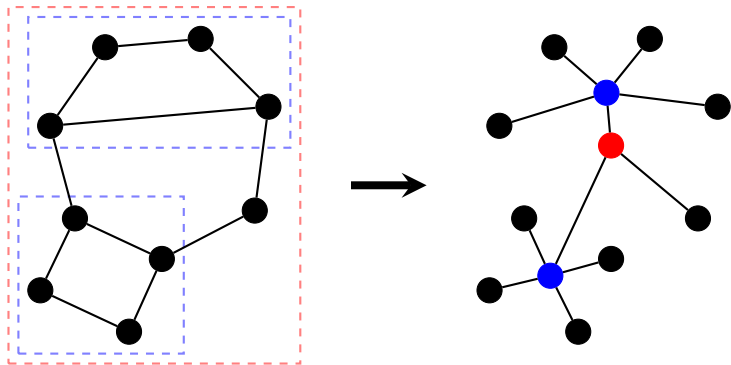
Contracting 2-connected components

Basic idea

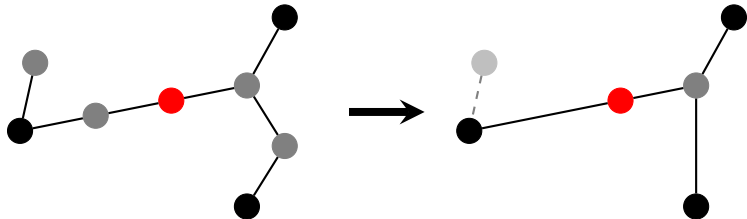


Contracting 2-connected components

SECT: structured contraction



Basics



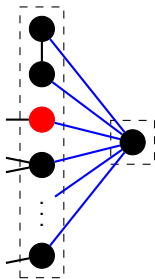
Keeping track of length

Two options:

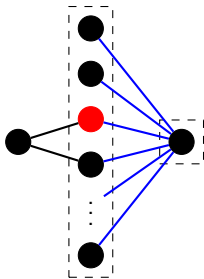
- ▶ Color active vertices such that it can be extended to a proper 2-coloring where possible.
- ▶ Label edges with weights to compute distance.

Main difficulty

The graph induced by a vertex class is not bounded, we can't describe its whole internal structure.



No forbidden cycle, but arbitrarily large class.



A forbidden cycle, using some vertex of another vertex class.

Solution

Guess in advance what will happen to the label, and block any transition that would contradict this guess.

Crucial part: Guessing if there will be a join.

- ▶ If not, the vertex class can be considered inactive in the same spirit as before.
- ▶ Otherwise, the vertex class can already be considered connected simplifying the structure to represent.

Previous results

Theorem (Bergougnoux, Bonnet, Brettell, Kwon 2020)

Under the ETH, SUBSET FEEDBACK VERTEX SET, SUBSET ODD CYCLE TRANSVERSAL, and EVEN CYCLE TRANSVERSAL cannot be solved in time $2^{o(\mathbf{pw} \log \mathbf{pw})} n^{\mathcal{O}(1)}$.

Our results

Theorem

There are algorithms solving SUBSET FEEDBACK VERTEX SET, SUBSET ODD CYCLE TRANSVERSAL, EVEN CYCLE TRANSVERSAL, and SUBSET EVEN CYCLE TRANSVERSAL in time $2^{\mathcal{O}(\text{tw} \log \text{tw})} n$.

Theorem

There is an algorithm solving SUBSET FEEDBACK VERTEX SET on a given clique-width expression of size n in time $2^{\mathcal{O}(\text{cw} \log \text{cw})} n$.

More results

Theorem

There is an algorithm solving ODD CYCLE TRANSVERSAL on a given clique-width expression of size n in time $\mathcal{O}(4^{\text{cw}} \cdot \text{cw} \cdot n)$, and the base of the exponent is optimal under the SETH.

Hegerfeld and Kratsch just gave a lower bound for DELETION TO r -COLORABLE and a matching algorithm.

Using our convolution computation, the algorithm runs in time $\mathcal{O}((2^r)^{\text{cw}} \cdot r \cdot \text{cw} \cdot n)$.

Open questions

- ▶ Tight algorithms for SUBSET ODD CYCLE TRANSVERSAL or EVEN CYCLE TRANSVERSAL on clique-width expressions ?
- ▶ ODD/EVEN CYCLE TRANSVERSAL are not equally hard, what about looking for optimal transversals of cycles with a given modular length ?