

Looking for an improved upper bound to the twin-width of planar graphs

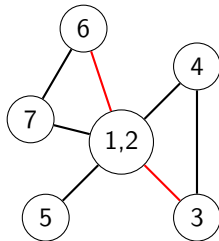
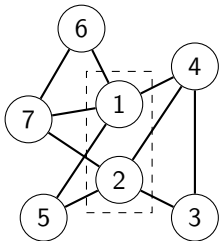
Hugo Jacob

Joint work with Marcin Pilipczuk

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Twin-width



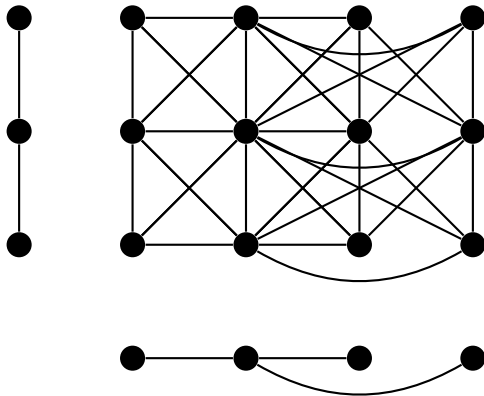
Strong product

Definition (Strong product)

Given two graphs G, H , their strong product is the graph $G \boxtimes H$ on vertex set $V(G) \times V(H)$ with edges $\{(u, u'), (v, v')\}$ such that:

- ▶ $u = v$ or $uv \in E(G)$; and
- ▶ $u' = v'$ or $u'v' \in E(H)$.

Strong product



Product structure of planar graphs

Using a construction of Pilipczuk and Siebertz [2], Dujmović et al [1] show:

Theorem

Every planar graph is a subgraph of:

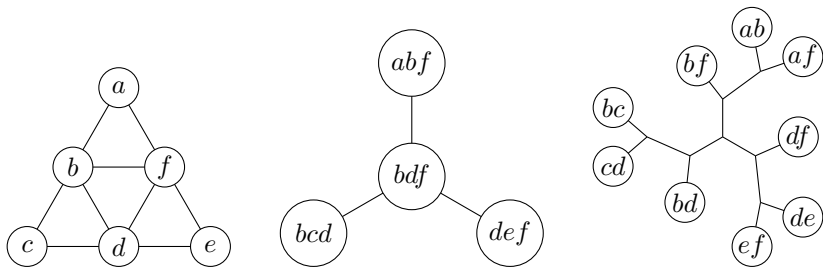
- 1. $H \boxtimes P$ for some planar graph H of treewidth at most 8 and some path P .*
- 2. $H \boxtimes P \boxtimes K_3$, for some planar graph H of treewidth at most 3 and some path P .*

Ueckerdt et al [3] improved the first result with:

Theorem

If G is planar then G is a subgraph of $H \boxtimes P$, where P is a path and H is a minor of G of treewidth at most 6.

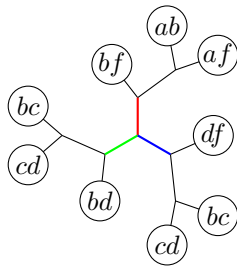
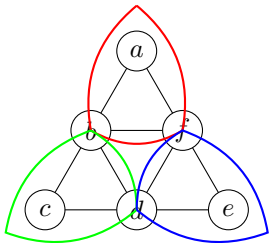
Treewidth and branchwidth



Theorem (Robertson and Seymour 1991)

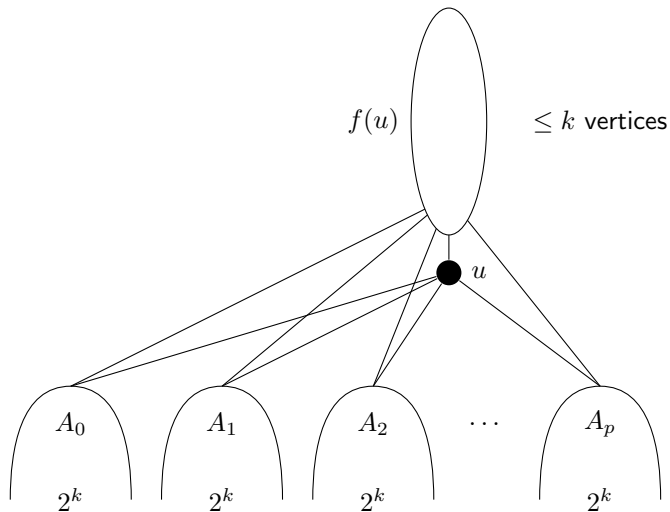
$$bw \leq tw + 1 \leq \lfloor \frac{3}{2}bw \rfloor$$

Sphere-cut decompositions



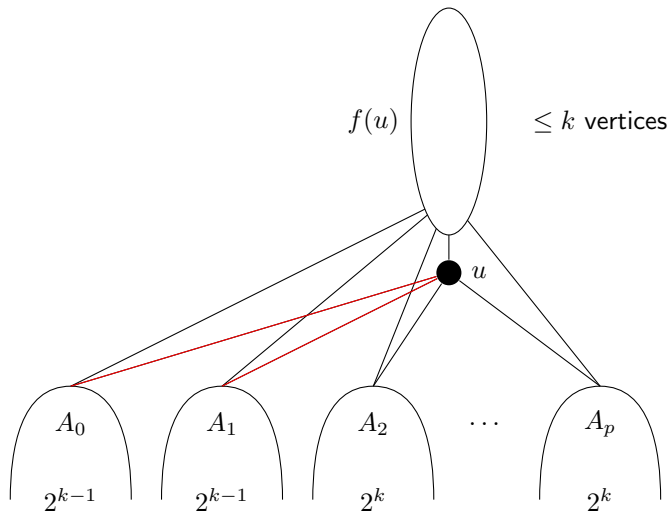
The case of bounded treewidth

Avoiding red edges upwards



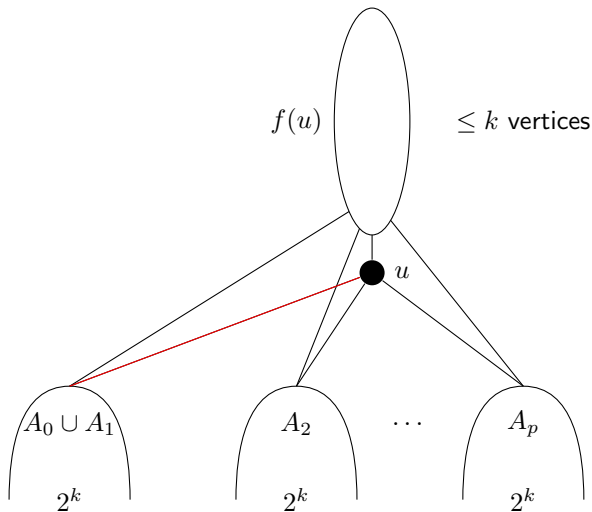
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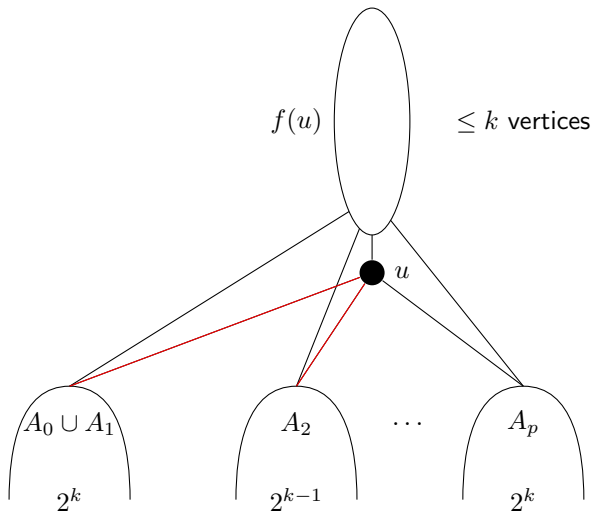
The case of bounded treewidth

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The case of bounded treewidth

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Neighbourhood complexity

Vertices of A_i with different neighbourhoods in $f(u)$ must not be contracted.

Definition (Neighbourhood complexity)

The *neighbourhood classes* with respect to Y in X is

$$\Omega(X, Y) = \{N(x) \cap Y : x \in X\}$$

We call *neighbourhood complexity* the number of classes.

In the general setting, we can bound the neighbourhood complexity by $2^{|Y|}$.

If we forbid $K_{s,s}$ as an induced subgraph and $|Y| > s$, we can bound it by $O(|Y|^s)$.

Concluding with treewidth

The previous construction gives:

Lemma

$$\text{tw}(G) \leq \text{ctw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$$

Construction by Hugues Déprés, PhD student at LIP (Lyon).

Lemma

There are graphs G with $\text{tw}(G) \geq 2^{\text{fvs}(G)} \geq 2^{\text{tw}(G)-1}$

Neighbourhood complexity in planar graphs

We count separately neighbourhoods of sizes less, equal, or more than 2:

< 2 : $|Y|$ singletons, 1 empty set

$= 2$: at most $3|Y| - 6$

> 2 : at most $2|Y| - 4$

$$|\Omega(X, Y)| \leq 6|Y| - 9.$$

Using structure

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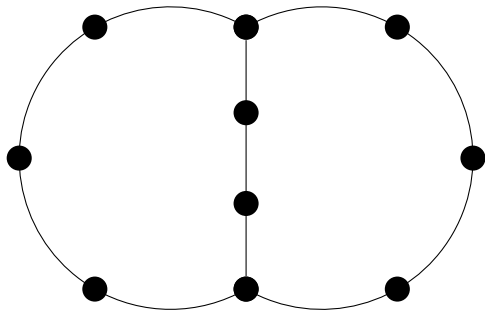
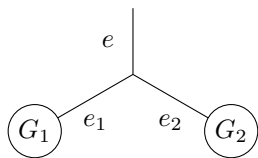
$= 2$: at most ~~$3|Y| - 6$~~ $2|Y| - 3$

> 2 : at most ~~$2|Y| - 4$~~ $|Y| - 2$

$|\Omega(X, Y)| \leq$ ~~$6|Y| - 9$~~ $4|Y| - 4$

We now assume that there is a closed curve going through Y that contains vertices of X .

Sphere-cut decompositions



Lemma

For an undirected connected planar graph G with $\mathbf{bw}(G) \geq 2$:

$$\mathbf{tww}(G) \leq \max(4\mathbf{bw}(G), \frac{9}{2}\mathbf{bw}(G) - 3)$$

An undirected graph with $\mathbf{bw}(G) \leq 1$ has $\mathbf{tww}(G) = 0$.

Bounds using previous results

Simple bound on the twin-width of a subgraph of $H \boxtimes P$:

$$5 \left(\frac{9 \cdot 7}{2} - 3 \right)^3 \leq 115746$$

But we could also use the fact that the subgraph is planar:

$$5(6 \cdot (3 \cdot 7) - 9) \leq 585$$

Using the embedding

To use our stronger neighbourhood complexity argument, we need an embedding.

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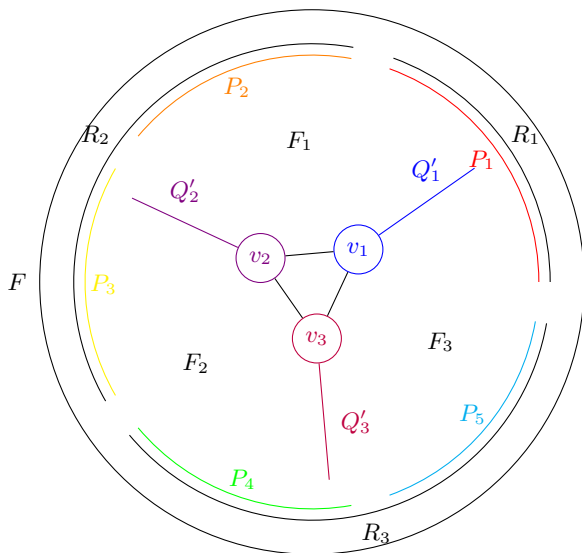
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For every edge $uv \in E$, $|d(u) - d(v)| \leq 1$

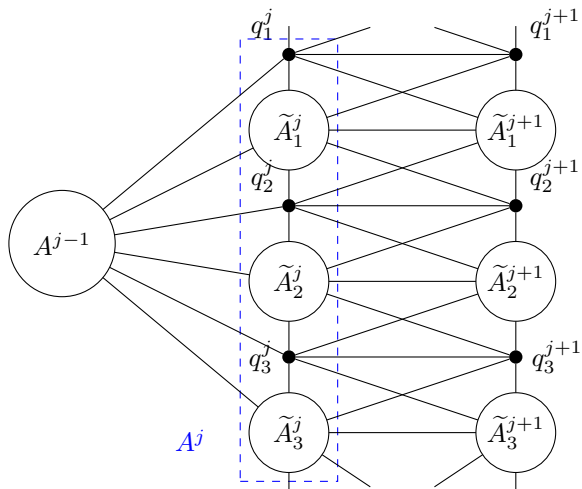
General planar case

Using the embedding



General planar case

Contraction sequence



We can conclude that twin-width of a planar graph is at most 183.

But also that its reduced-bandwidth is at most 131.

Our construction can also be translated to a polynomial time algorithm producing a contraction sequence with these guarantees.

Improving the planar bound

- ▶ Can we improve the decomposition further ?
- ▶ Can we improve the analysis of our constructions ?
- ▶ Can we improve the lower bound of 6 ?
- ▶ Using a different approach ?



V. Dujmovic, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood.

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CoRR, abs/2108.00198, 2021.